Heat equation

→ Simple partial differential equation that describes how heat propagates through space

$f(x,t)$ — temperature at x at time t

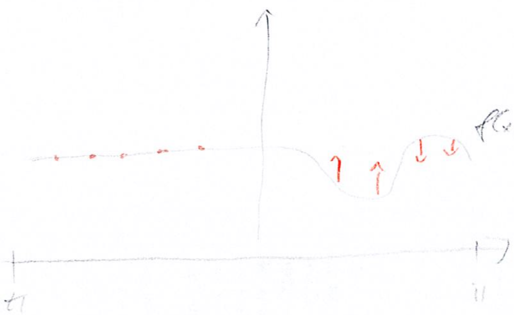
Heat equation

Follows from energy conservation and Fourier's law $\frac{dE}{dt} = -k \nabla f$

α → material constant

$$\frac{\partial f(x,t)}{\partial t} = \alpha \underbrace{\frac{\partial^2}{\partial x^2}}_{= \Delta} f(x,t)$$

→ change in time (at fixed x) is controlled by spatial behaviour of f (measured via curvature) — characteristic for partial diff. equation



How would $\frac{\partial f(x,t)}{\partial t}$ look like

We want to know heat distribution after 10 sec or 1 min or 1 hour

→ Why does one need numerical simulations?

↳ Effectively only means to obtain quantitative information about solution

↳ Proper discretization in general still open research problem.

→ We would like to exploit that we have eigenfuncts of differential operators

↳ We should be able to diagonalize this equation

Diagonalizing the heat equation

Ansatz

$$f(x,t) = \underbrace{\sum_{m\in\mathbb{Z}} f_m(t)}\, e^{imx}$$

coefficients have to be time-dependent
↳ think of time dependent vector in $\mathbb{R}^2$

→ Insert into heat equation

$$\frac{\partial}{\partial t} f(x,t) = \frac{\partial}{\partial t} \sum_{m\in\mathbb{Z}} f_m(t)\, e^{imx}$$

$$= \sum_{m\in\mathbb{Z}} \left(\frac{\partial}{\partial t} f_m(t)\right) e^{imx}$$

→ $$\Delta f(x,t) = \frac{\partial^2}{\partial x^2} \sum_m f_m(t)\, e^{imx}$$

$$= \sum_m f_m(t) \underbrace{(im)^2}_{= -m^2} e^{imx}$$

⇒ $$\sum_{m\in\mathbb{Z}} \left(\frac{\partial}{\partial t} f_m(t)\right) e^{imx} = \sum_m f_m(t)(-m)\, e^{imx}$$

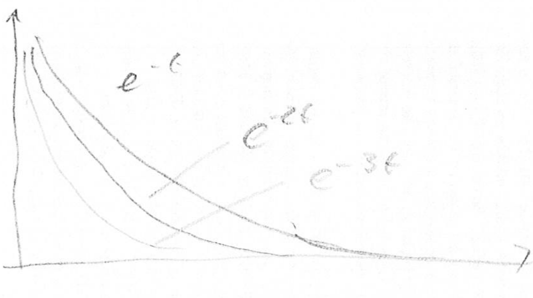
$$\underbrace{\frac{\partial}{\partial t} f_m(t) = -m\, f_m(t)}$$

Diagonalization of equation
↳ The evolution of m^{th} coeff only depends on current value

→ This has the form

$$\frac{d}{dx} f(x) = \alpha f(x) \quad \to \quad f(x) = e^{\alpha x}$$

⇒ $f_m(t)$ has to have the form $f_m(t) = e^{-mt}$



$\to f(x,t) = \sum \underbrace{f_n(t)}\, e^{inx}$

$f_n(t)$ decays faster if n is large

$\Rightarrow f(x,t)$ gets "smoother over time"

$\to$ What happens if $t \to \infty$