Tensor Analysis and Applications 2019

Exercise 12

Due 24/6/2019

1.) Let $F = B_x \, \mathrm{d}y \wedge \mathrm{d}z + B_y \, \mathrm{d}z \wedge \mathrm{d}x + B_z \, \mathrm{d}x \wedge \mathrm{d}y + E_x \, \mathrm{d}x \wedge \mathrm{d}t + E_y \, \mathrm{d}y \wedge \mathrm{d}t + E_z \, \mathrm{d}z \wedge \mathrm{d}t \in \bigwedge^2(\mathbb{R}_t \times \mathbb{R}^3)$ be the Faraday 2-form in space-time, with E, B, corresponding to the electric and magnetic fields.

   i.) Compute $\mathrm{d} \star F$. (You can use results for the Hodge dual of the basis functions in space-time from the literature).

   ii.) Identify the resulting expression with the classical vector calculus form of Maxwell's equations.

2.) Compute the Hodge dual of all differential form basis functions in $\mathbb{R}^2$.

3.) The classical continuity equation is[1]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, u) = 0 \tag{1}$$

where u is some vector field along which the the quantity ρ is transported. Re-write the equation using the operators of exterior calculus.

4.) Let $\eta : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism and $\mu = \mu \, \mathrm{d}x^1 \wedge \mathrm{d}x^2$ a volume form.[2]

   i.) Compute the pullback $\eta^* \mu$ of a volume form $\mu = \mu \, dx^1 \wedge dx^2$.

   ii.) The resulting coordinate expression can be identified with a classical functional of the Jacobian matrix. Which functional is this?

5.) Let $\psi \in \bigwedge^2(\mathbb{R}^2)$. Compute $\delta\psi \in \bigwedge^1(\mathbb{R}^2)$ where $\delta = \star \mathrm{d} \star$ is the codifferential.

6.) Let $\alpha = (3x_1^2 + 2x_2 - 1) \, \mathrm{d}x^1 + (7x - 2) \, \mathrm{d}x^2$. Integrate α over S^1.

[1] Cf. https://en.wikipedia.org/wiki/Continuity_equation

[2] Consult [MRA, Ch. 7.2] if you have trouble with the question.