Mathematical Tools for Computer Graphics 2017

Tutorial 9

In this tutorial we will investigate functions on the sphere. Our motivation is the shading equation

$$\bar{\ell}_x(\bar{\omega}) = \int_{H_x^2} \left(\ell_x(\omega) \cos\theta\right) \, \rho_x(\omega, \bar{\omega}) \, d\omega \tag{1}$$

that describes how light is reflected off a surface, where $\ell_x(\omega)$ is the incident light intensity at x from direction $\omega \in H_x^2$ in the hemisphere above x, $\bar{\ell}_x(\bar{\omega})$ is the outgoing one, and $\rho_x(\omega, \bar{\omega})$ is the shading kernel (or BRDF) that determines the fraction of incident light from ω that is scattered towards $\bar{\omega}$.

A common choice for representing the functions in the shading equation are spherical harmonics, which are the generalization of the Fourier series to the sphere S^2 and form an orthonormal basis for $L_2(S^2)$. The basis functions are given by

$$y_{lm} = C_{lm} \, P_l^m(\cos\theta) \, e^{im\phi} = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \, P_l^m(\cos\theta) \, e^{im\phi} \tag{2}$$

with $l = 0, 1, \cdots$ and $-l \leq m \leq l$ and where (θ, ϕ) are the standard coordinates on S^2 with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The P_l^m are associated Legendre polynomials, which coincide with regular Legendre polynomials when $m = 0$.

1.) Plot the associated Legendre polynomials $P_l^m(z)$ for $l \in \{0, 1, 2, 3, 4\}$ with one plot for each l. Use that

$$P_l^{-m}(z) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(z) \tag{3}$$

Taking the factor of $e^{im\phi}$ into account, how do the oscillations for fixed l change as $|m|$ increases from 0 to l? What happens as l increases?

Solution: For fixed l, the oscillations are shifted from the θ direction to the ϕ direction with the total oscillations being constant. As l increases the functions become more and more oscillatory, in analogy to the Fourier series.

2.) Implement the function `ylm()` in the provided skeleton code. Then use `plot_spherical_harmonics.py()` to plot the spherical harmonics for $l = 3$ and $m = 0, 1, 2, 3$.

3.) The Phong BRDF is one of the most popular shading functions used in rendering. It is given by

$$\rho(\omega, \bar{\omega}) = (r(\omega) \cdot \bar{\omega})^p \tag{4}$$

where ω is the incident direction of light and $\bar{\omega}$ the outgoing one, and $r(\omega)$ is the direction of perfect specular reflection with respect to the local surface normal n. The exponent $p > 0$ controls the specularity that increases as p increases. W.l.o.g, let $\omega = n$. Compute the spherical harmonics coefficients for the Phong BRDF for $p = 16$ and $p = 32$.

Solution: The reflection vector is given by

$$r = -\omega + 2\langle \omega, n \rangle n \tag{5}$$

Hence, when $\omega = n$ then $r = \omega$. The spherical harmonics coefficients are thus

$$\rho_{lm} = \big\langle \rho_\omega(\bar{\omega}), y_{lm}(\bar{\omega}) \big\rangle \tag{6a}$$

$$= \int_{H^2} \rho_\omega(\bar{\omega}), y_{lm}(\bar{\omega}) \, d\bar{\omega} \tag{6b}$$

$$= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (\cos\theta)^p \, y_{lm}(\theta, \phi) \, \sin\theta \, d\theta \, d\phi. \tag{6c}$$

The integral has an analytic solution. Only the $m = 0$ coefficients are non-zero since otherwise the integral over ϕ vanishes. The integral over θ then only involves regular Legendre polynomials. These can be written as

$$P_l(\cos\theta) = 2^l \sum_{k=0}^{l} \binom{l}{k} \binom{(l+k-1)/2}{l} \cos\theta^k. \tag{6d}$$

Using linearity we thus have

$$\rho_{lm} = 2\pi \, 2^l \sum_{k=0}^{l} \binom{l}{k} \binom{(l+k-1)/2}{l} \int_{\theta=0}^{\pi/2} \cos\theta^k \, \cos\theta^p \sin\theta \, d\theta \tag{6e}$$

The remaining integral over θ is easily resolved because

$$\frac{d}{d\theta} \left(-\frac{1}{n+1} \cos\theta^{n+1} \right) = \frac{n+1}{n+1} \cos\theta^n \sin\theta = \cos\theta^n \sin\theta \tag{6f}$$

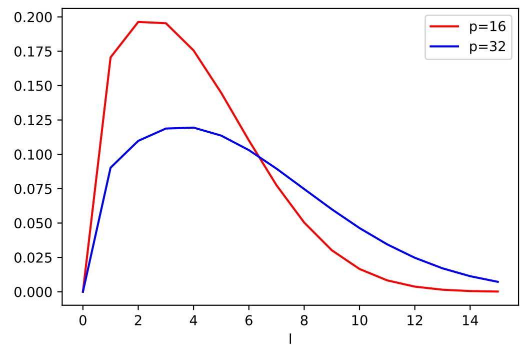

Figure 1: Spherical harmonics coefficients for Phong BRDF; only $m = 0$ coefficients are non-zero.

which follows from the chain rule. Thus, the anti-derivative for the integral is $\frac{1}{n+1} \cos \theta^{n+1}$ with $n = k + p$. Inserting we obtain

$$\rho_{lm} = 2\pi \, 2^l \sum_{k=0}^{l} \binom{l}{k} \binom{(l+k-1)/2}{l} \frac{1}{1+k+p} \tag{6g}$$

See Fig. 1 for resulting coefficients. Apparent is the decay as l increases that also follows from the last equation above.