Mathematical Tools for Computer Graphics 2017

Tutorial 5

In this tutorial we will consider an important application of the Riesz representation theorem. Specifically, we will study the point evaluation functional $\delta_{\bar{x}}(\cdot)$ for function spaces where it is continuous. Such spaces are known as Reproducing Kernel Hilbert Spaces (RKHS) and they play a critical role for interpolation and integration.

1.) Show that the point evaluation functional $\delta_{\bar{x}}(\cdot)$ is linear. Why does this property hold?

Solution: Linearity holds since:

$$\delta_{\bar{x}}(f+g) = \delta_{\bar{x}}(f) + \delta_{\bar{x}}(g) = f(\bar{x}) + g(\bar{x}) \tag{1a}$$

$$\delta_{\bar{x}}(af) = a\delta_{\bar{x}}(f) = af(\bar{x}) \tag{1b}$$

where the right hand side holds by definition of the addition and scalar multiplication for function spaces. The functional is naturally compatible with the linear structure of the space, which is defined pointwise.

2.) Apply the Riesz representation theorem to the point evaluation functional $\delta_{\bar{x}}(\cdot)$ assuming it is continuous in a Hilbert space $\mathcal{H}$.

Solution: The Riesz representation theorem establishes that for every $\delta_{\bar{x}}(\cdot)$, with different $\bar{x}$ yielding different functionals, there exists a function $k_{\bar{x}} \in \mathcal{H}$ such that

$$f(\bar{x}) = \delta_{\bar{x}}\big(f\big) = \big\langle k_{\bar{x}}(x), f(x) \big\rangle \tag{2}$$

for all $f \in \mathcal{H}$. The function $k_{\bar{x}}(x)$ is known as *reproducing kernel* .

3.) Consider the Hilbert space $\mathcal{H}_\chi^n([0,1]) \subset L_2([0,1])$ given by

$$\mathcal{H}_\chi^n([0,1]) = \operatorname{span}\Big(\big\{\chi_i(x)\big\}_{i=1}^n\Big)$$

where $\chi_i(x)$ is the characteristic function

$$\chi_i(x) = \begin{cases} 1 & x \in [(i-1)/n, i/n) \\ 0 & \text{otherwise} \end{cases}$$

equipped with the standard L_2 inner product. Is the point evaluation functional for $\mathcal{H}_\chi^n([0,1])$ continuous?

Solution: For continuity to hold we have to have that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\|f - \bar{f}\| < \delta \;\Rightarrow\; |\delta_{\bar{x}}(f) - \delta_{\bar{x}}(\bar{f})| < \epsilon \tag{3}$$

for all $\bar{x} \in [0,1]$.

Let us fix an arbitrary $\bar{x}$ and an arbitrary $\epsilon > 0$. Furthermore, f and $\bar{f}$ as well as $f - \bar{f}$ can be written using their respective basis representations using the characteristic basis. Then

$$\epsilon > |\delta_{\bar{x}}(f) - \delta_{\bar{x}}(\bar{f})| \tag{4a}$$

$$= \left| \delta_{\bar{x}} \left(\sum_{i=1}^n f_i \, \chi_i(x) \right) - \delta_{\bar{x}} \left(\sum_{i=1}^n \bar{f}_i \, \chi_i(x) \right) \right| \tag{4b}$$

and by linearity we have

$$\epsilon > \left| \sum_{i=1}^n f_i \, \delta_{\bar{x}} \left(\chi_i(x) \right) - \sum_{i=1}^n \bar{f}_i \, \delta_{\bar{x}} \left(\chi_i(x) \right) \right| \tag{4c}$$

and since $\delta_{\bar{x}}\left(\chi_i(x)\right) = 1$ when $\bar{x}$ is in the support of the j-th basis function and zero otherwise we have

$$\epsilon > \left| f_j - \bar{f}_j \right| . \tag{4d}$$

On the other hand, the difference of f and $\bar{f}$ is given by

$$\|f - \bar{f}\| = \|\mathrm{f} - \bar{\mathrm{f}}\| \tag{4e}$$

where $\mathrm{f} = (f_1, \cdots, f_n)$ is the vector of basis function coefficients and the norm on the right hand side is the ℓ_2 norm, i.e.

$$\|\mathrm{f} - \bar{\mathrm{f}}\| = \left(\sum_{i=1}^n \left(f_i - \bar{f}_i \right)^2 \right)^{1/2} . \tag{4f}$$

By our derivation $\delta_{\bar{x}}(\bar{f})| < \epsilon$ means that the basis function coefficients f_j and $\bar{f}_j$ differ by less than epsilon, which can indeed by satisfied for arbitrary

$\epsilon > 0$. One can now for example choose $|f_i - \bar{f}_i| < \sqrt{\epsilon}/N$ for all i, so that $\delta = \epsilon$, for Eq. 3 to hold.

In fact, it can be shown that the point evaluation functional is continuous in *every* finite dimensional function space. This is of considerable importance numerically where we are inherently confined to finite dimensional spaces.

4.) Let $\mathcal{H}(X)$ be a reproducing kernel Hilbert space, i.e. a Hilbert space where the point evaluation functional is continuous, and $\{\varphi_i\}_{i=1}^n$ an orthonormal basis for the space. Derive the basis expansion for the reproducing kernel $k_{\bar{x}}(x)$ with respect to $\{\varphi_i\}_{i=1}^n$.

Solution: By definition the basis function coefficients $k_{\bar{x}}^i$ are:

$$k_{\bar{x}}^i = \big\langle k_{\bar{x}}(x), \varphi_i(x) \big\rangle. \tag{5a}$$

Applying the reproducing (or point evaluation) property of $k_{\bar{x}}(x)$ immediately yields

$$k_{\bar{x}}^i = \big\langle k_{\bar{x}}(x), \varphi_i(x) \big\rangle = \varphi_i(\bar{x}). \tag{5b}$$

5.) Using Numpy, construct a reproducing kernel $k_{\bar{x}}(x)$ for the space $P_5([-1,1])$ spanned by the first five Legendre polynomials (note that Legendre polynomials are a priori *not* orthonormal). Plot the reproducing kernel and verify the reproducing property numerically.

Solution: See Fig. 1.

6.) The color of visible light is described by its spectrum, which is a function

$$f : [400 \text{ nm}, 700 \text{ nm}] \to \mathbb{R}$$

that describes the energy at each wavelength in $[400 \text{ nm}, 700 \text{ nm}]$. In computer graphics, the spectrum is typically represented by three discrete values f_r, f_g, f_b, which are associated with red, green, and blue, respectively. Can these three values reproduce f? If yes, how can this be accomplished?

Solution:

- f_i are in general basis function coefficients
- They can either be interpreted as general coefficients or as pointwise samples. The latter one is how rendering systems commonly work since f_i is typically interpreted as the amount of light in the respective color.
- Then f can in principle be reconstructed from the values using the basis representation.

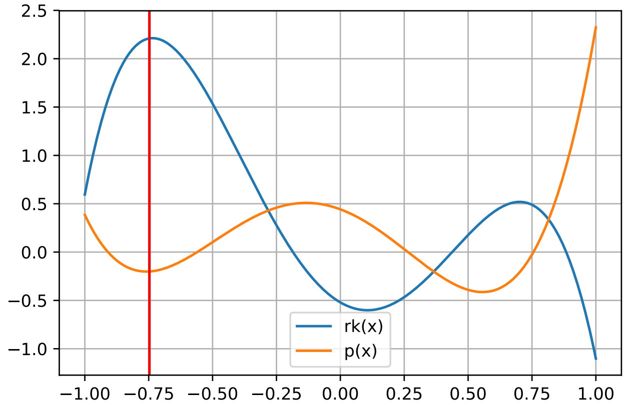

Figure 1: Reproducing kernel for $P_5([-1, 1])$ for $\bar{x} = -0.747$.

- Critical question: what is the function space that contains f. Is is, at least to good approximation, three dimensional?