Solution of Evolution Equations

- Heat equation

$$\frac{\partial f(x,t)}{\partial t} = \underbrace{\Delta}_{\Delta = \left(\frac{\partial^2}{\partial^2 x_1} + \frac{\partial^2}{\partial^2 x_2}\right)} f(x,t) \qquad x \in [0,1]^2 = \Omega$$

Subject to suitable boundary conditions, e.g.

$$\underbrace{f\big|_{\partial\Omega}}_{\text{at boundary of domain } \Omega} = 0$$

→ $f(x,t)$ depends on space and time

⇓

heat equation is evolution equation

- Last week we studied the spectral method for the Poisson equation

$$\Delta f = g$$

that diagonalized the Laplace operator Δ using it's eigenfunctions that were given by the Fourier series.

In 2D, we have analogously

$$\underbrace{\frac{1}{2\pi} e^{i m_1 x} e^{i m_2 x_2}}_{\substack{\text{tensor product,}\\ \text{i.e. each dimension}\\ \text{is treated independently}}} = e^{i(m_1 x_1 + m_2 x_2)} = e^{i\langle m, x\rangle}$$

A direct computation shows

$$\Delta e^{i\langle m,x\rangle} = \underbrace{(m_1^2+m_2^2)}_{\text{eigenvalue } \lambda_i} e^{i\langle m,x\rangle}$$

The Fourier series spans $L_2(\Omega)$. Hence

$$f(x,t) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}^2} \underbrace{f_m(t)}_{} e^{i\langle m,x\rangle}$$

basis coefficients are time dependent

→ Time is an independent dimension

For Δf we thus obtain

$$\Delta f = \Delta\left(\frac{1}{2\pi}\sum_{m\in\mathbb{Z}^2} f_m(t)\, e^{i\langle m,x\rangle}\right)$$

$$= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}^2} f_m(t)\, \underbrace{\left(\Delta e^{i\langle m,x\rangle}\right)}_{=\lambda_m e^{i\langle m,x\rangle} = -(m_1^2+m_2^2)\, e^{i\langle m,x\rangle}}$$

from complex exponentials

$$= \frac{1}{2\pi}\sum -\lambda_m f_m(t)\, e^{i\langle m,x\rangle}$$

The left hand side of the Laplace equation is

$$\frac{\partial}{\partial t}\left(\frac{1}{2\pi}\sum_{m\in\mathbb{Z}^2} f_m(t)\, e^{i\langle m,x\rangle}\right)$$

$$= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}^2}\left(\frac{\partial}{\partial t} f_m(t)\right) e^{i\langle m,x\rangle}$$

The Fourier series is a basis. Hence, the expansion coefficients are unique. Looking the left and right hand side of the heat equation in the eigen basis we thus have to have:

$$\frac{\partial}{\partial t} f_m(t) = -\lambda_m f_m(t)$$

→ Time evolution of the different dimensions is fully decoupled

→ Each $f_m(t)$ evolves according to

$$\frac{d}{dt} y(t) = -k\, y(t)$$

→ We look for a $y(t)$ that satisfies this equation: derivative has to equal fct times a constant

$$y(t) = e^{-kt}$$

$$\frac{d}{dt} e^{kt} = -k\, e^{kt}$$

$$\Rightarrow f_m(t) = \underbrace{f_m^{t=0}}_{\text{we can always introduce a multiplicative constant without affecting the equation}} e^{-\lambda t}$$

→ We have to match "input" functi at $t=0$

→ Putting this together we have

$$f_t(x) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}^2} f_m(t)\, e^{i\langle m,x\rangle}$$

$$= \frac{1}{2\pi} \sum_{m \in \mathbb{Z}^2} \Big(\underbrace{f_m^0}_{= \langle f(y, t=0),\, \frac{1}{2\pi} e^{-i\langle m,y\rangle}\rangle} e^{\lambda_m t} \Big)\, e^{i\langle m,x\rangle}$$

$$= \frac{1}{4\pi^2} \sum_{m \in \mathbb{Z}^2} \langle f(y, t=0),\, e^{-i\langle m,y\rangle}\rangle\, e^{i\langle m,x\rangle}\, e^{-\lambda_m t}$$

$$= \frac{1}{4\pi^2} \Big\langle f(y, t=0),\, \sum_{m \in \mathbb{Z}^2} e^{-\lambda_m t} e^{-i\langle m,y\rangle} e^{i\langle m,x\rangle} \Big\rangle$$

Hence;

$$f(x,t) = \int_{\Omega} f(y,t=0)\; G(x,y,t)\, dy$$

where the Green's function (or fundamental solution) is given by

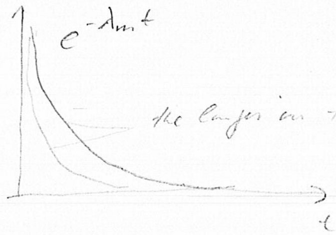
$$G(x,y,t) = \sum_{m \in \mathbb{Z}^2} e^{-\lambda_m t} \varphi_m(y)\, \varphi_m(x)$$

with the $\varphi_m(x)$ being the spatial eigenfunctions.

→ This provides an explicit solution to the heat equation for finite t.

↳ Conceptually possible for all linear PDEs (but construction can become very complex)

→ We have



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High-frequency oscillations decay much faster than low-freq. ones

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As $t \to \infty$ only $m=(0,0)$ component is preserved.

} This allows one to use heat-eq.-type flows for smoothing