Tensor Analysis and Applications 2019

Exercise 13

Due 27/6/2019

1.) Let $\alpha = (3x_1^2 + 2x_2 - 1)\,\mathrm{d}x^1 + (7x - 2)\,\mathrm{d}x^2$. Integrate α over S^1.

2.) Let $\alpha \in \bigwedge^k(V)$. Show that $\Delta\alpha = 0$, i.e. α is harmonic, implies $\mathrm{d}\alpha = 0$ and $\delta\alpha = 0$.

3.) Let $\alpha \bigwedge^3(\mathbb{R}^3)$ and $U \subset \mathbb{R}^3$ with non-empty boundary $\mathbb{R}^3$ (e.g. U can be the unit ball in $\mathbb{R}^3$).

   i.) Derive the coordinate expression for Stokes' theorem for this setting.

   ii.) Show that the result is equivalent to the classical divergence theorem.

4.) Let $\mathrm{d}\theta$ be the standard 1-form basis function for S^1. Show that $\mathrm{d}\theta$ is harmonic.

   It is important to note that $\mathrm{d}\theta$ *is* not *exact. Here the notation might lead to confusion.*

5.) Let $v \in \mathfrak{X}(\mathbb{R}^2)$ the velocity field of a fluid and $v^\flat$ be the associated 1-form.

   i.) Apply the Hodge-Helmholtz decomposition to $v^\flat$.

   ii.) Find the names for the resulting potentials in the classical literature.[1]

[1] See for example A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics*, third ed. Springer, 1993.