Mathematical Tools for Computer Graphics 2017

Tutorial 10/11

In this tutorial we will consider solution strategies for the shading equation

$$\bar{\ell}_x(\bar{\omega}) = \int_{H_x^2} \left(\ell_x(\omega) \cos\theta\right) \, \rho_x(\omega, \bar{\omega}) \, d\omega \tag{1}$$

where $\ell_x(\omega)$ is the incident light intensity at x from direction $\omega \in H_x^2$ in the hemisphere above x, $\bar{\ell}_x(\bar{\omega})$ is the outgoing one, and $\rho_x(\omega, \bar{\omega})$ is the shading kernel (or BRDF) that determines the fraction of incident light from ω that is scattered towards $\bar{\omega}$.

1.) Precomputed radiance transfer[1] is a popular technique to attain global illumination effects in real-time that derives from the Galerkin projection of the shading equation.

To obtain an approximate solution to the shading equation that can be computed in real-time, one only computes the diffuse component of the exitant light intensity and assumes the light sources to be (very) distant so that they can be assumed to not dependent on the location x. The shading equation then takes the form

$$\bar{\ell}(\bar{\omega}) = \rho \int_{H_x^2} \ell_x(\omega) \cos\theta \, d\omega \tag{2}$$

where the incoming light intensity depends on those on the light source $h(\omega)$ as

$$\ell_x(\omega) = V_x(\omega) h(-\omega) \tag{3}$$

with $V_x(\omega) \in \{0, 1\}$ being the visibility function.

[1] Sloan, Kautz, and Snyder, "Precomputed Radiance Transfer for Real-Time Rendering in Dynamic, Low-Frequency Lighting Environments"; Kautz, Sloan, and Snyder, "Fast, Arbitrary BRDF Shading for Low-Frequency Lighting using Spherical Harmonics".

i) Based on the Galerkin projection of the shading equation using spherical harmonics, derive an equation that determines the outgoing light intensity in precomputed radiance transfer. What is the complexity of the computations for each shading point?

Solution: To obtain an equation that can be resolved efficiently at runtime, one should separate the factors that dependent on the shading point x and those that do not,

$$\bar{\ell}(\bar{\omega}) = \rho \int_{H_x^2} h(-\omega) \big(V_x(\omega) \cos\theta \big) \, d\omega. \tag{4a}$$

Representing $h(-\omega)$ and the cosine-weighted visibility $t(\omega) \equiv V_x(\omega)\cos\theta$ in spherical harmonics yields

$$h(-\omega) \approx \sum_{l=0}^{L} \sum_{m=-l}^{l} h_{lm} \, y_{lm}(\omega) \tag{4b}$$

$$t(\omega) \approx \sum_{l=0}^{L} \sum_{m=-l}^{l} t_{lm} \, y_{lm}(\omega). \tag{4c}$$

Inserting we obtain

$$\bar{\ell} = \rho \int_{H_x^2} \left(\sum_{l=0}^{L} \sum_{m=-l}^{l} h_{lm} \, y_{lm}(\omega) \right) \left(\sum_{l'=0}^{L} \sum_{m=-l'}^{l} t_{l'm'} \, y_{l'm'}(\omega) \right) d\omega$$

and using linearity yields

$$\bar{\ell} = \rho \sum_{l=0}^{L} \sum_{m=-l}^{l} \sum_{l'=0}^{L} \sum_{m=-l'}^{l} h_{lm} \, t_{l'm'} \int_{H_x^2} y_{lm}(\omega) \, y_{l'm'}(\omega) d\omega. \tag{4d}$$

By the orthonormality of the spherical harmonics this equals

$$\bar{\ell} = \rho \sum_{l=0}^{L} \sum_{m=-l}^{l} h_{lm} \, t_{lm}. \tag{4e}$$

The last equation is the dot product of the coefficient vectors. For every shading point one has thus $L+1$ multiplications and $L-1$ additions to compute the exitant light intensity.

i) Compute the projection of the "Uffizi" light probe, see Fig. 1, which represent $\ell(\omega)$ using a (u,v) parametrization of S^2 with

$$\theta = \pi \, v \quad , \quad v \in [0,1] \tag{5a}$$

Figure 1: Uffizi light probe.

Figure 2: Uffizi light probe after projection into spherical harmonics with $L = 5$.

$$\phi = 2\,\pi\,u \quad , \quad u \in [0, 1] \tag{5b}$$

into spherical harmonics with $L = 5$ and reconstruct a suitable representation to assess the quality of the approximation.[2]

Solution: See Fig. 2.

i) What visual effects are afforded by precomputed radiance transfer?

Solution: Precomputed radiance transfer yields an occlusion and orientation dependent modulation of the light intensity.

[2]Original image from `http://gl.ict.usc.edu/Data/HighResProbes/`.

i) How does the equation for precomputed radiance transfer change when one considers a view dependent scattering function. How does this affect the computations?

Solution: Instead of a dot product at every shading point one obtains a matrix-vector product. Furthermore, the exitant light intensity into direction $\bar{\omega}$ needs to be reconstructed from the basis representation.

i) Read the paper by Sloan, Kautz and Snyder[3] that introduced precomputed radiance transfer.

2.) When one fixes the outgoing direction $\bar{\omega}$ then the shading equation is an integral. Since the integrand is complex and typically not even an analytic description is available one has to resort to numerical quadrature rules to determine the integral.

A quadrature rule takes, by definition, the form

$$\int_X f(x)\,dx = \sum_{i=1}^{n} w_i\, f(x_i) \tag{6}$$

with w_i being suitable weights and the x_i being quadrature nodes. where the equality holds only under appropriate conditions on the integrand that not necessarily satisfied in practice. Eq. 6 is very general since it only requires one to know the integrand at pointwise values $f(x_i)$.

In this task we will develop a very general methodology to obtain quadrature rules.

i) Assume the integrand $f(x)$ is given in an orthonormal basis, i.e.

$$f(x) = \sum_{i=1}^{n} f_i\, \phi_i(x) = \sum_{i=1}^{n} \langle f(y), \phi_i(y) \rangle\, \phi_i(x) \tag{7}$$

Use the basis representation to obtain an expression for the integral involving the basis function coefficients f_i. How can this be interpreted as a quadrature rule?

Solution: Inserting the basis representation for $f(x)$ and using linearity we immediately obtain

$$\int_X f(x)\,dx = \int_X \left(\sum_{i=1}^{n} f_i\, \phi_i(x) \right) dx \tag{8a}$$

[3] Sloan, Kautz, and Snyder, "Precomputed Radiance Transfer for Real-Time Rendering in Dynamic, Low-Frequency Lighting Environments".

$$= \sum_{i=1}^{n} f_i \underbrace{\int_X \phi_i(x)\, dx}_{\equiv w_i} . \tag{8b}$$

ii) Using an orthonormal basis we obtain an integration rule involving basis function coefficients f_i. Quadrature rules, however, use pointwise values. We saw previously that point evaluation functionals $\delta_{\bar{x}} : \mathcal{H} \to \mathbb{R}$ connect functions and their pointwise values. Furthermore, under suitable conditions these have a representation using reproducing kernels $k_{\bar{x}}(x)$,

$$f(\bar{x}) = \delta_{\bar{x}}(f) = \langle f(x), k_{\bar{x}}(x) \rangle \tag{9}$$

Recall the conditions for the existence of a reproducing kernel. Is it possible to construct a basis whose basis functions are reproducing kernels? If this is the case, how would the solution to the integration problem look like?

Solution: When H is a Hilbert space where the point evaluation functional is continuous then, by the Riez representation theorem, it has a representation as a reproducing kernel.

The reproducing kernel $k_{\bar{x}}(x)$ is a function in H and, moreover, different locations $\bar{x}$ yield different functions. We saw this when we constructed reproducing kernels for the space of polynomials. Hence, with $m \geq n$ different locations λ_k, and assuming corresponding functions are linearly independent, these could provide a basis for H. Any function $f \in H$ would then have the representation

$$f(x) = \sum_{i=1}^{n} \big\langle f(y), k_{\lambda_i}(y) \big\rangle \tilde{k}_i(x) \tag{10a}$$

$$= \sum_{i=1}^{n} f(\lambda_i)\, \tilde{k}_i(x) \tag{10b}$$

where the $\tilde{k}_i(x)$ the dual basis functions and the second line follows by the reproducing property of the $k_{\lambda_i}(y)$.

Following the same ansatz as with an orthonormal basis we obtain

$$\int_X f(x)\, dx = \int_X \left(\sum_{i=1}^{n} f(\lambda_i)\, \tilde{k}_i(x) \right) dx \tag{11a}$$

$$= \sum_{i=1}^{n} f(\lambda_i) \int_X \tilde{k}_i(x)\, dx \tag{11b}$$

$$= \sum_{i=1}^{n} w_i\, f(\lambda_i) \tag{11c}$$

with the weights w_i are given by the integrals of the dual basis functions. Eq. 11c provides a quadrature rule in the sense of Eq. 6 and all classical quadrature rules, such as Gauss-Legendre quadrature, can be obtained as outlined by first constructing a quadrature rule for the space under consideration and then using linearity of the integral.

iii) Let $\mathcal{P}_4([-1,1])$ be the space of polynomials up to degree 4 with the Legendre polynomials as orthonormal basis. Show experimentally that for $\{\lambda_i\}_{i=1}^5$ being uniformly distributed points in $[0,1]$ the set

$$\{k_{\lambda_i}(x)\}_{i=1}^5 \tag{12}$$

spans $\mathcal{P}_4([-1,1])$. Use that the reproducing kernel can be written as

$$k_{\bar{x}}(x) = \sum_{i=1}^{n} \phi_i(\bar{x})\, \phi_i(x) \tag{13}$$

where $\phi_i(x)$ is an orthonormal basis for the space under consideration. Begin by showing the last formula.[4]

Solution: We have

$$f(x) = \sum_{i=1}^{n} f_i\, \phi_i(x) \tag{14a}$$

$$= \sum_{i=1}^{n} \big\langle f(y), \phi_i(y) \big\rangle\, \phi_i(x) \tag{14b}$$

and by using linearity we obtain

$$f(x) = \left\langle f(y)\,,\, \sum_{i=1}^{n} \phi_i(y)\, \phi_i(x) \right\rangle \tag{14c}$$

Comparing to the definition of a reproducing kernel in Eq. 9 we immediately see that the basis representation of the reproducing kernel is Eq. 13 where the $\phi_i(\bar{x})$ are the basis function coefficients.

That the $k_{\lambda_i}(x)$ span the space can be verified by constructing the basis matrix

$$K = \begin{pmatrix} P_0(\lambda_1) & \cdots & P_4(\lambda_1) \\ \vdots & \ddots & \vdots \\ P_0(\lambda_5) & \cdots & P_4(\lambda_5) \end{pmatrix} \tag{15}$$

and checking that it has a non-vanishing determinant.

[4] We already worked with reproducing kernels for $\mathcal{P}_4([-1,1])$ in Tutorial 5.

iv) Derive a quadrature formula for $\mathcal{P}_4([-1,1])$ of the form

$$\int_X f(x)\,dx = \sum_{i=1}^{n} w_i\, f(\lambda_i) \tag{16}$$

where the $\{\lambda_i\}_{i=1}^5$ are the locations from the last question.

Solution: Since the $k_{\lambda_i}(x)$ span $\mathcal{P}_4([-1,1])$ there exist dual kernel functions $\tilde{k}_i(x)$ such that

$$f(x) = \sum_{i=1}^{n} \left\langle f(y), k_{\lambda_i}(y) \right\rangle \tilde{k}_i(x) = \sum_{i=1}^{n} f(\lambda_i)\, \tilde{k}_i(x) \tag{17}$$

with the crucial aspect being that the inner product determining the basis function coefficients is trivial and yields $f(\lambda_i)$. As for any biorthogonal basis the dual kernel functions can be obtained by the inverse of the kernel matrix in Eq. 15 whose columns are the coefficients of $\tilde{k}_i(x)$ with respect to the Legendre polynomials.

Inserting the basis representation into the integral and using linearity yields the desired quadrature formula with

$$w_i = \int_{-1}^{1} \tilde{k}_i(x)\,dx, \tag{18}$$

cf. Eq. 11.

v) Interpolation is the problem of determining a function $f(x)$ from pointwise values $f(x_i)$. The classical approach is via the Vandermonde matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & \ddots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \tag{19a}$$

that determines the coefficients a_i of a polynomials $f(x) = \sum_{i=1}^n a_i\, x^i$ interpolating the values $f(x_i)$ by

$$V\, a = f \tag{20}$$

with $a = (a_1, \cdots, a_n)$ and $f = (f(x_1), \cdots, f(x_n))$.

A second classical approach uses Lagrange polynomials

$$\ell_j(x) = \prod_{m \neq j} \frac{x - x_m}{x_j - x_m} \tag{21a}$$

so that the polynomial $f(x)$ interpolating the $f(x_i)$ is given by

$$f(x) = \sum_{i=1}^{n} f(x_i)\, \ell_i(x). \tag{21b}$$

How are these interpolation techniques related to reproducing kernel bases that yield a basis representation of the form

$$f(x) = \sum_{i=1}^{n} f(\lambda_i)\, \tilde{k}_i(x)\,? \tag{22}$$

Solution: The Lagrange polynomials $\ell_j(x)$ are just a representation for the reproducing kernel. The Vandermonde matrix is its basis representation with respect to monomials. Hence, these classical approaches to interpolation are also special instances of interpolation.

References

Kautz, J., P.-P. Sloan, and J. Snyder. "Fast, Arbitrary BRDF Shading for Low-Frequency Lighting using Spherical Harmonics". In: *EGSR 2002: Proceedings of the 13th Eurographics Workshop on Rendering*. Aire-la-Ville, Switzerland, Switzerland: Eurographics Association, 2002, pp. 291–296.

Sloan, P.-P., J. Kautz, and J. Snyder. "Precomputed Radiance Transfer for Real-Time Rendering in Dynamic, Low-Frequency Lighting Environments". In: *Proceedings of ACM SIGGRAPH 2002*. New York, NY, USA: ACM Press, 2002, pp. 527–536.