Mathematical Tools for Computer Graphics 2017

Assignment 2

Due date: 29/6/2017

1.) **Integration over the cone**

A parametrization of a circular cone C at the origin with height h and aperature 2γ is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (h-u)\,\tan\theta\,\cos\phi \\ (h-u)\,\tan\theta\,\sin\phi \\ u \end{pmatrix} \tag{1}$$

see Fig. 1. The angle $\theta \in [0,\gamma]$ measures the deflection from the z axis and $\phi \in [0, 2\pi]$ the angle in polar coordinates in each disc at height $u \in [0, h]$.

a.) Derive the Jacobi matrix J for the cone coordinates in Eq. 1. Sketch (or draw) the columns of the Jacobi matrix J for a few points on C.

b.) Determine the Jacobian determinant for integration over *the surface* of a cone. Write down the integral for a function $f(\phi, u)$ so that it would be suitable for a numerical quadrature routine or for determining an integral analytically using anti-derivatives.

c.) Compute the integral of the function

$$f(u, \phi) = u\,\cos(u)\,e^{-i\,m\phi} \tag{2}$$

over the entire cone analytically.

2.) **Galerkin projection and Discretization of Operator Equations**

a.) Let $(\mathcal{H}, \langle , \rangle)$ be a Hilbert space with biorthogonal basis

$$\Big(\{\varphi_i\}_{i=1}^n, \{\tilde{\varphi}_i\}_{i=1}^n\Big). \tag{3}$$

Formally derive the Galerkin projection for the operator equation

$$g = Af \tag{4}$$

for $A : \mathcal{H} \mapsto \mathcal{H}$ being linear (no weak formulation is needed).

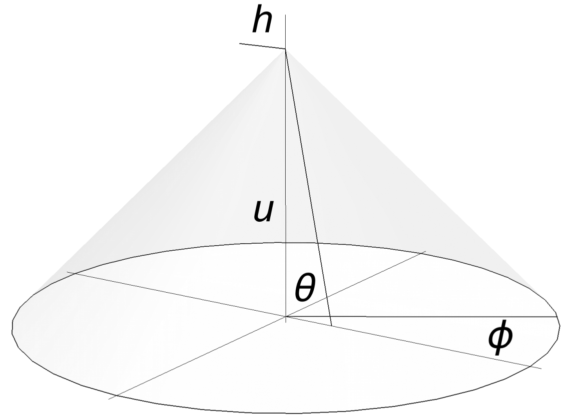

Figure 1: Parametrization of cone.

b.) Assume $\mathcal{H} \subset L_2(H^2)$ and let $\{k_j(\omega) \equiv k_{\lambda_j}(\omega)\}_{j=1}^n$ be an orthonormal reproducing kernel basis for $\mathcal{H}$. Using this basis, derive the Galerkin projection of the shading equation

$$\bar{\ell}_x(\bar{\omega}) = P\,\ell_x(\omega) = \int_{H_x^2} \rho(\omega, \bar{\omega}) \left(\ell_x(\omega)(n_x \cdot \omega)\right) d\omega \tag{5}$$

assuming $P : \mathcal{H} \mapsto \mathcal{H}$. From a practical perspective, how does the result differ from Galerkin projection using spherical harmonics?

3.) **Quadrature rules for the sphere**

a.) Let $\Lambda = \{\lambda_k\}_{k=1}^n$ with $\lambda_k \in S^2$ be locations on the sphere. Derive a quadrature rule for the space spanned by all spherical harmonics $y_{lm}(\omega)$ up to degree L.

b.) Generate well distributed random points $\Lambda = \{\lambda_k\}_{k=1}^n$ on S^2 by mapping van der Corput points from the square to S^2, see the provided code. For this use the area preserving mapping $\eta : [0,1] \times [0,1] \to [0,\pi] \times [0,2\pi]$ given by

$$\theta = \operatorname{asin}(2\,x_1 - 1) + \frac{\pi}{2} \tag{6a}$$

$$\phi = 2\pi\,x_2. \tag{6b}$$

Use these locations to implement a python function that computes the quadrature for $\mathcal{H}_5$, that is the space spanned by all spherical harmonics up to $l = 5$.

c.) Test your routine for random functions $f \in \mathcal{H}_5$ by determining the average error. How does your function perform for $f \in \mathcal{H}_6$. How could its performance potentially be improved?