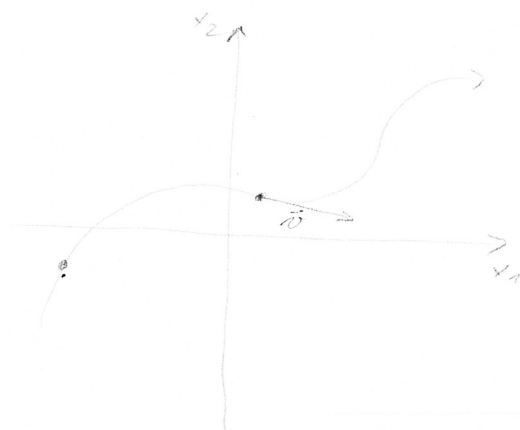


# Linear Algebra

11.4.2017

• What is a vector?

Here's immediately a deep connection between geometry - vectors and calculus - derivatives.



Newton's world view:

particle moving through space

→ what's the particle's velocity: vector  $\vec{v}$



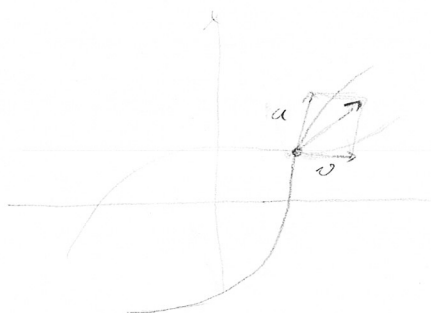
$\vec{v}$  is located or anchored at some point

→ If this is the "natural" interpretation then operations on vectors should also make sense from this perspective

→ What happens when the particle traverses the curve in half the time (assuming uniform motion)

$v \rightarrow 2v$  : scaling of vector by real number

→ What happens if two particles collide inelastically (assuming there's no absorption, i.e. conversion to internal energy)



$$\vec{w} = \vec{u} + \vec{v}$$

Direction the collided super-particle will move

You cannot add vectors that "live" at different points

→ What happens if the particle returns to where it came from?

$v \rightarrow -v$  : every vector has an inverse

$$v + (-v) = \vec{0}$$

zero vector: neutral element

Remark: We will talk about  $\mathbb{R}^2$  today but everything applies to  $\mathbb{R}^n$

We could analogously make sense of the remaining properties of vector spaces by considering the underlying application. ( $a, b \in \mathbb{R}$ ,  $u, v, w \in \mathbb{R}^2$ )

- Associativity of addition:  $(v+u)+w = v+(u+w)$
- Commutativity of addition:  $u+v = v+u$
- Distributivity of scalar multiplication:  $a(u+v) = au + av$   
 $(a+b)u = au + bu$

### Remark

The above discussion shows a principle we will try to adhere to for the rest of the course: we develop mathematics as a tool or language for modelling or describing the "real" world.

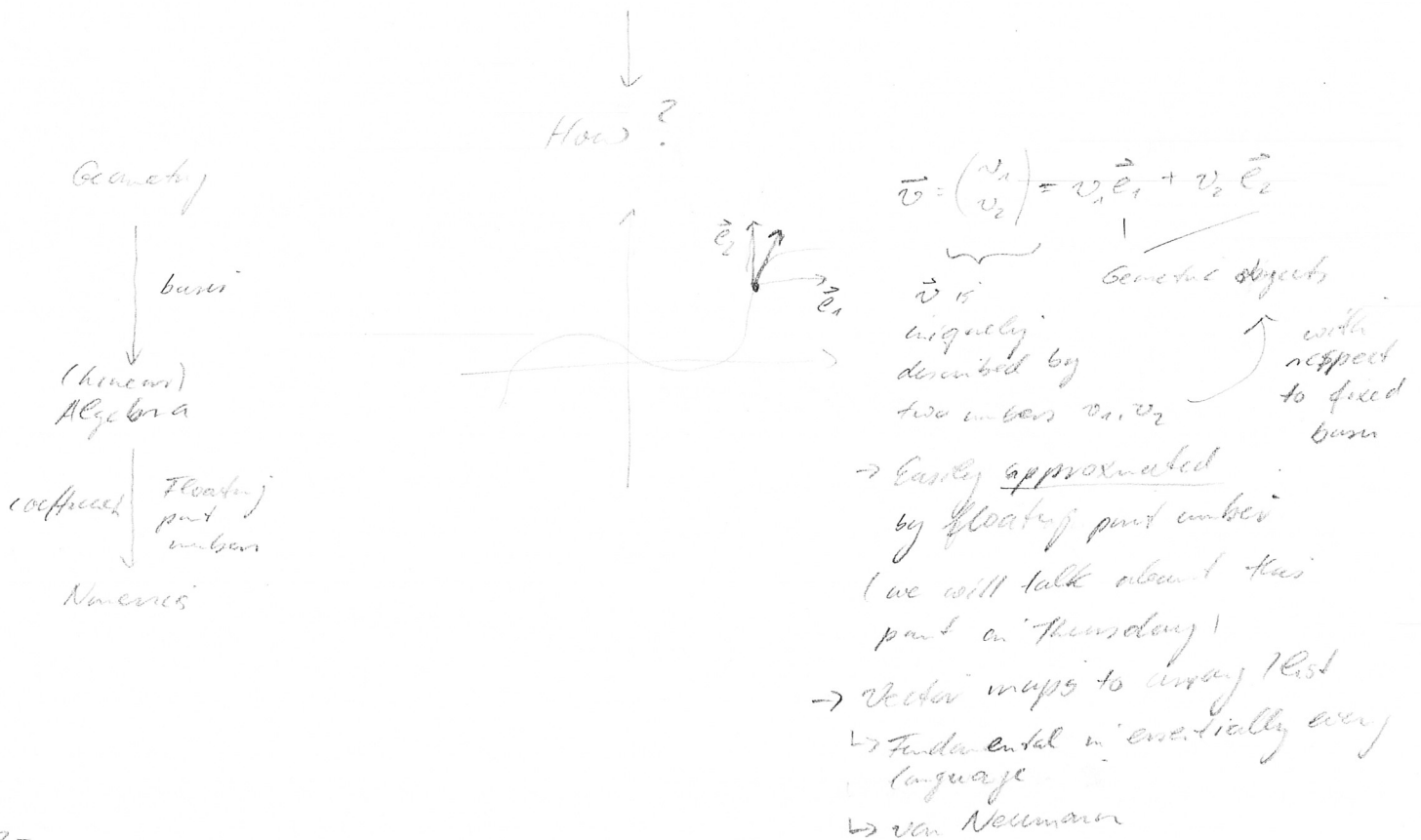
### Important

For the remainder of the course our models will typically be reduced or natural consist of vectors

→ See object that adheres to the axioms of vector spaces

Why? Because vectors are convenient to work with on a computer

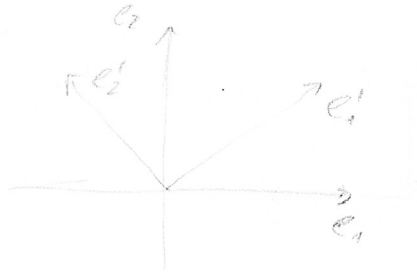
How?



## Bases

→ Tool to work algebraically with "geometric" vector representation of vector

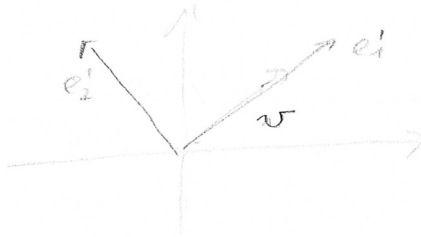
⇒ Result has to be independent of the basis that is used



⇒ Often much more complex than it might sound now, as we will see in the next weeks

↳ What does this mean when we use floating point numbers

→ It allows us to choose a convenient coordinate system



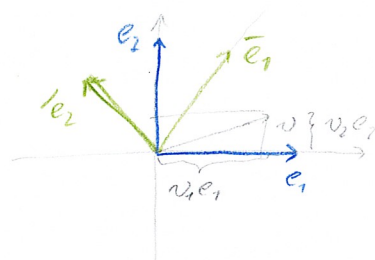
$v \rightarrow a v$   
is a linear problem in  $(e_i, e_j)$  basis

## Change of basis

→ Geometric object is invariant under basis

↳ We just change how we look at it

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}$$



$$v = \begin{pmatrix} 0.3 \\ 0.1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

known as  
given

→ How do we obtain  $\bar{v}_1, \bar{v}_2$ ?

$$v = \bar{v}_1 \bar{e}_1 + \bar{v}_2 \bar{e}_2$$

↳ By definition

$$\bar{v}_1 = \bar{e}_1 \circ v = \langle \bar{e}_1, v \rangle$$

$$\bar{v}_2 = \bar{e}_2 \circ v = \langle \bar{e}_2, v \rangle$$

$\bar{v}_i$  are scaling factors  
for basis vectors



We need a measuring  
device for scaling

Note how  
we exploit  
basis  
properties  
(or axes)  
of linear  
space, here

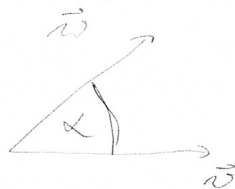
Scalar or dot product

→ What does it do geometrically?

↳ Measures correlation or similarity  
between  $v$  and  $\bar{e}_1, \bar{e}_2$

$$\langle v, w \rangle = \frac{\cos \alpha}{|v| |w|} = \cos \alpha \quad |v|, |w| = 1 = \begin{cases} 1 & \alpha = 0 \\ 0 & \alpha = \frac{\pi}{2} = 90^\circ \end{cases}$$

orthogonality



Check how  
this are  
natural  
requirements  
for an  
inner product  
device

• Basic Properties:  $(a, b \in \mathbb{R}; u, v \in \mathbb{R}^n)$

$$\bullet \langle au, bv \rangle = ab \langle u, v \rangle$$

(linearity) w.r.t  $(\mathbb{R}, +)$

$$\bullet \langle v, u+w \rangle = \langle v, u \rangle + \langle v, w \rangle$$

(linearity) u.r.t  $(\mathbb{R}^n, +)$

$$\bullet \langle v, v \rangle \geq 0$$

(positivity)

$$\bullet \langle v, v \rangle = 0 \Rightarrow v = 0$$

→ We are interested in:

$$\vec{v}_1 = \langle \vec{e}_1, v \rangle$$

This is just a vector, so we can write it with respect to  $\{\vec{e}_1, \vec{e}_2\}$

$$\vec{e}_1 = \vec{e}_1^1 \vec{e}_1 + \vec{e}_2^1 \vec{e}_2$$

just real-valued coefficients

And  $v$  naturally also has a representation w.r.t.  $\{\vec{e}_1, \vec{e}_2\}$

$$v = v_1 \vec{e}_1 + v_2 \vec{e}_2$$

→ inserting into our question:

$$\vec{v}_1 = \langle \vec{e}_1^1 \vec{e}_1 + \vec{e}_2^1 \vec{e}_2, v_1 \vec{e}_1 + v_2 \vec{e}_2 \rangle$$

and using the properties of the dot product we obtain:

$$\vec{v}_1 = \langle \vec{e}_1^1 \vec{e}_1, v_1 \vec{e}_1 \rangle + \langle \vec{e}_1^1 \vec{e}_1, v_2 \vec{e}_2 \rangle$$

$$+ \langle \vec{e}_2^1 \vec{e}_1, v_1 \vec{e}_1 \rangle + \langle \vec{e}_2^1 \vec{e}_1, v_2 \vec{e}_2 \rangle$$

Linearity  
w.r.t. vector  
addition

$$= \vec{e}_1^1 v_1 \langle \vec{e}_1, \vec{e}_1 \rangle + \vec{e}_1^1 v_2 \langle \vec{e}_1, \vec{e}_2 \rangle$$

$$+ \vec{e}_2^1 v_1 \langle \vec{e}_2, \vec{e}_1 \rangle + \vec{e}_2^1 v_2 \langle \vec{e}_2, \vec{e}_2 \rangle$$

Linearity  
w.r.t. scalar  
multiplication  
+ orthogonality

$$= \vec{e}_1^1 v_1 + \vec{e}_2^1 v_2$$



$$\vec{e}_1^1 = \langle \vec{e}_1, \vec{e}_1 \rangle = \cos \alpha$$

$$\vec{e}_2^1 = \langle \vec{e}_1, \vec{e}_2 \rangle = \sin \alpha$$

$$\vec{v}_1 = \cos \alpha v_1 + \sin \alpha v_2$$

And analogously we obtain:

$$\vec{v}_2 = \vec{e}_1^2 v_1 + \vec{e}_2^2 v_2$$

$$= -\sin \alpha v_1 + \cos \alpha v_2$$

Convenient to write as mapping

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rightarrow \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix}$$

$$\begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Rotation matrix

