Tensor Analysis and Applications 2019

# Exercise 8

Due 28/5/2019

1.) Let the flow map $F_t : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$F_t(x) = x + t\,v \tag{1}$$

for some fixed $v = (v^1, v^2, v^3)$. Formally compute the vector field that generates the flow.

2.) An (ideal) vortex in fluid dynamics is a spatially localized, radially symmetric vector field $v(x) \in \mathfrak{X}(\mathbb{R}^n)$. In two dimensions it can be described by

$$v(x) = \zeta(|x|) \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} \tag{2}$$

where $\zeta(|x|)$ is a radial profile, e.g.

$$\zeta(|x|) = \begin{cases} \cos\left(\frac{\pi}{2}\log_2\left(\frac{2|\xi|}{\pi}\right)\right) & \frac{\pi}{4} < |\xi| < \pi \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and $\theta$ is the angle to the positive $x$-axis.

i.) Plot the vector field $v(x)$.

ii.) Explicitly describe the integral curves of $v(x)$.

iii.) Explicitly describe the flow map $F_t$ generated by $v(x)$.

iv.) Describe the transport of the density

$$\rho(x) = \begin{cases} 1 & 0 \leq \theta \leq \text{ and } 1/2 \leq |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

by $v(x)$.