# Robustness of frames

→ Let $H$ be a $n$-dimensional Hilbert space and

$$\{(\tilde{u}_i, u_i)\}_{i=1}^m$$

an $m$-frame for $H$ with $m > n$ so that

$$v = \sum_{i=1}^m \langle v, \tilde{u}_i \rangle u_i = \sum_{i=1}^m v_i \tilde{u}_i \qquad \forall v \in H$$

→ Erasures:

- A coefficient $v_i = \langle v, \tilde{u}_i \rangle$ is completely lost
- Clearly, as long as not more than $m-n$ coefficients are lost we can still recover $v \in H$ from the remaining ones

→ Corruption by additive Gaussian noise

- Each frame coefficient is corrupted by a random Gaussian "offset" $\varepsilon \in N(0, \sigma)$, that is

$$\bar{v}_i = v_i + \varepsilon_i$$

↳ Standard (and simplest) model for an analogue channel / signal transmission

↳ Also useful for measurements, which play an important role in graphics

- The signal reconstructed from the noise corrupted coefficients is given by

$$\bar{v} = \sum_{i=1}^m \bar{v}_i u_i = \sum_{i=1}^m (v_i + \varepsilon_i) u_i$$

and by linearity

$$\bar{v} = \underbrace{\sum_{i=1}^m v_i u_i}_{= v} + \sum_{i=1}^m \varepsilon_i u_i$$

The $\varepsilon_i$ are random variables, since they are generated by a stochastic process, and hence we have to analyze the error in a statistical sense

- Expected value:

$$E(\tilde{v}) = E\Big(\underbrace{v}_{\substack{\text{Deterministic so} \\ E(v)=v \\ \text{also linear}}} + \sum_{i=1}^{m} \varepsilon_i u_i\Big)$$

$$= v + E\Big(\sum_{i=1}^{m} \varepsilon_i u_i\Big)$$

using again linearity

$$= v + \sum_{i=1}^{m} \underbrace{E(\varepsilon_i)}_{\substack{=0 \text{ since from a} \\ \text{normal distribution} \\ \text{with mean } 0}} u_i$$

Independent of $m$ → $m$ should increase robustness

- Variance:

→ To simplify the analysis let the frame be unit norm and tight so that

$$v = \frac{n}{m} \sum_{i=1}^{m} \langle v, u_i \rangle u_i$$

Then

$$V(\tilde{v}) = V\Big(\frac{n}{m} \sum_{i=1}^{m} \langle \tilde{v}_i, u_i \rangle u_i\Big)$$

$$= \frac{n^2}{m^2} V\Big(\sum_{i=1}^{m} \langle \tilde{v}_i, u_i \rangle u_i\Big)$$

and by linearity

$$V(\tilde{v}) = \frac{n^2}{m^2} \Big(\underbrace{V\Big(\sum_{i=1}^{m} \langle v, u_i \rangle u_i\Big)}_{=0} + V\Big(\sum_{i=1}^{m} \varepsilon_i u_i\Big)\Big)$$

$$= \frac{n^2}{m^2} \sum_{i=1}^{m} \underbrace{V(\varepsilon_i)}_{=\sigma^2} = \frac{n^2}{m^2} (\sigma^2 m) = \underbrace{\frac{n^2}{m} \sigma^2}_{\text{Variance decays as } \frac{1}{m}}$$