Mathematical Tools for Computer Graphics 2017

Tutorial 6

In this tutorial we will study how to work numerically with frames.

1.) The Mercedes-Benz frame is given by

$$u_1 = \big(0.0, 1.0\big)^T \tag{1a}$$

$$u_2 = \big(\sqrt{3}/2, -1/2\big)^T \tag{1b}$$

$$u_3 = \big(-\sqrt{3}/2, -1/2\big)^T \tag{1c}$$

Use the function `plotVectors2D()` in the provided source code to plot the frame.

Solution: See Fig. 1.

2.) Construct the (canonical) dual frame to the Mercedes Benz frame given in Eq. 1. Plot the dual frame vectors.

Solution: The dual can be constructed as

$$\tilde{u}_i = S^{-1} u_i \tag{2}$$

where $S = BB^T$ is the frame operator. See Fig. 1 for a visualization.

3.) The Mercedes Benz frame in Eq. 1 is an example of a tight frame. These can be considered as generalizations of orthogonal bases. Explain the last statement.

Solution: For an orthogonal basis the dual basis vectors are, up to a rescaling, identical to the primary basis vectors. This property also holds for tight frames.

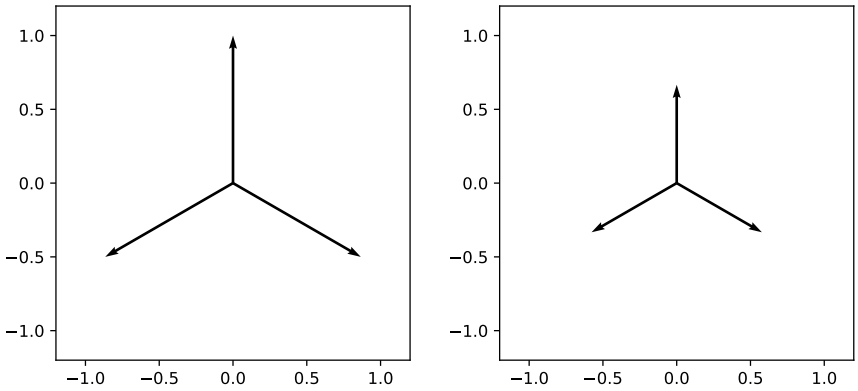

Figure 1: The Mercedes Benz frame (left) and its dual (right).

4.) The overcomplete generalization of an orthonormal basis is a Parseval (tight) frame. Modify the Mercedes Benz frame in Eq. 1 so that it becomes Parseval tight.

Solution: The dual vectors of the tight frame in Eq. 1 are given by

$$\tilde{u}_i = \frac{2}{3} u_i. \tag{3}$$

This scaling factor has to be "distributed" equally onto the primary and dual frame vectors. This yields for the Parseval frame vectors

$$\bar{u}_i = \sqrt{\frac{2}{3}} u_i. \tag{4}$$

Incidentally, $2/3$ is the reciprocal redundancy of the frame and the reciprocal frame bound. This holds more generally for any unit norm tight frame.

5.) In the following we will study how the redundancy of a frame affects the reconstruction error that can be attained. Consider $\mathbb{R}^{100}$ and generate random frames with a redundancy ranging from 1 to 3. For each redundancy determine the average reconstruction error for a set of random vectors in the space. Plot the average errors as a function of the redundancy. How could the reconstruction error for fixed redundancy be improved?

Solution: See Fig. 2.

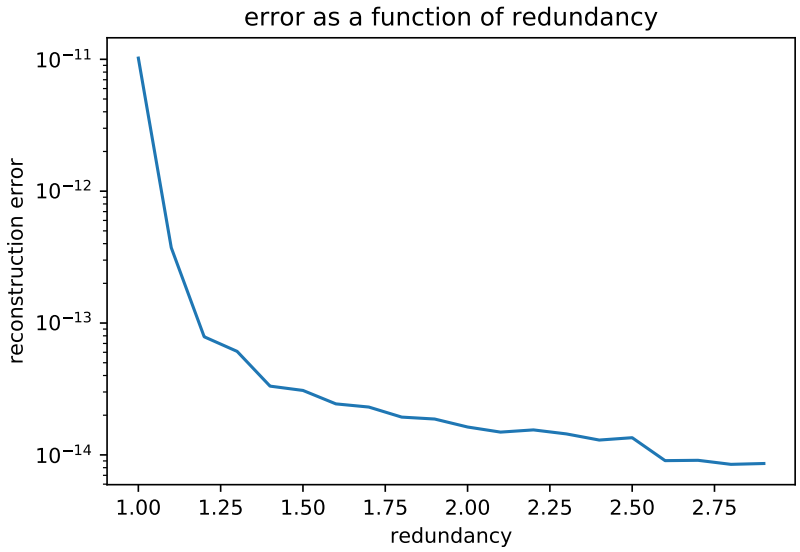

Figure 2: Average reconstruction error for random vectors in $\mathbb{R}^{100}$ as a function of the redundancy of random frames.