Tensor Analysis and Applications 2019

Exercise 7

Due 20/5/2019

1.) Let V be a vector space with basis $\{e_i\}$ and assume that this basis is also used in the fibers T_vV. Derive the transformation law for the velocity vectors $\dot{c}(t)$ of a curve

$$c(t) = \begin{pmatrix} c^1(t) \\ c^2(t) \\ c^3(t) \end{pmatrix} : [a, b] \to \mathbb{R}^3 \tag{1}$$

2.) In this exercise we will derive the property that characterizes the finite time transformations generated by Hamiltonian vector fields. Recall that a Hamiltonian vector field on an even-dimensional vector space V is given by

$$X_H^j = \omega^{ij}(\nabla H)_i \tag{2}$$

where $H : V \to \mathbb{R}$ is the energy function and $\omega \in T_2^0(V)$ the symplectic form, i.e. a covariant, non-degenerate and anti-symmetric tensor of rank 2. Let $z(t) : [a, b] \to V$ be the curve whose velocity vector is the Hamiltonian vector field, i.e.

$$\dot{z}(t) = X_H \tag{3}$$

Furthermore, let the "teleporting map" $\varphi : V \to V$ be

$$\bar{z}(t) = \varphi(z(t)), \tag{4}$$

where $z(t), \bar{z}(t)$ are different points along the curve $z(t)$, and let A_i^j be its coordinate representation.

i.) Why is the map φ independent of time?

ii.) Use the chain rule to write $\dot{\bar{z}}(t)$ as a function of $\dot{z}(t)$.

iii.) Use the definition of the Hamiltonian vector field to introduce $(\nabla H)_i$.

iv.) Relate the Hamiltonian vector fields at $z(t)$ and $\bar{z}(t)$ to write $\dot{\bar{z}}(t)$ only in terms of quantities at $\bar{z}$.

v.) Interpret the last expression geometrically, i.e. in a coordinate free language.