Summer-2019

WR4 Final Project

1 Goals:

We intend to implement the shallow water model for the torus. We use a exterior calculus approach to describe the equations and Fourier differential forms for the spectral decomposition of the main variables.

2 Shallow Water model on Torus.

In vector calculus notation the Euler equation in a fixed domain T is given by

$$\dot{\vec{u}} = \xi(\vec{k} \times \vec{u}) - \nabla \left(\frac{\vec{u} \cdot \vec{u}}{2} + Gh \right) \tag{1}$$

where $\vec{u}$ is the velocity of the fluid, $\xi = \nabla \times \vec{u}$ its vorticity, G is the gravity acceleration and h is the height of the fluid. For the exterior calculus interpretation of the variables we consider vorticity a 2-form $\xi(x, y) = \xi(x, y)\mathrm{d}x \wedge \mathrm{d}y$ (we are going to omit the volume form $\mathrm{d}x \wedge \mathrm{d}y$ when there is no risk of confusion) and the height h is a 0-form (scalar function). Using the flat operator the equation becomes

$$\dot{u}^\flat = - \star \xi \wedge \star u^\flat - \mathrm{d} \left(\frac{i_{\vec{u}} u^\flat}{2} + Gh \right). \tag{2}$$

Consider the Helmhotz decomposition

$$u^\flat = u^\flat_{\mathrm{d}} + u^\flat_\delta \tag{3}$$

where the exact part of u is $u^\flat_{\mathrm{d}} = \mathrm{d}(\star\chi)$, its co-exact part is $u^\flat_\delta = \delta\psi$, χ is the velocity potential 2-form and ψ is the stream function 2-form. These differential are related with the concepts of divergence D and vorticity ξ by

$$\begin{aligned} D &= \mathrm{d}(i_u vol) = \mathrm{d}(\star u^\flat) && (4) \\ &= \mathrm{d}(\star u^\flat_{\mathrm{d}} + \star u^\flat_\delta) = \mathrm{d}(\star u^\flat_{\mathrm{d}}) && (5) \\ &= \mathrm{d}(\star \mathrm{d}(\star\chi)) = \mathrm{d}\delta\chi && (6) \end{aligned}$$

$$= (\mathrm{d}\delta + \delta\mathrm{d})\chi = \Delta\chi \tag{7}$$

and

$$\xi = \Delta\psi = \mathrm{d}\delta\psi. \tag{8}$$

Applying the exterior derivative d and d$\star$ to the Euler equation one has the momentum equations for the shallow water model

$$\dot{\xi} = -g(\mathrm{d}(\star\xi), u^\flat) - \star\xi \wedge D \tag{9a}$$

$$\dot{D} = g(\mathrm{d}(\star\xi), \star u^\flat) + \star\xi \wedge \xi - \star\Delta\left(\frac{i_{\vec{u}}u^\flat}{2} + Gh\right) \tag{9b}$$

where g is the co-metric for 1-forms.

Finally, the vector calculus version of the continuity equation is

$$\dot{h} = -\nabla \cdot (h\,(\vec{u})) \tag{10}$$

$$= -\nabla h \cdot \vec{u} - h\nabla \cdot \vec{u} \tag{11}$$

which is translated to the exterior calculus notation as

$$\dot{h} = -\star\, d i_{h\vec{u}}(\mathrm{d}_{x_1} \wedge \mathrm{d}_{x_2}) \tag{12}$$

$$= -g(\mathrm{d}h, u) - h \wedge \star D. \tag{13}$$

Here we consider $Q = [0, 2\pi] \times [0, 2\pi]$ with identified edges to define the torus T. This interpretation of the problem avoid some issues with boundary conditions.

3 Fourier differential forms

We define our basis of differential forms as follows,

$$\alpha^{0,\delta}_{k_1,k_2}(x_1, x_2) = \frac{1}{(k_1^2 + k_2^2)^{\frac{1}{2}}}\, e^{i\,(k_1x_1 + k_2x_2)} \tag{14}$$

$$= \frac{1}{C_{k_1,k_2}}\, e^{i\,(k_1x_1 + k_2x_2)} \tag{15}$$

$$\alpha^{1,\mathrm{d}}_{k_1,k_2}(x_1, x_2) = \mathrm{d}\left(\alpha^{0,\delta}_{k_1,k_2}\right) \tag{16}$$

$$= \frac{1}{C_{k_1,k_2}}\, [(i\,k_1 e^{i\,(k_1x_1+k_2x_2)})\mathrm{d}x_1 + (i\,k_2 e^{i\,(k_1x_1+k_2x_2)})\mathrm{d}x_2] \tag{17}$$

$$\alpha^{1,\delta}_{k_1,k_2}(x_1, x_2) = \star\left(\alpha^{1,\mathrm{d}}_{k_1,k_2}\right) \tag{18}$$

$$= \frac{1}{C_{k_1,k_2}}\, [-(i\,k_2 e^{i\,(k_1x_1+k_2x_2)})\mathrm{d}x_1 + (i\,k_1 e^{i\,(k_1x_1+k_2x_2)})\mathrm{d}x_2] \tag{19}$$

$$\alpha^{2,\mathrm{d}}_{k_1,k_2}(x_1,x_2) = \mathrm{d}\left(\alpha^{1,\delta}_{k_1,k_2}\right) \tag{20}$$

$$= -C_{k_1,k_2}\, e^{i\,(k_1x_1+k_2x_2)}\mathrm{d}x_1 \wedge \mathrm{d}x_2 \tag{21}$$

where $k_1, k_2 \in \mathbb{Z}$. For the sake of completeness we define $\alpha^{0,\delta}_{0,0}(x_1,x_2) = 1$.

3.1 Some Properties:

(i) $\star\alpha^{\mathrm{d},2}_{k1,k2} = \star\mathrm{d}\star\mathrm{d}\alpha^{\delta,0}_{k1,k2} = \Delta\alpha^{\delta,0}_{k1,k2} = -(k_1^2+k_2^2)\alpha^{\delta,0}_{k1,k2}$

(ii) $\star\alpha^{\delta,0}_{k1,k2} = \frac{-1}{k_1^2+k_2^2}\star\star\alpha^{\mathrm{d},2}_{k1,k2} = \frac{-1}{k_1^2+k_2^2}\alpha^{\mathrm{d},2}_{k1,k2}$

(iii) $\delta\alpha^{d,2}_{k1,k2} = \star\mathrm{d}\star\alpha^{d,2}_{k1,k2} = -(k_1^2+k_2^2)\star\mathrm{d}\alpha^{\delta,0}_{k1,k2} = -(k_1^2+k_2^2)\alpha^{\delta,1}_{k1,k2}$

4 Tasks

1. Derive Eq. 9 from Eq. 2.
2. Explain the methods reconstructForm, reconstruct1Form and projectForm.
3. Implement the Shallow-Water equations Eq. 9 and Eq. 12 by completing the given skeleton code. Please complete the method infinitesimalAdvection that computes the derivatives of the main variables (vorticity, divergence and height).