Integration revisited

- Covariance of integration:

  - Energy

$$E = \int_{t=0}^{T} P(t)\, dt$$

where $P(t)$ is the power

$$E = \int_{0}^{10} 1\, dt = 10$$

↳ needs unit to be physically meaningful

$$P: \left[\frac{kg\, m^2}{s^3}\right] = W$$

$$E: \left[\frac{kg\, m^2}{s^2}\right] = J$$

→ Evidently, the energy has to be independent of the units we use to measure time

↳ Fully analogous to our foregoing discussion about the invariance of vectors to the coordinate system that is used to describe it

→ What we really meant

$$E = \int_{t=0s}^{10} 1W\, dt = 10J$$

- But we could equally well have used minutes to measure the same interval

$$E = \underbrace{\int_{t=0\,min}^{1/6\,min} 1W\,dt}$$

Evidently, for our result to be independent of the time units we have to adjust our integrand

$$P(t) = 1\,W \longrightarrow 60\,\frac{kg\,m^2}{s^2\,min}$$

→ We need a systematic way to do this without units

$$J = \int_{x_1=a}^{x_2=b} f(x)\,dx$$

Change of coordinates

$$y = \alpha x \qquad , \alpha \in \mathbb{R}^+$$

$$y_1 = \alpha \cdot x_1 = \alpha a$$

$$y_2 = \alpha x_2 = \alpha b$$

$$f(x) = f(\alpha x)$$

$$\frac{dx}{dy} = \frac{d}{dy}\left(x = \frac{1}{\alpha}\right)$$

$$\Rightarrow \underbrace{dx = \frac{1}{\alpha}\,dy}$$

$f(\alpha x)$ and $dx = \frac{1}{\alpha} dy$ compensate each other

$$J = \int_{y_1}^{y_2} f(y)\,\frac{1}{\alpha}\,dy$$

This calculation is abuse of notation!

↳ But yields correct result and avoids the quite complex theory of differential forms that is needed in general

- In our example:

$$t_{min} = \frac{1}{60}\,t_s$$

$$t_1 = \frac{1}{60} \cdot 0 = 0$$

$$t_2 = \frac{1}{60} \cdot 10 = \frac{1}{6}$$

$$P(t_s) = P\left(\frac{1}{60} t_s\right) = 1$$

$$\frac{dt_s}{dt_{min}} = \frac{d}{dt_{m}} 60\,t_{min} \Rightarrow \underbrace{dt_s = 60\,dt_{min}}$$

the factor of 60 really comes from here!

Rem.:

$\frac{\partial}{\partial x}$ = infinitesimal step: velocity vector that we use when we integrate from left to right

dx is a functional covector

↳ $dx\left(\frac{\partial}{\partial x}\right)$

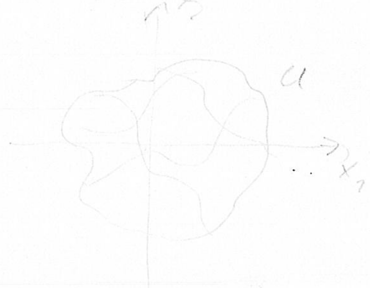
↳ When we rescale $\frac{\partial}{\partial x}$ then dx has to be scaled inversely for the result to be independent

↳ the dx measures the size of an infinitesimal step

This is exactly the concept of covariance we considered before

- Integration in $\mathbb{R}^2$:

$$J = \int_{U \subset \mathbb{R}^2} f(x)\, dx$$



$$= \int_{U \subset \mathbb{R}^2} f(x_1, x_2)\, dx_1\, dx_2$$

→ When U is axis-aligned this can be written as

$$J = \int_{x_1=a_1}^{b_1} \int_{x_2=a_2}^{b_2} f(x_1, x_2)\, dx_1\, dx_2$$

→ x_2 is fixed when we integrate over x_1

→ J can be determined by solving two 1-dimensional integration problems

Ex:

$$f(x) = 3x_1 + 4x_2^2 + 5x_1 x_2$$

$$I = \int_{[0,1]^2} f(x)\, dx$$

$$= \int_{x_1=0}^{1} \underbrace{\int_{x_2=0}^{1} 3x_1 + 4x_2^2 + 5x_1 x_2 \, dx_2}\, dx_1$$

$$= \left[3x_1 x_2 + \frac{4}{3} x_2^3 + \frac{5}{2} x_1 x_2^2 \right]_0^1$$

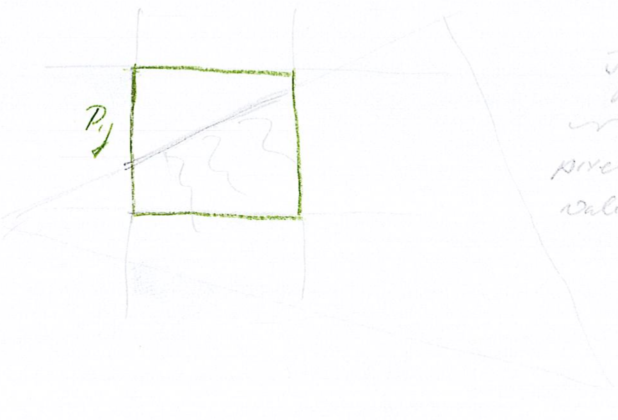
$$= 3x_1 + \frac{4}{3} + \frac{5}{2} x_1$$

$$= \left[\frac{3}{2} x_1^2 + \frac{4}{3} x_1 + \frac{5}{4} x_1^2 \right]_0^1$$

$$= \frac{3}{2} + \frac{4}{3} + \frac{5}{4} = \frac{18 + 16 + 15}{12} = \underline{\underline{\frac{49}{12}}}$$

→ Unfortunately, many problems are not axis aligned

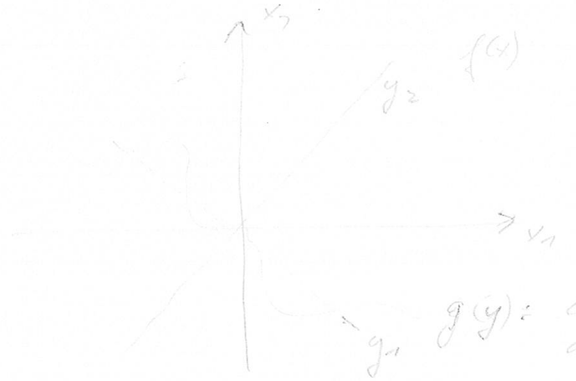
↳ Ex: Computation of pixel value



$$\underbrace{I_{ij}}_{\text{pixel value}} = \int_{P_{ij}} \underbrace{f(x)}_{\text{texture etc}}\, dx$$

↳ Not aligned with coordinate axis but with pixel boundary

More generally:



$g(y)$: can be computed analytically

→ constant in second direction

$$J = \int_{\mathbb{R}^2} f(x)\, dx$$

→ We can choose a convenient coordinate system for our problem!

↳ This is the key for many multi-variate problems

→ In our case:

$$y = \underbrace{R}\, x$$

rotation that aligns g with coordinate axes: say x_1

→ This looks like the change of coordinates that we already considered on $\mathbb{R}$

↳ Do we have to change our domain of integration?

- No, since we integrate over all of $\mathbb{R}$

↳ Do we need scaling factor for dx?

- No, since rotation does not "stretch" or "squeeze"

$$J = \int_{\mathbb{R}^2} f(x)\, dx = \int_{\mathbb{R}^2} \underbrace{f(R^T y)}\, dy = \int \bar{g}(y_1)\, 1\, dy_1\, dy_2$$

$$= \bar{f}(y) = \bar{g}(y_1)\, 1_{y_2}$$

$$= \bar{f}(Ry)$$

$$= f(x) = \bar{f}(R R^T y)$$