Mathematical Methods

for Computer Graphics

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“It is through science that we prove,
but through intuition that we discover.”

Henri Poincaré

Abstract

Mathematics plays an important role in computer graphics. These notes ...

We don't learn how to prove things.

Scientists approach: use mathematics to express ideas and model things ... and through this to solve the problems of interest.

What we do not talk about: - measure theory. - probabilistic techniques. - We also only discuss groups and Lie groups in passing.

Reproducible science: like in other science also in computer science experiments should be reproducible. Since we have only very few and highly standardized experimental setups, known as programming languages or environments, this is simpler than in any other science. For all experiments that are presented in the following source code is provided and the reader is encouraged to experiment with the experiments her self.[1]

[1] Source code is available at `http://www.dgp.toronto.edu/people/lessig/teaching/math_for_cg/`. See (Donoho, Maleki, Rahman, Shahram, and Stodden, "Reproducible Research in Computational Harmonic Analysis") for a more detailed discussion on reproducible computational science.

Contents

Chapter 1

Linear Algebra

Linear algebra is an elementary pillar of computer graphics, as is evident by the central place it takes in introductory computer graphics classes. In the following, however, we will consider some aspects that are usually not emphasized in computer graphics but that play important roles for us in subsequent chapters.

1.1 The Fabric of Vector Spaces

In the last lecture we saw that function spaces can also have a linear structure. Let us begin by considering this structure more formally, see again Def. **??** for the definition.

1.1.1 The Linear Structure of Function Spaces

In the examples in the previous section we defined vector addition and scalar multiplication through the monomial and harmonic bases. When we look at the examples together, cf. Eq. and Eq. , it can be seen that we used in fact the same definition. Let $f(x), g(x) : \mathbb{R} \to \mathbb{R}$ be two functions from either of the previous examples. Then we defined their addition pointwise as

$$(f + g)(x) = f(x) + g(x), \tag{1.1}$$

which is just the addition of real numbers since for fixed x the function values $f(x)$ and $g(x)$ are just that. Correspondingly, scalar multiplication is also defined pointwise,

$$(a\, f)(x) = a\, f(x). \tag{1.2}$$

These pointwise definitions can be considered natural for the vector addition and scalar multiplication of functions since they immediately imply that the standard laws that hold for real numbers, such as associativity or commutativity, also hold for functions. This, in turn, then directly yields that a set of functions forms a vector space, satisfying the properties in Def. **??**, *under the condition* that closure is satisfied, that is that the addition of two functions from the set and the multiplication with a scalar yields again an element in the set. In the foregoing, the closure was apparent from our use of a basis but later on we will also establish it without requiring this structure. We summarize the foregoing discussion in the following definition.

Definition 1.1. *A **function space** $\mathbf{F}(X)$ over a set X is a vector space whose elements are functions, $f : X \to \mathbb{R}$, with addition of elements of $\mathbf{F}(X)$ defined by pointwise addition*

$$(f + g)(x) = f(x) + g(x)$$

and scalar multiplication by

$$(a\, f)(x) = a(f(x))$$

for $f, g \in F(X)$ and $a \in \mathbb{R}$.

Although, as we just mentioned, the linear structure of a function space is a priori independent of a basis, our preferred means to work with a function space will be through a basis, not the least because it provides a means for numerical computations through the basis function coefficients and the "correspondence principle" we introduced in the last lecture.

1.1.2 Measurement Devices on Vector Spaces

In most applications a plain vector space is not very useful for applications. Hence, one typically equips the spaces with devices to measure lengths and angles.

Definition 1.2. *Let $\mathbf{E}$ be a vector space. A **norm** on $\mathbf{E}$ is a function*

$$\| \cdot \| : \mathbf{E} \to \mathbb{R}$$

into the real numbers that satisfies for all $x, y \in \mathbf{E}$, $a \in \mathbb{R}$:

i.) positivity, $|x| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$;

ii.) triangle inequality, $\|x + y\| \leq \|x\| \, \|y\|$;

iii.) homogenity, $\|a\, x\| = |a| \, \|x\|$.

*A linear space $\mathbf{E}$ together with a norm $\| \cdot \|$ is a **normed linear space** $(\mathbf{E}, \| \cdot \|)$.*

The classical example for a norm is the Euclidean length that, with respect to the canonical basis for $\mathbb{R}^n$, is given by

$$\|\mathrm{v}\| = \sqrt{\sum_{i=1}^{n} v_i^2}. \tag{1.3}$$

The classical examples of normed function spaces are defined as follows.

Definition 1.3. *Let $f : [0,1] \to \mathbb{R}$ and consider*

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p} \tag{1.4}$$

for $1 \leq p < \infty$ and[1]

$$\|f\|_\infty = \text{ess sup}_{x \in [0,1]}(|f(x)|). \tag{1.5}$$

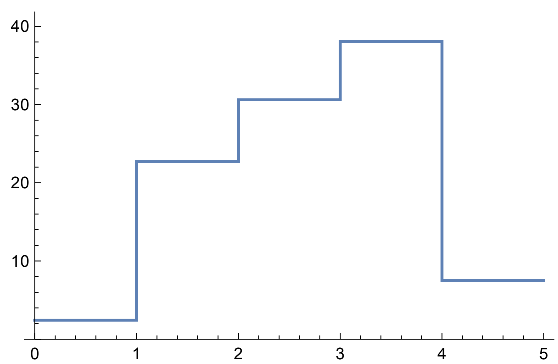
*Then $\| \cdot \|_p$ defines a norm, the so called L_p **norm**, and satisfies the properties in Def. 1.2. Using this family of norms we can define associated L_p **spaces***

$$L_p([0,1]) = \left\{ f : [0,1] \to \mathbb{R} \,\middle|\, \|f\|_p < \infty \right\}. \tag{1.6}$$

These places play an important role in analysis. The definition is easily extended from $[0,1]$ to more general domains where an appropriate notion of integration is defined.

To gain some intuition for the effect of the parameter p consider the following example.

Example 1. To gain some intuition on the parameter p we consider the following piecewise constant function



whose values are $f \cong \{f_1, \cdots, f_5\} = \{2.43, 22.69, 30.61, 38.08, 7.49\}$. Since the function is piecewise constant its L_p norm is given by

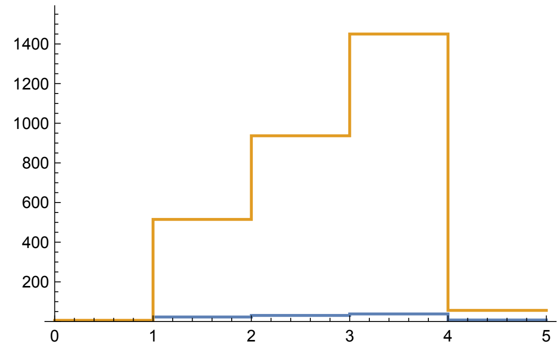
$$\|f\|_p = \left(\int_0^5 |f(x)|^p \right)^{1/p} = \sum_{i=1}^{5} |f_i|^p. \tag{1.7}$$

The L_1 norm is hence given by

$$\|f\|_1 = 2.434, 22.69, 30.61, 38.08, 7.492, 7.492 = 54.96 \tag{1.8a}$$

The following plot shows $|f(x)|^2$ that determines the L_2 norm (yellow, in blue again $f(x)$:

[1] The essential supremum is defined as $\text{ess sup}(|f|) \equiv \{C \geq 0 \,|\, |f(x)| \leq C\}$.



which explicitly is given by

$$\|f\|_2 = (5.92 + 514.96 + 936.82 + 1450.09 + 56.12)^{1/2} = 54.96 \tag{1.8b}$$

Similarly, the L_3 norm is given by

$$\|f\|_2 = (14.41 + 11686.1 + 28674.0 + 55219.4 + 420.487)^{1/3} = 45.85 \tag{1.8c}$$

Comparing Eq. 1.8a, Eq. 1.8a and Eq. 1.8a we see that as p grows the value of the norm is disproportionally determined by the large values f_i. In particular f_1 has a negligible effect on the L_2 and L_3 norms that is dominated by f_2,f_3,f_4. At the extreme, for $p = \infty$ the norm is entirely determined by the largest value of $|f(x)|$.

The definition of the L_p spaces demonstrates that one can equip a vector space with different norms and thereby obtain different structures. The different L_p norms also yield different restrictions on the set of functions that are in the corresponding L_p spaces.

Example 2. Consider functions over the real line. Then

$$\frac{1}{x} \in L_2(\mathbb{R}) \notin L_1(\mathbb{R}) \tag{1.9a}$$

$$\sin{(x)} \in L_\infty(\mathbb{R}) \notin L_1(\mathbb{R}) \notin L_2(\mathbb{R}). \tag{1.9b}$$

Remark 1. In a finite dimensional vector space $\mathbf{E}$, all norms, and hence in particular all ℓ_p norms are *equivalent* in the sense that there are constants $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|x\|_p \leq \|x\|_p \leq \beta\|x\|_p \tag{1.10}$$

for all $x \in \mathbf{E}$ where $1 \geq p, q \leq \infty$ are arbitrary. Despite this equivalence, even in the finite dimensional case it does make in practice typically a difference which norm is used, optimization problems being a point in case.

Next to measuring length, the natural operation we perform on vectors in Euclidean space is measuring angles between them. This is accomplished using the inner product $\langle \cdot , \cdot \rangle$. As we have seen before, the inner product is for us of considerable importance because it enables us to compute expansion coefficients for bases.

Definition 1.4. *Let* $\mathbf{E}$ *be a vector space. A bilinear function*

$$\langle \cdot , \cdot \rangle : \mathbf{E} \times \mathbf{E} \to \mathbb{R}$$

on $\mathbf{E}$ *is called an* ***inner product*** *when it satisfies for all* $x, y, z \in \mathbf{E}$ *and* $a, b \in \mathbb{R}$*:*

i) linearity, $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ *and* $\langle a\,x, b\,y \rangle = a\,b \langle x, y \rangle$*;*

ii) symmetry $\langle x, y \rangle = \langle y, x \rangle$ *;*

iii) positivity, $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0 \Leftrightarrow x = 0$ *.*

A vector space $\mathbf{E}$ *with an inner product is a* ***inner product space*** $(\mathbf{E}, \langle \cdot , \cdot \rangle)$*.*

Naturally, $\mathbb{R}^n$ with the dot product is an inner product space. The most important example for an inner product for functions is the following, which we already introduced before when we considered the space of harmonic functions.

Example 3. Let $f, g : [0,1] \to \mathbb{R}$ with the usual vector space structure. Then

$$\big\langle f, g \big\rangle = \int_0^1 f(x)\, g(x)\, dx \tag{1.11}$$

is an inner product, as can be seen easily from the properties of the integral. As in the Euclidean case, where the inner product is the squared length of a vector, the inner product in Eq. 1.11 also induces a natural norm

$$\|f\|_2^2 = \big\langle f, f \big\rangle \tag{1.12}$$

and $\| \cdot \|$ is the L_2 norm introduced in Ex. 1.3. The space $L_2([0,1], \langle \cdot , \cdot \rangle)$ with the inner product in Eq. 1.11 is an inner product space and the inner product is known as L_2 inner product. The domain $[0,1]$ can again be generalized to other sets over which there is an appropriate notion of integration.

From the properties of the inner product in Def. 1.4, in particular iii.), it is not difficult to see that in fact every inner product induces a compatible norm in the way in Eq. 1.11. The following definitions is a straightforward generalization from the Euclidean case.

Definition 1.5. *Let* $(\mathbf{E}, \langle \cdot , \cdot \rangle)$ *be an inner product space. Then two vectors* $x, y \in \mathbf{E}$ *are called* ***orthogonal*** *when* $\langle x, y \rangle = 0$*. A vector* $x \in \mathbf{E}$ *is called* ***normalized*** *when* $\|x\| = 1$*.*

Normed vector spaces and inner product spaces typically arise in a form with one additional assumption known as completeness.

Definition 1.6. *A normed vector space* $\mathbf{E}$ *is* ***complete*** *if every Cauchy sequence in* $\mathbf{E}$ *has a limit in* $\mathbf{E}$.

Recall that a Cauchy sequence is a convergent sequence, that is a sequence $\{x_i\}$ with $x_i \in \mathbf{E}$ such that

$$\|x_i - x\| \xrightarrow[i \to \infty]{0}, \tag{1.13a}$$

so that for every $\epsilon > 0$ there exist $n \in \mathbb{N}$ such that

$$\|x_k - x_l\| < \epsilon \ , \forall l, k > n. \tag{1.14}$$

For us such sequences are of importance because the sum in a basis expansion

$$v = \sum_{i=1}^{k} a_i \, u_i \tag{1.15}$$

can be considered as a sequence

$$\left\{ \sum_{i=1}^{1} a_i \, u_i \, , \, \sum_{i=1}^{2} a_i \, u_i \, , \, \sum_{i=1}^{3} a_i \, u_i \, , \cdots \right\} \tag{1.16}$$

and in fact when k is infinity then the equality in Eq. 1.15 will be defined in the sense of convergence of this sequence, see Def. 1.9. Hence, for the basis expansions that are of crucial importance for us to make vector spaces amenable to numerical computations the completeness of the spaces is required to ensure that the expansions converge. We will in the following always assume that our vector spaces are complete. This is justified because every vector space has an (essentially) unique completion[2], analogous to how the "incomplete" rational numbers $\mathbb{Q}$ are completed using the real numbers $\mathbb{R}$.

Because of their practical importance, complete normed vector space and complete inner product spaces have dedicated names.

Definition 1.7. *A complete normed vector space is called a* ***Banach space****.*[3]

Definition 1.8. *A complete inner product space is called a* ***Hilbert space****.*[4]

Redraw figure for our purposes.

Banach and in particular Hilbert spaces will be the principal setting for our subsequent work. The relationship between these spaces is summarized in Fig. 1.1.

[2] See for example Lax, *Linear Algebra and Its Applications*, Ch. 5.

[3] Stefan Banach (1892-1945) was perhaps the most important Polish mathematician. He was one of the founders of modern functional analysis.

[4] David Hilbert (1862-1943) was one of the central figures of mathematics in the first half of the 20$^{\text{th}}$ century. He made numerous contributions to mathematics and mathematical physics but he had his most profound impact on the field arguably through Hilbert's programme which aimed at establishing a rigorous foundation for mathematics. With his incompleteness theorems Gödel showed that this objective was untenable.

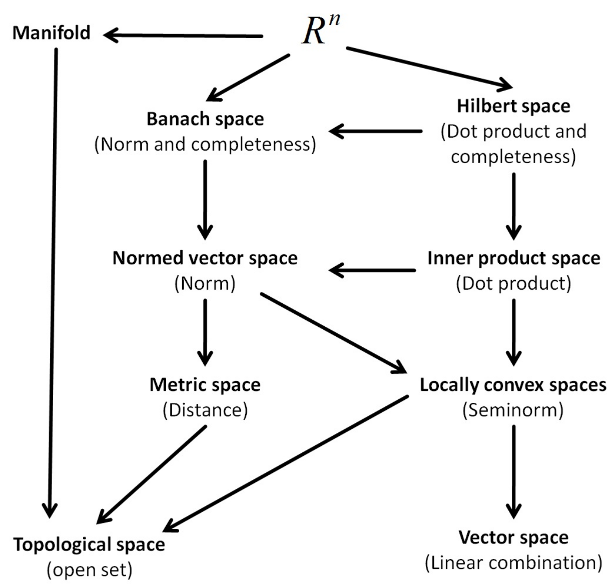


Figure 1.1: Relationship between different spaces (from https://upload.wikimedia.org/wikipedia/commons/6/60/Mathematical_implication_diagram_eng.jpg)

1.1.3 Bases for Vector Spaces

The most general notion of basis we will consider is the following.

Definition 1.9. *Let* $(\mathbf{E}, \|\cdot\|)$ *be a Banach space. A sequence* $\{f_i\}_{i=1}^k$ *of elements in* $\mathbf{E}$*, with* k *possibly being infinity, is a* ***Schauder basis*** *for* $\mathbf{E}$ *if for every* $x \in \mathbf{E}$ *there exists a* unique *sequence of coefficients* $\{a_i\}_{i=1}^k$ *such that*

$$v = \sum_{i=1}^{k} a_i \, f_i$$

with equality to be understood as convergence with respect to the norm when $k = \infty$*. If* $\{f_{\sigma(i)}\}_{i=1}^k$ *provides a basis for every permutation of* $1, \cdot, k$ *then it provides an* ***unconditional Schauder basis****.*

This definition captures what is for us the essential structure of a basis: it provides a set of scalar coefficients that enables us to uniquely identify a vector, be it a geometric vector or a continuous function, with a sequence of coefficients. What it leaves open, however, is how the sequence $\{a_i\}_{i=1}^n$ is obtained for a given v. We already know that the a_i can be obtained using the inner product in the Euclidean case when we consider the canonical basis, that is

$$\mathrm{v} = \langle \mathrm{v}, \mathrm{e}_1 \rangle \mathrm{e}_1 + \langle \mathrm{v}, \mathrm{e}_2 \rangle \mathrm{e}_2 \tag{1.17}$$

and we also used the L_2 inner product to obtain the coefficients for the harmonic basis in Eq. XXX. However, for the monomials a different approach is needed. Indeed, let $p(x) = 1 + 3/2x + x^2$. Evidently, when the coefficients could be obtained as in the Euclidean case and for the harmonic basis then the inner product

$$\int_{-1}^{1} p(x)\, x\, dx = \frac{3}{2} \tag{1.18a}$$

should equal 3/2 since this is exactly the coefficient a_i for the monomial x. A simple calculation shows, however, that

$$\int_{-1}^{1} \left(1 + 3/2x + x^2\right) x\, dx = 1 \neq \frac{3}{2}. \tag{1.19}$$

What distinguishes the monomials from the Euclidean case and the harmonic basis is that the x^k are not orthogonal with respect to the inner product. But orthogonality is no requirement in the definition of a Schauder basis in Eq. 1.9 and the monomials clearly provide an example for such a basis. We will discuss how to obtain the expansion coefficients for basis vectors that are not orthogonal in the next section. Before that, however, let us summarize the known result for orthonormal bases.

Definition 1.10. *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\{e_i\}_{i=1}^k$ be a Schauder basis for $\mathcal{H}$. Then $\{e_i\}_{i=1}^k$ is an orthonormal basis when*[5]

$$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & i == j \\ 0 & i \neq j \end{cases} . \tag{1.20}$$

The expansion coefficient for any $v \in \mathcal{H}$ are then given by $v_i = \langle v, e_i \rangle$ so that one has

$$v = \sum_{i=1}^{k} v_i\, e_i = \sum_{i=1}^{k} \langle v, e_i \rangle e_i. \tag{1.21}$$

The second part of the "definition" is currently merely supported by experience through the examples we considered. We will provide a formal justification in the next section.

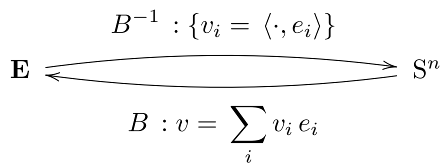
Sequence spaces and vector space isomorphisms Before we turn to the computation of expansion coefficients we will consider an important alternative perspective on what a basis is. For this we need the following notion.

Definition 1.11. *The space S^n of sequences $(x_1, \cdots, x_n)$ with $x_i \in \mathbb{R}$ is called a **sequence space**. It has a vector space structure when for $\mathrm{x} = (x_1, \cdots, x_n), \mathrm{y} = (y_1, \cdots, y_n) \in \mathrm{S}^n$ and $a \in \mathbb{R}$ one defines*

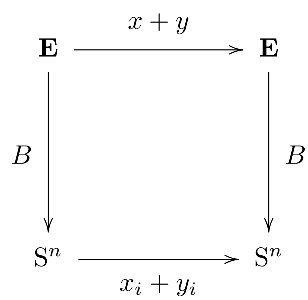
[5] The symbol "δ_{ij}" is known as Kronecker delta.

- *vector addition,* $\mathrm{x} + \mathrm{y} = (x_1 + y_1, \cdots x_n + y_n)$*;*
- *scalar multiplication,* $a\mathrm{x} = (ax_1, \cdots, ax_n)$ *.*

The importance of sequence spaces for our purposes arises from the following diagram.



where we assumed our basis to be orthogonal so that our map is invertible; a Schauder basis as in Def. 1.9 only provides a map from S^n to V. A basis hence provides a mapping from a sequence space S into a vector space V. Moreover, the vector space structures of the two spaces are compatible in that



that is addition or scalar multiplication of vectors in the sequence space followed by reconstruction is equivalent to first reconstructing the vectors and then performing the operations in V. This is just what we called previously the "correspondence principle". Hence, the importance of sequence spaces for us lies in the following: a sequence space is a numerically tractable vector space and a basis provides us a means to map an arbitrary vector space into this setting. This more conceptual point of view helps to understand the structure that we use to work with vector spaces and numerically and we will often employ it in the following.

Formally, the above diagram provides an example of a *vector space isomorphism*, that is we have two vector spaces and a map, known as isomorphism, between them, so that any vector space operation in one vector space can also be performed in the other. This can be extended to Banach and Hilbert space isomorphisms, where also the norm and inner product is preserved under the mapping, when we introduce appropriate correspondents of the L_p norms and L_2 inner product in Ex. 1.3 and Ex. 3, respectively, for sequence spaces.

Definition 1.12. *Let* S^n *be a sequence space. Then the* ℓ_p *norm on* S^n *is for* $x \in \mathrm{S}^n$ *defined as*

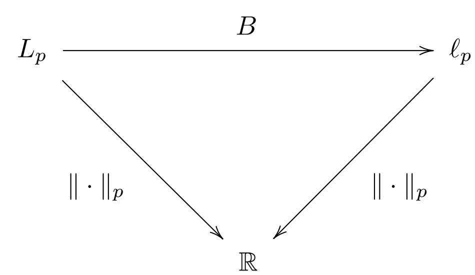
$$\|x\| = \left(\sum_{i=1}^{n} x_i^p \right)^{1/p} \tag{1.22}$$

and $(S^n, \|\cdot\|_p)$ *is a Banach space known as* ℓ_p *space. The* ℓ_2 *inner product on* S^n *is for* $x, y \in \mathrm{S}^n$ *defined as*

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \, y_i \tag{1.23}$$

and $(S^n, \langle \cdot, \cdot \rangle)$ *is a Hilbert space.*

For the intuition of the parameter p see again Ex. 1.
Using the ℓ_p spaces we have the following isomorphism:



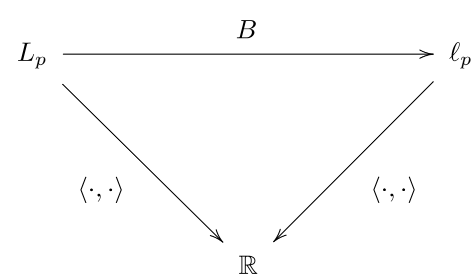
Such a "correspondence principle" for the norm is known as an isometry.

Definition 1.13. *Let* $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ *and* $(\mathbf{F}, \|\cdot\|_{\mathbf{F}})$ *be Banach spaces. An* ***isometry*** *is a map* $\varphi : \mathbf{E} \to \mathbf{F}$ *that for all* $x \in \mathbf{E}$ *satisfies*

$$\|\varphi(x)\|_{\mathbf{F}} = \|x\|_{\mathbf{E}}$$

Note that the isometry $\varphi : \mathbf{E} \to \mathbf{F}$ is a priori not necessarily linear. This only holds when φ maps the origin in $\mathbf{E}$ to the origin in $\mathbf{F}$.

Similarly, to the last diagram we have the following equivalence for the inner product:



We now recognize that the dot product is the "standard" inner product in the sequence space ℓ_2 and that a basis provides a means to map our problem to the numerically tractable setting. Such isormorphisms for Hilbert spaces have the following name.

Definition 1.14. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ and $(\mathcal{U}, \langle\cdot,\cdot\rangle_{\mathcal{U}})$ be Hilbert spaces. A* ***unitary transformation*** *$\varphi : \mathcal{H} \to \mathcal{U}$ is a Hilbert space isomorphism satisfying*

$$\big\langle \varphi(x), \varphi(y) \big\rangle_{\mathcal{U}} = \big\langle x, y \big\rangle_{\mathcal{H}}$$

for all $x, y \in \mathcal{H}$.

Isomorphisms and unitary transformations play an important role in physics because these are the transformations that preserve energy, one of the fundamental symmetries of nature. A classical example for a unitary transformation are rotations.

Example 4. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation in $\mathbb{R}^2$. Then R is unitary.

In fact, as we will see later, in some sense all unitary transformations can be considered as rotations.

1.2 Beyond Orthogonality

In the foregoing, in the context of the Fourier basis, we already introduced an inner product for functions,

$$\big\langle f, g \big\rangle = \int_X f(x)\, g(x)\, dx \tag{1.24}$$

and we showed that, analogous to the situation in $\mathbb{R}^2$ and $\mathbb{R}^3$, we can use it to obtain the expansion coefficient for a signal, that is

$$f_{j,k} = \int_0^{2\pi} f(x)\, \phi_{j,k}(x)\, dx. \tag{1.25}$$

so that

$$f(x) = \sum_{j,k} f_{j,k}\, \phi_{j,k}(x) \tag{1.26}$$

where $\phi_{j,k}(x)$ is either a dilated cosine or sine wave as defined in Eq. **??**.

We already saw that we cannot use the L_2 inner product for the monomial basis. For the sake of completeness let us repeat the argument. Let

$$f(x) = 2 + 5x + 3x^2. \tag{1.27}$$

Since $f(x)$ is already written in the monomial basis, as usual, applying the inner product has to yield corresponding coefficient. However,

$$\big\langle f(x), x \big\rangle = \int_{-1,1} f(x)\, x\, dx = \frac{10}{3} \tag{1.28}$$

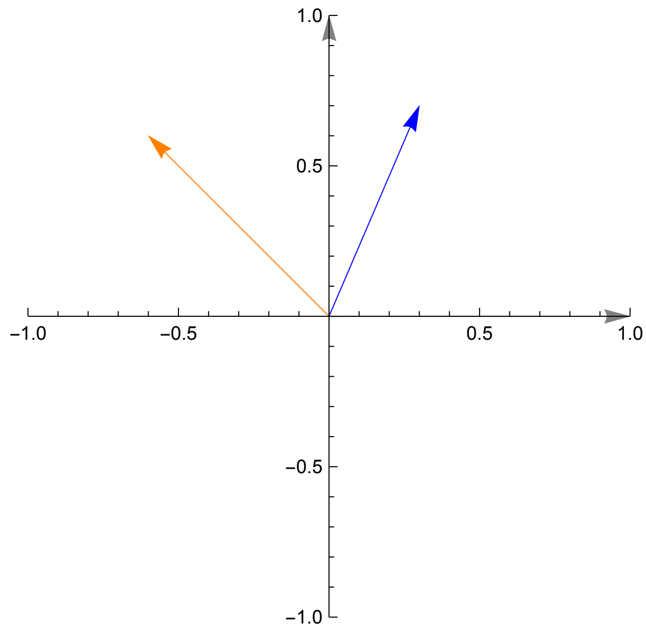


Figure 1.2: Analogue of monomial basis in $\mathbb{R}^2$ given by two non-orthogonal vectors u_1 and u_2 (blue). Also shown is the canonical basis $\{e_1, e_2\}$.

whereas the expansion coefficient is 5. One possibility is clearly that the appropriate inner product for the space of polynomials Π^n is not those in Eq. 1.24—after all, our initial justification for the space of harmonic functions was rather ad-hoc. However, assuming the inner product is appropriate, there is also an issue with the monomials:

$$\langle 1, x \rangle = 0 \tag{1.29}$$

$$\langle 1, x^2 \rangle = \frac{2}{3} \tag{1.30}$$

$$\langle x, x^2 \rangle = 0 \tag{1.31}$$

The monomials, for all their convenience, are not orthogonal! We clearly can use the monomials as basis functions but how do we obtain the expansion coefficients when these are not orthogonal? We will consider this question first in $\mathbb{R}^2$ and then return to function spaces.

Biorthogonal bases in $\mathbb{R}^2$ An analogue of the monomial basis in $\mathbb{R}^2$ is shown in Fig. 1.2. We have a set of vectors in the space that is non-orthogonal. The first question we need to answer is if these vectors can be used as a basis for $\mathbb{R}^2$, that is for every $\mathrm{v} \in \mathbb{R}^2$ there exists coefficients v_1, v_2 such that

$$\mathrm{v} = v_1^u \mathrm{u}_1 + v_2^u \mathrm{u}_2. \tag{1.32}$$

More formally, the question is if the span of $\{\mathrm{u}_1, \mathrm{u}_2\}$ is $\mathbb{R}^2$.

To understand if this holds let us perform a little calculation. Since, u_1 and u_2 are vectors in $\mathbb{R}^2$ they do have a basis representation with respect to the canonical basis. In our concrete example,

$$\mathrm{u}_1 = 0.3\,\mathrm{e}_1 + 0.7\,\mathrm{e}_2 \tag{1.33a}$$

$$\mathrm{u}_1 = -0.6\,\mathrm{e}_1 + 0.6\,\mathrm{e}_2 \tag{1.33b}$$

Thus, Eq. 1.32 can be written as

$$\mathrm{v} = v_1^u \mathrm{u}_1 + v_2^u \mathrm{u}_2 \tag{1.34a}$$

$$= v_1^u \big(0.3\,\mathrm{e}_1 + 0.7\,\mathrm{e}_2\big) + v_2^u \big(- 0.6\,\mathrm{e}_1 + 0.6\,\mathrm{e}_2\big) \tag{1.34b}$$

and by using linearity we can write the last expression as

$$\mathrm{v} = (0.3 v_1^u - 0.6 v_2^u)\mathrm{e}_1 + (0.7 v_1^u + 0.6 v_2^u)\mathrm{e}_2. \tag{1.34c}$$

By definition, this implies that the basis function coefficients v_1, v_2 with respect to the canonical basis $\{\mathrm{e}_1, \mathrm{e}_2\}$ are given by

$$v_1 = 0.3 v_1^u - 0.6 v_2^u \tag{1.35a}$$

$$v_2 = 0.7 v_1^u + 0.6 v_2^u \tag{1.35b}$$

We know that any vector $v \in \mathbb{R}^2$ can be written with respect to the canonical basis $\{\mathrm{e}_1, \mathrm{e}_2\}$. But does this also hold for the set $\{\mathrm{u}_1, \mathrm{u}_2\}$? This is easily seen when we write Eq. 1.35 in component form

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0.3 & -0.6 \\ 0.7 & 0.6 \end{pmatrix} \begin{pmatrix} v_1^u \\ v_2^u \end{pmatrix} . \tag{1.36}$$

Any component vector $(v_1, v_2)^T$, that we know uniquely identifies a vector v, is mapped onto a unique $(v_1^u, v_2^u)^T$ when the matrix

$$B = \begin{pmatrix} u_1^1 & u_1^2 \\ u_2^1 & u_2^2 \end{pmatrix} = \begin{pmatrix} 0.3 & -0.6 \\ 0.7 & 0.6 \end{pmatrix} \tag{1.37}$$

is invertible. Computing the determinant we obtain

$$\det(B) = \frac{3}{5} \tag{1.38}$$

and hence B is invertible. This establishes that $\{\mathrm{u}_1, \mathrm{u}_2\}$ indeed spans $\mathbb{R}^2$, where the span is formally defined as follows.

Let us consider when B becomes singular and the determinant vanishes. For a 2×2 matrix the determinant is given by

$$\det(B) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\,d - b\,c \tag{1.39}$$

We are thus interested in

$$u_1^1 u_2^2 - u_1^2 u_2^1 = 0 \tag{1.40}$$

where (u_1^1, u_2^1) are the basis function coefficients of u_1 with respect to $\{\mathrm{e}_1, \mathrm{e}_2\}$ and (u_1^2, u_2^2) the corresponding coefficients of u_2. Assuming the coefficients with respect to u_2 as being fixed this can be written as

$$u_1^1 u_2^2 - u_2^1 u_1^2 = 0 \tag{1.41}$$

where we are interested in geometric conditions on the coefficients u_i^1 for the above equation to hold. However, this is just the definition of linear independence.

We should define linear independence.

Hence, Eq. 1.41 holds, and the determinant of the matrix B vanishes, when the coefficients (u_1^1, u_2^1) and (u_1^2, u_2^2) are linearly dependent, and by the "correspondence principle", when the vectors u_1 and u_2 are linearly dependent. Geometrically, this means that u_1 and u_2 are along the same direction, possibly up to a assign, so that $\mathrm{u}_1 = a\,\mathrm{u}_2$ for some $a \in \mathbb{R}$. In terms of the inner product, the vectors have to be parallel for $\det(B) = 0$. Obviously, this is the condition we would have expected: two vectors span $\mathbb{R}^2$ when these are not co-linear.

This established that we can represent any vector using $\{\mathrm{u}_1, \mathrm{u}_2\}$. However, we still do not know how to obtain the expansion coefficients v_1^u, v_2^u. Or do we? Considering again Eq. 1.36, we see that we already have an equation that relates v_1^u, v_2^u to the canonical coefficients v_1, v_2 which we know are given by

$$v_1 = \langle \mathrm{v}, \mathrm{e}_1 \rangle \tag{1.42a}$$

$$v_2 = \langle \mathrm{v}, \mathrm{e}_2 \rangle \tag{1.42b}$$

Hence, we can obtain v_1^u, v_2^u by first computing v_1, v_2 and then solving the linear system in Eq. 1.36. By the "correspondence principle" we expect that this two step computation also has a geometric interpretation. In fact, Eq. 1.36 can be written as

$$\begin{pmatrix} v_1^u \\ v_2^u \end{pmatrix} = B^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \bar{b}_{11} & \bar{b}_{12} \\ \bar{b}_{21} & \bar{b}_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{1.43}$$

which can be written as

$$v_1^u = \bar{b}_{11} v_1 + \bar{b}_{12} v_2 \tag{1.44}$$

$$v_1^u = \bar{b}_{21} v_1 + \bar{b}_{22} v_2 \tag{1.45}$$

and inserting the definition of v_1, v_2 we obtain

$$v_1^u = \bar{b}_{11} \langle \mathrm{v}, \mathrm{e}_1 \rangle + \bar{b}_{12} \langle \mathrm{v}, \mathrm{e}_2 \rangle \tag{1.46}$$

$$v_1^u = \bar{b}_{21} \langle \mathrm{v}, \mathrm{e}_1 \rangle + \bar{b}_{22} \langle \mathrm{v}, \mathrm{e}_2 \rangle. \tag{1.47}$$

We can use linearity to rearrange terms, which yields

$$v_1^u = \langle \mathrm{v}\,,\, \bar{b}_{11} \mathrm{e}_1 \rangle + \langle \mathrm{v}\,,\, \bar{b}_{12} \mathrm{e}_2 \rangle \tag{1.48}$$

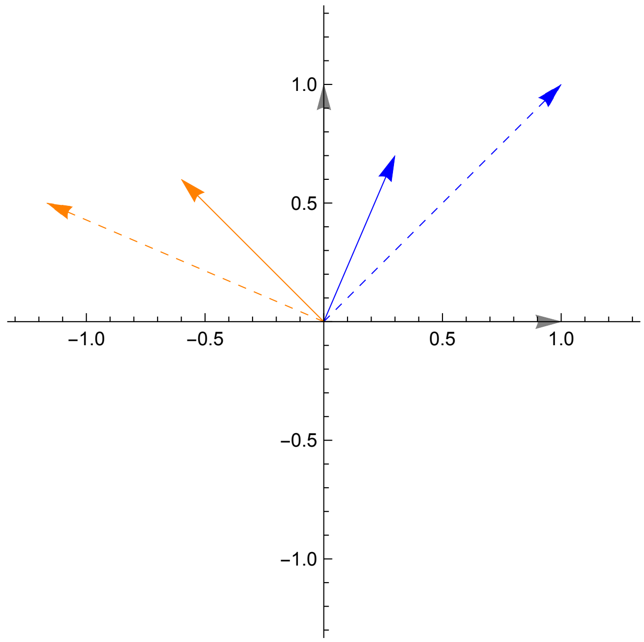


Figure 1.3: Basis vector u_1 (blue) and u_2 (orange) and their duals $\tilde{u}_1$ and $\tilde{u}_1$ (dashed). Also shown is the canonical basis $\{e_1, e_2\}$.

$$= \left\langle v\, , \, \bar{b}_{11} e_1 + \bar{b}_{12} e_2 \right\rangle \tag{1.49}$$

$$v_1^u = \left\langle v\, , \, \bar{b}_{21} e_1 \right\rangle + \left\langle v\, , \, \bar{b}_{22} e_2 \right\rangle \tag{1.50}$$

$$= \left\langle v\, , \, \bar{b}_{21} e_1 + \bar{b}_{22} e_2 \right\rangle . \tag{1.51}$$

The last equations can be interpreting as stating that v_1^u, v_1^u are obtained by computing the inner products

$$v_1^u = \left\langle v, \tilde{u}_1 \right\rangle \tag{1.52}$$

$$v_2^u = \left\langle v, \tilde{u}_2 \right\rangle . \tag{1.53}$$

with the vectors

$$\tilde{u}_1 = \bar{b}_{11} e_1 + \bar{b}_{12} e_2 \tag{1.54}$$

$$\tilde{u}_2 = \bar{b}_{21} e_1 + \bar{b}_{22} e_2 \tag{1.55}$$

The vectors $\tilde{u}_1$ and $\tilde{u}_3$ are dual basis vectors and, by construction, these are the unique vectors to obtain the expansion coefficients for u_1,u_2. The dual vectors for our example are shown in Fig. 1.3.

Instead of working with the canonical basis $\{e_1, e_2\}$ we can hence also work with a biorthogonal basis, which for $\mathbb{R}^2$ can be given by any two vectors that are not co-linear. Such bases also provide us with a means to work with vectors numerically since we can obtain expansion coefficients, using the dual basis functions, and the geometric vector can be reconstructed from these. In other words, the "correspondence principle" also applies to biorthogonal bases. The ability to work with bases formed by almost any set of vectors will prove very useful in the context of function spaces where it will enable to enforce properties that are not available when one requires orthogonality (that orthogonality can be very restrictive is already apparent in $\mathbb{R}^2$ where, up to rotation, only one orthogonal basis exists).

The dual basis vectors satisfy an important property which can be established from

$$\mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{B}^{-1}\,\mathrm{B} = \begin{pmatrix} \tilde{u}_1^1 & \tilde{u}_2^1 \\ \tilde{u}_1^2 & \tilde{u}_2^2 \end{pmatrix} \begin{pmatrix} u_1^1 & u_1^2 \\ u_2^1 & u_2^2 \end{pmatrix} \tag{1.56}$$

The matrix matrix product can be interpreted as four individual dot products between the rows of B^{-1}, corresponding to the dual vectors $\tilde{\mathrm{u}}_1$ and $\tilde{\mathrm{u}}_2$, and the columns of B, corresponding to the primary vectors u_1 and u_2, that satisfy

$$\sum_{a=1}^{2} \tilde{u}_a^i \, u_a^j = \delta_{ij} \tag{1.57}$$

where δ_{ij} is again the Kronecker delta. By the correspondence principle, this equals

$$\big\langle \tilde{u}_i, \tilde{u}_j \big\rangle = \delta_{ij} \tag{1.58}$$

which is a basis independent definition of the dual basis vectors. The previous derivations are summarized in the next definition that introduces the concepts just constructed in $\mathbb{R}^2$ for arbitrary vector spaces.

Definition 1.15. *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle$ be a Hilbert space and $\{u_i\}_{i=1}^n$ be a basis for $\mathcal{H}$. A **biorthogonal sequence** with respect to $\{u_i\}_{i=1}^n$ is a sequence $\{\tilde{u}_i\}_{i=1}^n$ of elements $\tilde{u}_i \in \mathcal{H}$ satisfying the **biorthogonality condition***

$$\langle \tilde{u}_i, u_j \rangle = \delta_{ij}.$$

*The sequence $\{\tilde{u}_i\}_{i=1}^n$ is then also a basis for $\mathcal{H}$ and $\{(u_i, \tilde{u}_i)\}_{i=1}^n$ forms a **biorthogonal basis pair**. Any $v \in \mathcal{H}$ then has the representation*

$$v = \sum_{i=1}^{n} \langle v, \tilde{u}_i \rangle \, u_i = \sum_{i=1}^{n} \langle v, u_i \rangle \, \tilde{u}_i.$$

Example 5 (The dual basis for monomials)**.** Let us return to our initial motivation for considering non-orthoognal bases: the monomial basis. In the case of biorthogonal bases for $\mathbb{R}^2$ we obtained the dual basis using the canonical (and orthonormal) basis $\{e_1, e_2\}$ by first determining B, which was given by the basis expansion of the primary basis vectors u_1 and u_2 with respect to $\{e_1, e_2\}$. The dual basis vectors $\tilde{u}_i$ was then given by the rows of the inverse

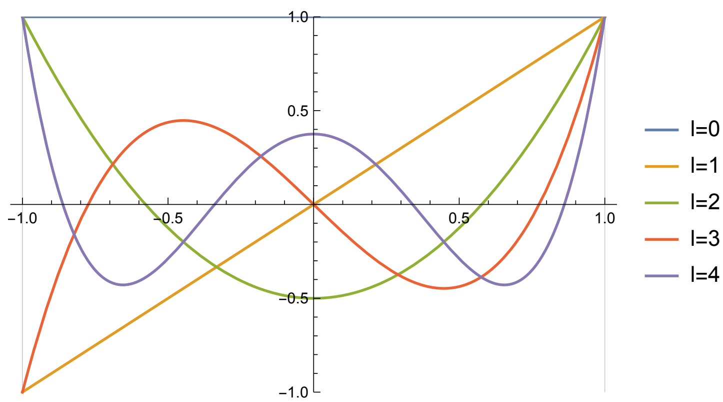


Figure 1.4: The first few Legendre polynomials $P_l(x)$.

B^{-1}. Tracing the argument we used in $\mathbb{R}^2$ we can see that this "recipe" does not rely on any special property of $\mathbb{R}^2$ but is applicable whenever we have an orthonormal "reference" basis for the space of interest.

The classical orthonormal basis for the space of polynomials $\Pi^n([-1,1])$ is given by Legendre[6] polynomials $P_l(x)$ where l denotes the maximum degree of the function. The first few (unnormalized) Legendre polynomials are given by

$$P_0(x) = 1 \tag{1.59a}$$

$$P_1(x) = x \tag{1.59b}$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \tag{1.59c}$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{1.59d}$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \tag{1.59e}$$

and a plot is shown in Fig. 1.4.

One might ask why we not directly use the Legendre polynomials, or in general the orthonormal reference basis, instead of first constructing the dual basis pair and then working with the two bases. As we will see in the following, the dual basis pair can have properties that make it much more convenient to work with then the orthonormal reference basis. In the case of monomials we have, for example, that these can be evaluated directly whereas an expansion in Legendre polynomials uses the expansion of these in the monomial basis.

[6] Adrien-Marie Legendre (1752-1833) was a French mathematician who made important contributions to mathematics and physics and was one of the first to develop the method of least squares.

Bibliography

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"My work always tried to unite
the truth with the beautiful,
but when I had to choose
one or the other,
I usually chose the beautiful."

Hermann Weyl