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Galerkin projection and Weak Formulation

- Weak formulation:

$$\cdot\ Au = v \longrightarrow \underbrace{\langle Au, f\rangle}_{a(u,f)} = \langle v, f\rangle$$

→ Differential equations
→ Integral equations

→ Idea: solution has to hold for all measurements

→ (Babuška) Lax-Milgram theorem establishes existence and uniqueness of weak formulation

Thm (Lax-Milgram):

Let $(H, \langle , \rangle)$ be a Hilbert space and $a(\cdot,\cdot)$ a bilinear form on H with $0 \leq \gamma \leq \Gamma < \infty$ satisfying

- $a(v,v) \geq \gamma \|v\|^2$ (H-ellipticity)
- $a(u,v) \leq \Gamma \|u\| \|v\|$ (boundedness)

Then

$$a(u,v) = f(v) \qquad \forall f \in H'$$

has a <u>unique</u> solution $u \in H$.

Ex

The weak form for an integral equation

$$\int_X k(x,y)\, u(x)\, dx = v(y)$$

is

$$\int_Y \int_X k(x,y)\, u(x)\, f(y)\, dx\, dy = \int_Y v(y)\, f(y)\, dy$$

→ H ellipticity: $k(x,y)$ does not have a nullspace

→ boundedness: $k(x,y)$ yields finite result

Ex

The Heaviside step function is defined as

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We have

$$\int_{-\infty}^{\infty} H(x) f'(x)\, dx = \underbrace{\Big[f(x)\, H(x) \Big]_{-\infty}^{\infty}}_{\substack{\text{Assume } f(x) \text{ is smooth} \\ \text{and has compact support} \\ \Rightarrow\ = 0}} - \int f(x)\, H'(x)\, dx$$

$$= \int_0^{\infty} f'(x)\, dx$$

$$= \Big[f(x) \Big]_0^{\infty}$$

$$= f(\infty) - f(0)$$

$$= -f(0)$$

$$\Rightarrow \quad \int f(x)\, H'(x)\, dx = f(0)$$

$$\Rightarrow \quad H'(x) = \delta_0(x)$$

in a weak sense

→ Generalizing and systemizing this idea yields to Laurent Schwartz's theory of distributions (which was first developed by Sobolev)

The Lax Milgram theorem implies that

$$a(u_h, v_h) = f(v_h)$$

where $u_h, v_h \in H_h \subset H$ is a finite sub-space.

↳ Using basis representation yields a matrix vector equation for the above weak formulation.

→ How good is the approximation?

→ How large has n to be to attain a prescribed error?

→ Which basis should one use?

The discrete and continuous solutions satisfy

$$a(u_h, v_h) = f(v_h) \qquad \forall v_h \in H_h$$

$$a(u, v) = f(v) \qquad \forall v \in H$$

Since $H_h \subset H$ this implies

$$a(u, v_h) = f(v_h) \qquad \forall v_h \in H$$

Subtracting this from the discrete weak form we obtain

$$a(u, v_h) - a(u_h, v_h) = f(v_h) - f(v_h)$$

and using the bi-linearity of $a(\cdot,\cdot)$ we have

$$a(u - u_h, v_h) = 0.$$

→ $a(\cdot,\cdot)$ satisfies

$$a(v,v) \geq \gamma \|v\|$$

$$a(u,v) \leq \Gamma \|u\| \|v\|$$

Hence, if $a(\cdot,\cdot)$ is symmetric it defines an inner product

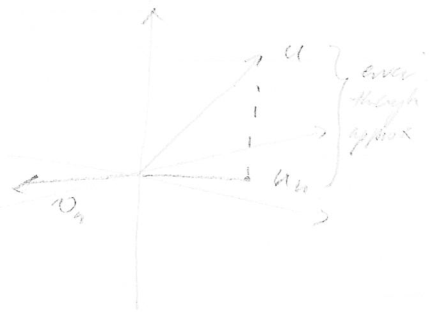
The equation

$$a(u - u_n, v_n) = 0$$

is thus a generalization of

$$\langle u - u_n, v_n \rangle = 0$$

→ To gain some intuition let $H = \mathbb{R}^3$ and $H_n = \mathbb{R}^2$



The largest vector u_n that satisfies the above equation is the orthogonal projection of u

→ An example for a non-trivial inner product is

$$\langle u, v \rangle_A = u^T A v = u^T \begin{pmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & a_{33} \end{pmatrix} v \quad , a_{ii} > 0$$

↳ As we will see later this example is in some sense generic, i.e. all symmetric $a(\cdot,\cdot)$ have this form

→ Can we make the observation that the optimal u_n is the $a(\cdot,\cdot)$-orthogonal projection onto H_n

By H-ellipticity we have

$$\|u - u_n\|^2 \leq \frac{1}{\gamma}a(u - u_n, u - u_n)$$

$$= \frac{1}{\gamma^c} a(u - u_n, u - v_n + v_n - u_n)$$

$$= \frac{1}{\gamma^c}\Big(a(u - u_n, u - v_n) + \underbrace{a(u - u_n, v_n) + a(u - u_n, -u_n)}_{=0 \text{ by } a(\cdot,\cdot) \text{ orthogonality}}\Big)$$

$$= \frac{1}{\gamma^c} a(u - u_n, u - v_n)$$

$$\leq \frac{\Gamma'}{\gamma^c} \|u - u_n\| \, \|u - v_n\|$$

Since v_n is arbitrary the optimal bound is

$$\|u - u_n\| \leq \frac{\Gamma'}{\gamma^c} \underbrace{\min_{v_n \in H_n} \|u - v_n\|}_{\text{The } v_n \text{ that minimizes this is the } a(\cdot,\cdot)\text{-orthogonal projection of } u \text{ onto } H_n}$$

→ The above result is known as Cea's Lemma

→ Closely related to the error is the convergence rate that measures how fast the error decays as n increases

↳ Depends on the choice of basis