A Primer on Geometric Mechanics

Variational principles and Hamiltonian Mechanics

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Outline

Course Outline

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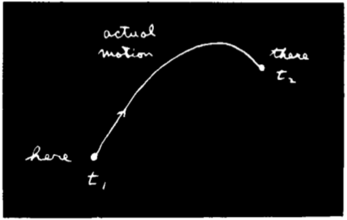
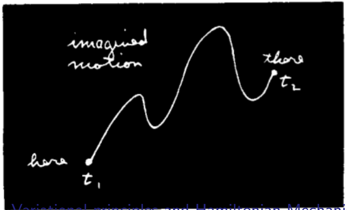
The principle of least action

Feynman's lectures on Physics, vol. I – Lecture 19

"In other words, the laws of Newton could be stated not in the form $F = ma$ but in the form: the average kinetic energy less the average potential energy is as little as possible for the path of an object going from one point to another.





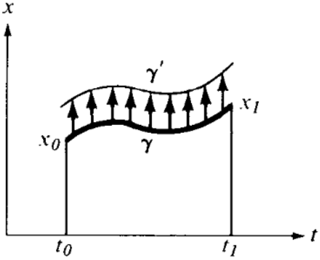
Lagrange's equations

Notation: $\mathbf{q}, \dot{\mathbf{q}} \in \mathbb{R}^d$ and $\mathbf{q(t)}$ is a smooth path in $\mathbb{R}^d$.
Given a Lagrangian $L(q, \dot{q})$, Lagrange's equation of motion is

$$\frac{d}{dt}\nabla_{\dot{\mathbf{q}}} L(\mathbf{q}, \dot{\mathbf{q}}) - \nabla_{\mathbf{q}} L(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{0}.$$

This equation is the Euler-Lagrange equation minimizing the action integral (functional) $S[\mathbf{q(t)}] := \int_{t_0}^{t_1} L(\mathbf{q}(t), \dot{\mathbf{q}}(t))dt$.

Euler-Lagrange equations



$$S(q(t)+h(t)) - S(q(t)) =$$

$$= \int_{t_0}^{t_1} (L(q+h, \dot{q}+\dot{h}) - L(q,\dot{q})dt$$

$$= \int (\frac{\partial L}{\partial \dot{q}}\dot{h} + \frac{\partial L}{\partial q} h)dt + O(h^2)$$

$$= \int (-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q})hdt$$

$$+ \text{boundary term} + O(h^2)$$

$$\Rightarrow \delta S = 0 \Rightarrow -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q} = 0$$

Hamilton's principle

Most famous action integral from classical mechanics is $S = \int (T - U)dt$, where

$$\text{T} = \text{kinetic energy}$$

$$\text{U} = \text{potential energy}$$

For a particle of mass m in a constant gravitational field $g\hat{\mathbf{k}}$,

$$S = \int_{t_1}^{t_2} [\frac{1}{2} m (\frac{dq}{dt})^2 - mgq],$$

where q is the height measured from ground level. E.-L. eqn.'s:

$$\ddot{q} = -g/m.$$

Hamilton stated his principle in 1834-35.

Example/exercise:

Consider a particle moving in a constant force field (e.g. gravity near earth, $g\hat{\mathbf{k}}$) and starting at (x_1, y_1) (rest) and descending to some lower point (x_2, y_2). Find the path that allows the particle to accomplish the transit in the least possible time.

Hint. Compute the Euler-Lagrange equations for the transit time functional given by

$$\text{time} = \int_{x_1}^{x_2} \sqrt{(1+y'^2)/2gx}\,dx.$$

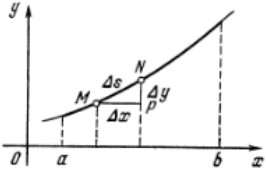
Can you describe the solution curves geometrically?

Calculus of variations

For us,

calculus of variations = calculus with functionals

A *functional* is a scalar field whose domain is a certain space of functions (e.g. C^k paths $\gamma(t)$ on $[0,1]$ plus bdry. conditions). E.g. (calculus): arc length, area, time to travel etc.



$$\Delta s \approx \sqrt{\Delta x^2 + \Delta y^2}$$

$$\Rightarrow s = \int_a^b \sqrt{1 + y'(x)}dx.$$

An important remark

The condition that $\mathbf{q}(t)$ be an extremal of a functional does not depend on the choice of a coordinate system.
For example, arc length of $\mathbf{q}(\mathbf{t})$ is given in different coordinates by different formulas

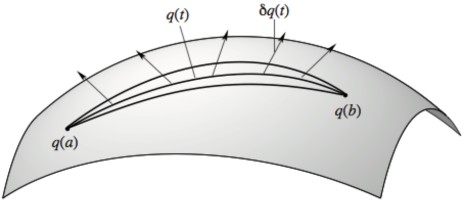
$$s = \int_{t_0}^{t_1} \sqrt{\dot{x}_1^2 + \dot{x}_2^2}\,dt \text{ (cartesian)},$$

$$s = \int_{t_0}^{t_1} \sqrt{\dot{r}^2 + r^2\dot{\phi}^2}\,dt \text{ (polar)}.$$

However, extremals are the same: straight lines in the plane.

Modern take on Variational Calculus

Hamilton's principle has been generalized to various nonlinear/curved contexts (e.g. constraints, optimal control, Lie groups (matrix groups), field theories etc.). **Focus later on will be on motion on Lie Groups (Henry Jacobs).**



Dynamics on Lie groups:

- tops,
- fluids,
- plasma,
- Maxwell-Vlasov equations,
- Maxwell's equations etc.

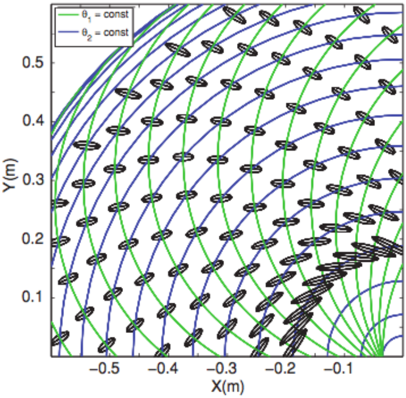
Variational Calculus on Manifolds

Let (Q^n, g_{ij}) be a Riemannian manifold.

Let $\mathbf{V}(t) = \dot{\mathbf{q}}(\mathbf{t})$, $\mathbf{A}(t) = \frac{D}{dt}\mathbf{V}(t)$ and $\mathbf{J}(t) = \frac{D}{dt}\mathbf{A}(t)$. Examples of functionals on a Riemannian manifold:

1. $S_1 = \int_{t_0}^{t_1} g_{\mathbf{q}(t)}(\dot{\mathbf{q}}(t), \dot{\mathbf{q}}(t))dt$ $\Rightarrow$ E-L: $\frac{D^2\mathbf{q}}{dt^2} = 0$ (geodesic motion).
2. $S_2 = \int_{t_0}^{t_1} g_{\mathbf{q}(t)}(\ddot{\mathbf{q}}(t), \ddot{\mathbf{q}}(t))dt$ $\Rightarrow$ E-L: $\frac{D\mathbf{J}}{dt} - R(\mathbf{V}, \mathbf{A})\mathbf{V} = 0.$

RIEMANNIAN GEOMETRIC APPROACH TO HUMAN ...



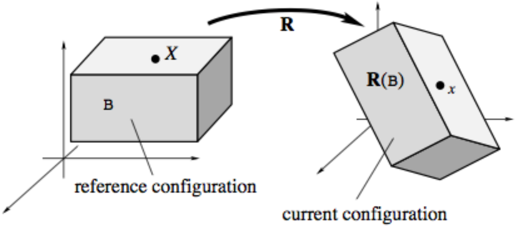
Motion on a potential field

We can generalize geodesic motion to include potentials $V : Q \to \mathbb{R}$. The action functional is now

$$S = \int_{t_0}^{t_1} (\frac{1}{2} g_{\mathbf{q}(t)}(\dot{\mathbf{q}}(t), \dot{\mathbf{q}}(t))dt - V(\mathbf{q}(t))dt$$

$$\Rightarrow \text{ E.L. : } \frac{D^2}{dt}\mathbf{q}(t) - \operatorname{grad} V(\mathbf{q}(\mathbf{t})) = 0.$$

Reduced variational principles: Euler-Poincaré I



$\mathbf{X}$	reference configuration
$\mathbf{x}$	spatial configuration
$\mathbf{R(t)}$	motion
$\dot{\mathbf{R}}(\mathbf{t}) = \frac{d}{d\epsilon}\Big\vert_{\epsilon=0} R(t+\epsilon)$	gen. velocity
$SO(3)$ = orthogonal matrices	configuration space
$TSO(3)$ = tangent bundle	state space

Reduced variational principles: Euler-Poincaré II

$$\mathbf{x}(\mathbf{t}) = \mathbf{R}(\mathbf{t})\mathbf{X},$$

X is a point on the reference configuration. Therefore,

$$\dot{\mathbf{x}} = \dot{\mathbf{R}}\mathbf{R}^{-1}\mathbf{x}(\mathbf{t}).$$

Exercise. $\dot{\mathbf{R}}\mathbf{R}^{-1} = \hat{\omega}$ is an anti-symmetric matrix.
The kinetic energy of the body is:

$$L(\mathbf{R}, \dot{\mathbf{R}}) = \text{kinetic} = \int_{\text{body}} \frac{1}{2} dm\ \dot{x}^2$$

$$= \frac{1}{2} \int_{\text{body}} m||\Omega \times X||^2 dX = \frac{1}{2}\langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle := l(\boldsymbol{\Omega}).$$

Reduced variational principles: Euler-Poincaré III

Theorem (Poincaré(1901-02): Geometric Mechanics is born)

Hamilton's principle for rigid body action $\delta S = \delta \int_{t_0}^{t_1} L(\mathbf{R}, \dot{\mathbf{R}})dt = 0$ *is equivalent to*

$$\delta S_{red} = \delta \int_{t_0}^{t_1} l(\mathbf{\Omega})dt = 0,$$

with $\mathbf{\Omega} \in \mathbb{R}^3$ *and for variations of the form* $\delta\mathbf{\Omega} = \dot{\mathbf{\Sigma}} + \mathbf{\Omega} \times \mathbf{\Sigma}$, *and bdry. conditions* $\mathbf{\Sigma}(a) = \mathbf{\Sigma}(b) = 0$.

How do they look like for the rigid body equation?

Reduced Lagrange's equations are called Euler-Poincaré equations. Euler-Poincaré equations occur for many systems: fluids, plasma dynamics etc.

What's next? Lagrangian Reduction and other bargains.

Kummer equations, Lagrange-Poincaré equations etc.

Example/exercise: discrete variational mechanics

Consider the Lagrangian function $L(q, \dot{q})$ and the action integral

$$S[q(t)] := \int_{t_0}^{t_1} L(q(t), \dot{q}(t))dt.$$

We replace the integral by a finite sum (discrete action)

$$S_{\text{dis}}[\{q^n\}] = \sum_n L(q^n, \frac{q^{n+1} - q^n}{\Delta t})\Delta t$$

and find the local minimizer from the condition

$$\frac{\partial}{\partial q^n} S[\{q^n\}] = 0.$$

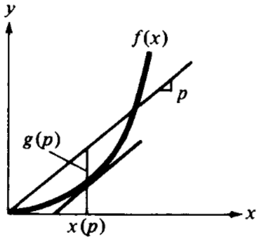
What numerical scheme do you obtain by explicitly evaluating the previous formula for a density $L(q, \dot{q}) = \dot{q}^2/2 - V(q)$? This derivation is a simple example of a simple **discrete variational principle.**

Outline

The Legendre Transform; Hamiltonian mechanics I

Let $y = f(x)$ be convex. Define

$$g(p) = \max_x (p\,x - f(x)).$$



Exercise. Experiment to compute the Legendre transform of a convex function that's a broken line.

The Legendre Transform; Hamiltonian mechanics II

Take the Legendre transform w.r.t. $v = \dot{q}$ of $L(q, v)$ and obtain $H(q, p)$ called Hamiltonian function.

After passing to the Hamiltonian side of the picture (on S-S. Chern's word's: "The sophisticated side.") we obtain that Lagrange's equations become:

$$\dot{\mathbf{q}} = \frac{\partial H(\mathbf{q}, \mathbf{p})}{\partial \mathbf{p}},$$

$$\dot{\mathbf{p}} = -\frac{\partial H(\mathbf{q}, \mathbf{p})}{\partial \mathbf{q}}.$$

$\mathbf{q}$	generalized coordinates
$\dot{\mathbf{q}}$	generalized velocities
$\mathbf{p} := \frac{\partial L}{\partial \dot{\mathbf{q}}}$	generalized momentum
$\frac{\partial L}{\partial \mathbf{q}}$	generalized force field

Canonical and non-canonical Hamiltonian structures I

Let $\mathbf{z} = (\mathbf{q}, \mathbf{p})^{\mathsf{T}} \in \mathbb{R}^d$ where $d =$ no. of D.O.F. The space of positions and generalized momenta is called phase space. It often has the structure of a cotangent bundle.

Define $\mathbf{J} := \begin{bmatrix} \mathbf{0_d} & \mathbf{I_d} \\ -\mathbf{I_d} & \mathbf{0_d} \end{bmatrix}$. The block-matrices $\mathbf{0_d}, \mathbf{I_d}$ are $d \times d$ matrices.

A canonical Hamiltonian system is an O.D.E. system of the form

$$\dot{\mathbf{z}} = \mathbf{J} \nabla_{\mathbf{z}} H(\mathbf{z}).$$

For a mechanical system with Lagrangian $L(\mathbf{q}, \mathbf{v})$, the Hamiltonian function is

$$H(\mathbf{q}, \mathbf{p}) = (\mathbf{p}, \mathbf{v}) - L(\mathbf{q}, \mathbf{v})$$

Canonical and non-canonical Hamiltonian structures II

and $\mathbf{v} = \mathbf{v}(\mathbf{p})$ by the Legendre transform.

Canonical and non-canonical Hamiltonian structures III

Example 1. $H = (\mathbf{z}^{\mathsf{T}}\mathbf{L}\mathbf{z})/2 \Rightarrow \dot{\mathbf{z}} = (\mathbf{J}\mathbf{L})\mathbf{z}$. Matrices of the form $\mathbf{J}\mathbf{L}$ w/ $\mathbf{L}$ symmetric are called **Hamiltonian matrices.** They generate the algebra of infinitesimally symplectic matrices. (More about symplectic transformations later on.)

Take for example the harmonic oscillator Hamiltonian $H = \frac{1}{2}[q, p]\begin{bmatrix} \omega^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix} = p^2/2 + \omega^2 q^2/2.$

Re-scaling: $p = \hat{p}\sqrt{\omega}$ and $q = \hat{q}/\sqrt{\omega}$ we can re-write Hamiltonian as,

$$\hat{H} = \omega(\hat{p}^2 + \hat{q}^2)/2.$$

Canonical and non-canonical Hamiltonian structures IV

In the new (canonical) coordinates:

$$\frac{d}{dt}\begin{bmatrix} \hat{q} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 1 \end{bmatrix}\begin{bmatrix} \hat{q} \\ \hat{p} \end{bmatrix}$$

and exponentiating

$$\begin{bmatrix} \hat{q}(t) \\ \hat{p}(t) \end{bmatrix} = \exp(t\begin{bmatrix} 0 & \omega \\ -\omega & 1 \end{bmatrix})\begin{bmatrix} \hat{q}_0 \\ \hat{p}_0 \end{bmatrix} = \mathbf{\Phi}_{\omega \mathbf{t}}\begin{bmatrix} \hat{q}_0 \\ \hat{p}_0 \end{bmatrix},$$

and $\mathbf{\Phi}_{\omega \mathbf{t}}$ is a rotation matrix in the plane.

Spectral structure of Hamiltonian matrices

λ eigenvalue $\Rightarrow -\lambda, \bar{\lambda}, -\bar{\lambda}$ are also eigenvalues.

Proof: $\mathbf{JLv} = \lambda \mathbf{v} \Rightarrow \mathbf{JL}(\mathbf{Jw}) = \lambda \mathbf{Jw} \Rightarrow -\mathbf{L}(\mathbf{Jw}) = -\lambda \mathbf{w}$, but $(LJ) = -(JL)^{\mathsf{T}}$ and therefore $-\lambda$ is eigenvalue of the transposed Hamiltonian matrix. $\square$

Examples I

(1) One D.O.F. problems.
(2) Central forces.
(3) Charged particle in a magnetic field (non-canonical Hamiltonian system):
$\frac{d}{dt}\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{0_d} & \mathbf{I_d} \\ \mathbf{I_d} & \hat{\mathbf{b}} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix}$, the matrix $\mathbf{J}_{\hat{\mathbf{b}}}$ is an example of a non-canonical Hamiltonian structure.
For this non canonical structure, the charged particle Hamiltonian is written as:

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2m}|\mathbf{p}|^2 - \gamma/|\mathbf{q}|.$$

Examples II

(4) N-body problems.

$$\mathbf{F_{ij}} = -\frac{\partial \phi_{ij}(\mathbf{r_{ij}})}{\partial \mathbf{r_{ij}}}$$

and
$H = \frac{1}{2}\sum_{i=1}^{N} |\mathbf{p_i}^2|/m_i + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \phi(\mathbf{r_{ij}}) = T + U.$

(5) Rigid-body motion, other types of Lie-Poisson dynamics etc.

First Integrals and Poisson Brackets I

A first integral (or integral of motion) is a function $G : \mathbb{R}^{2d} \to \mathbb{R}$ such that $G(\mathbf{z}(t; \mathbf{z_0}) = G(\mathbf{z_0})$. This physical quantities are conserved along the trajectories (solution curves) of a Hamiltonian system

$$\dot{\mathbf{z}} = \mathbf{J}\nabla_{\mathbf{z}} H(\mathbf{z}).$$

First integrals usually lead to geometric reduction of the problem: solution curves will live in $\{G = \text{constant}\}$.
Problem: find a practical way to determine a function is an integral of motion.

First Integrals and Poisson Brackets II

Use the chain rule:

$$\frac{d}{dt}(G(\mathbf{z}(t;\mathbf{z_0})) = \nabla_{\mathbf{z}} G(\mathbf{z}(t;\mathbf{z_0}))^{\mathsf{T}} \frac{d}{dt}\mathbf{z}(t;\mathbf{z_0}),$$

but $\frac{d}{dt}\mathbf{z}(t;\mathbf{z_0}) = \mathbf{J}\nabla_{\mathbf{z}} H(\mathbf{z}(t;\mathbf{z_0}))$ and therefore

$$\frac{d}{dt}(G(\mathbf{z}(t;\mathbf{z_0})) = \nabla_{\mathbf{z}} G(\mathbf{z}(t;\mathbf{z_0}))^{\mathsf{T}} \mathbf{J}\nabla_{\mathbf{z}} H(\mathbf{z}(t;\mathbf{z_0})).$$

This leads us to introduce the following bilinear operation on scalar fields defined on phase space:

$$\{F, G\}(\mathbf{z}) := \nabla_{\mathbf{z}} F(\mathbf{z})^{\mathsf{T}} \mathbf{J}\nabla_{\mathbf{z}} G(\mathbf{z}) \text{ (**Poisson bracket**)} .$$

First Integrals and Poisson Brackets III

1. antisymmetry $\{F, G\} = -\{G, F\}$.
2. A fundamental property of the Poisson bracket is the Jacobi identity:

$$\{H, \{F, G\}\} = -\{G, \{H, F\}\} - \{F, \{G, H\}\}.$$

Exercise. Check that the components of the vector $\mathbf{m} = \mathbf{q} \times \mathbf{p}$ is a conserved quantity for the system with Hamiltonian $H(\mathbf{p}, \mathbf{q}) = \frac{|\mathbf{p}|^2}{2} - \frac{1}{|\mathbf{q}|}$.

The matrix J does not need to be constant. Non-canonical structures are very common all over physics and mechanics. Take for instance the following poisson structure in $\mathbb{R}^3$:

$$\{F(\mathbf{M}), G(\mathbf{M})\}_{\mathrm{EP}} := \langle \nabla F(\mathbf{M}), \nabla C(\mathbf{M}) \times \nabla G(\mathbf{M}) \rangle,$$

First Integrals and Poisson Brackets IV

$C = (M_1^2 + M_2^2 + M_3^2)/2 = |\mathbf{M}|^2/2.$
Take $H = \langle \mathbf{M}, \mathbb{I}^{-\mathbf{1}}\mathbf{M}\rangle/2$ (rigid body kinetic energy), therefore

$$\dot{\mathbf{M}} = \nabla C \times \nabla H = \mathbf{M} \times \mathbb{I}^{-1}\mathbf{M}.$$

Our non-constant Poisson structure is

$$\mathbf{J}_{\text{EP}} := \begin{bmatrix} 0 & -M_3 & M_2 \\ M_3 & 0 & M_1 \\ -M_2 & M_1 & 0 \end{bmatrix},$$

and the Poisson bracket becomes

$$\{F(\mathbf{M}), G(\mathbf{M})\}_{\text{EP}} = \nabla \mathbf{F}^{\mathsf{T}} \mathbf{J}_{\text{EP}} \nabla \mathbf{G}.$$

First Integrals and Poisson Brackets V

The inertia tensor can be made diagonal by a orthogonal change of basis, and

$$\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3),$$

$I_1 > I_2 > I_3$.
We need to check that surfaces $\{H = \text{const}\}$ and $\{C = \text{const}\}$ are invariant manifolds.

First Integrals and Poisson Brackets VI

By intersecting different ellipsoids

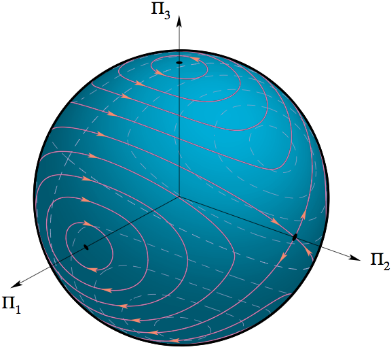
$$\{H = \frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3} = \text{const}\}$$

and the sphere

$$\{C = M_1^2 + M_2^2 + M_3^2 = \text{const}\}$$

we obtain the reduced solution curves depicted in the blue sphere. Using reconstruction formulas we can compute the associated motion in $SO(3)$.

The famous picture from JEM's book cover.



Applications of Poisson structures

Lie-Poisson integrators, Lie-transformation methods in bifurcation theory, field theories, constrained Hamiltonian systems etc.

Hamiltonian flows I

Flows generated by Hamiltonian vector fields possess many useful geometric properties.

The Hamiltonian vector field $\mathbf{X}_H(\mathbf{z}) = \mathbf{J}\nabla H(\mathbf{z})$ generates a flow on the manifold M^{2d}, often T^*Q with coordinates $(\mathbf{q}, \mathbf{p})$.

For example, consider a free particle moving in space

$$\ddot{\mathbf{q}} = 0.$$

Its equations of motion in Hamiltonian form are

$$\dot{\mathbf{q}} = \mathbf{p},$$
$$\dot{\mathbf{p}} = \mathbf{0},$$

and the corresponding Hamiltonian is $H_{\text{part}}(\mathbf{q}, \mathbf{p}) = |\mathbf{p}|^2/2$.

The flow map is

$$\Phi_H^t(\mathbf{q_0}, \mathbf{p_0}) = (q_0 + tp_0, p_0).$$

Hamiltonian flows II

The mapping Φ_H^t is a 1-parameter family of transformations of $\mathbb{R}^{2d}$.

Another useful example is the harmonic oscillator, $H_{\text{osc}} = p^2/2 + \omega^2 q^2/2$. The flow generated by the Hamiltonian vector field in this case is

$$\Phi_H^t(p_0, q_0) = \begin{bmatrix} \cos(\omega t) & \omega^{-1}\sin(\omega t) \\ \omega \sin(\omega t) & \sin(\omega t) \end{bmatrix} \begin{bmatrix} q_0 \\ p_0 \end{bmatrix},$$

which is conjugate to a rotation matrix in the plane.

An important property of Hamiltonian flows is that they infinitesimally preserve the symplectic (resp. Poisson) structure.

Hamiltonian flows III

This means that $\Phi_{\mathbf{z}}(\mathbf{z})\mathbf{J}(\mathbf{z})\Phi_{\mathbf{z}}(\mathbf{z})^{\mathsf{T}} = \mathbf{J}(\mathbf{z})$

Examples/exercises

1. A particle in a central field

$$L = \frac{1}{2}||\dot{\mathbf{q}}||^2 - (-1/||\mathbf{q}||),$$

   and since $m = 1$, $\mathbf{p} = \dot{\mathbf{q}}$.

2. A charged particle in a magnetic field

$$L = \frac{m}{2}||\dot{\mathbf{q}}||^2 - (-\gamma/||\mathbf{q}|| - \frac{1}{2}\mathbf{B}(\mathbf{q}, \dot{\mathbf{q}})),$$

   $\mathbf{B}$ is an anti-symmetric matrix representing a constant magnetic field. In this case, $\mathbf{p} = m\dot{\mathbf{q}} - \frac{1}{2}\mathbf{B}(\mathbf{q})$. **Example of a non-canonical hamiltonian system. More later.**

The braquistocrhone problem: Pontryagin Principle

Our next goal is to connect the calculus of variations with Hamiltonian mechanics.

Outline

Bibliography