Exterior Calculus, its Fourier transform based discretization, and Applications

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Motivation

- Objective: simulation with qualitatively correct behavior
  - Structure preserving discretization
  - PDEs: requires a discretization of exterior calculus
  - Adaptive simulation to capture relevant details

De Rham complex

$$0 \xrightarrow{\mathrm{d}} \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \rightarrow \cdots \rightarrow \Omega^{n-1} \xrightarrow{\mathrm{d}} \Omega^n \xrightarrow{\mathrm{d}} 0$$

De Rham complex

$$0 \xrightarrow{\mathrm{d}} \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \rightarrow \cdots \rightarrow \Omega^{n-1} \xrightarrow{\mathrm{d}} \Omega^n \xrightarrow{\mathrm{d}} 0$$

Forms as fields covariant under differentiation

De Rham complex

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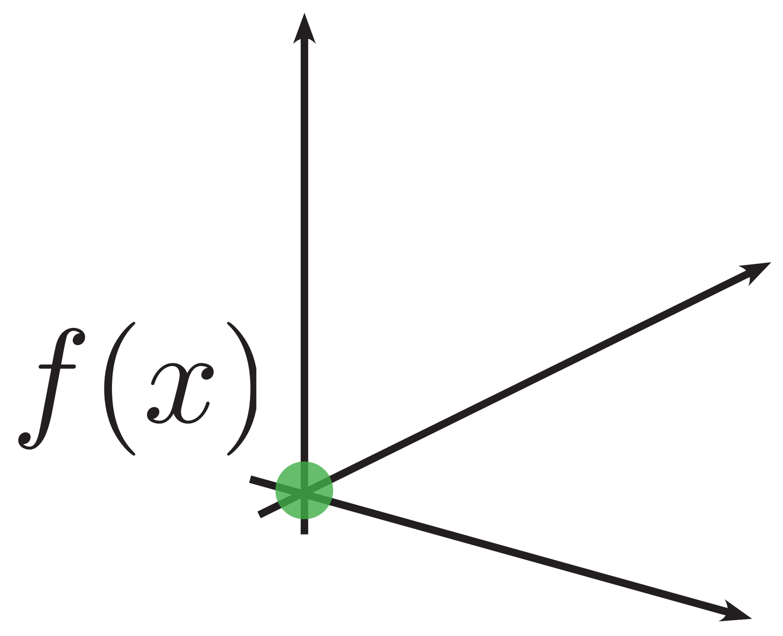

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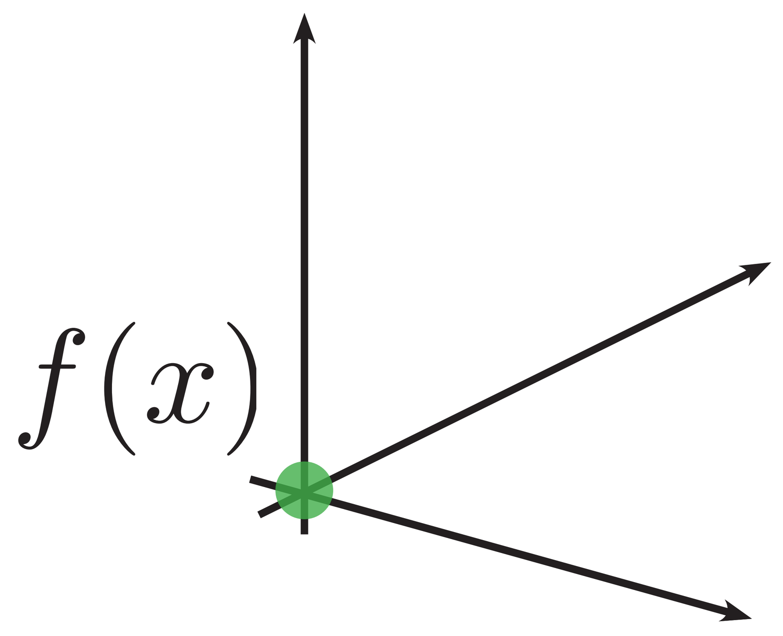

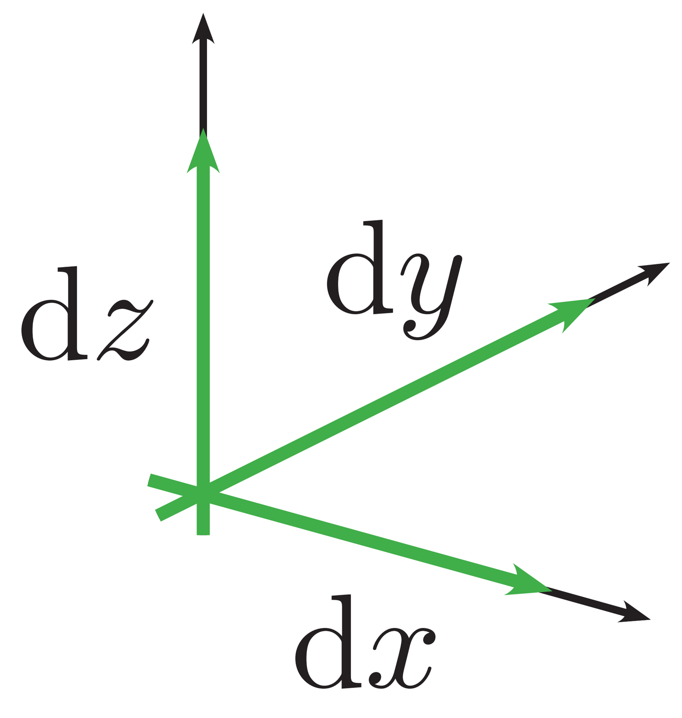

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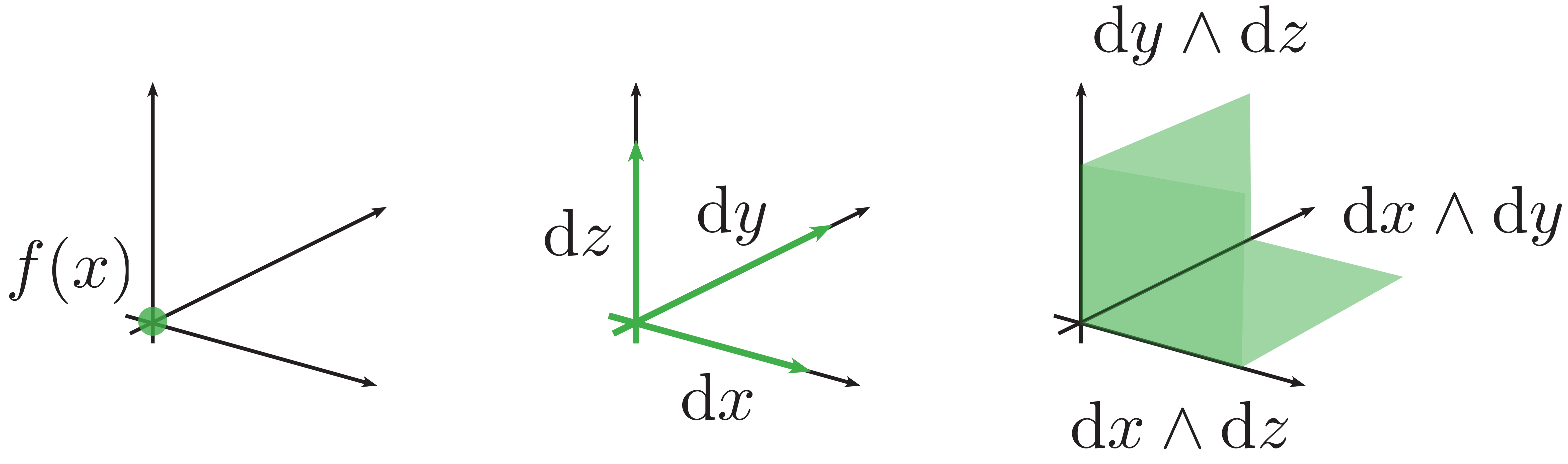

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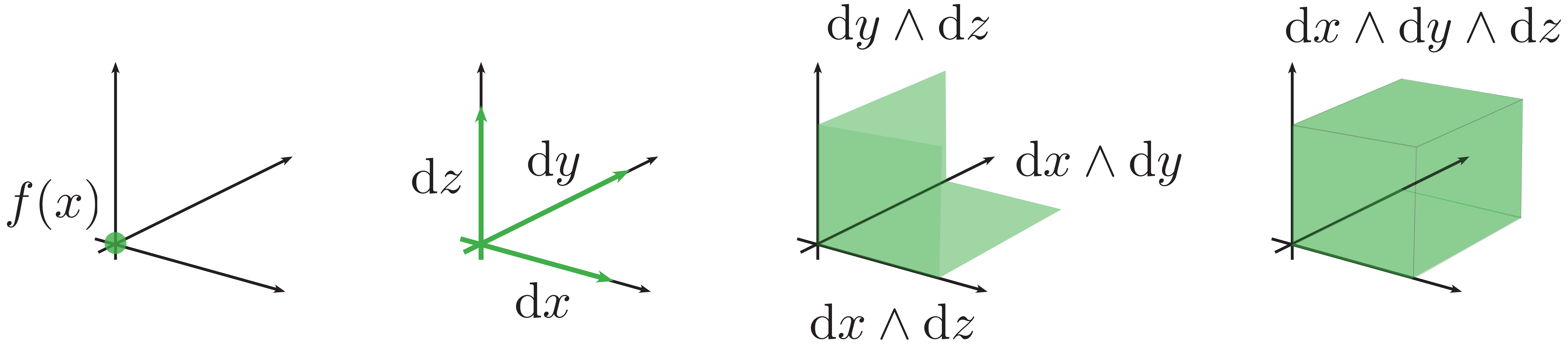

Forms as infinitesimal integrals (measurements)

De Rham complex (with metric)

$$0 \xrightarrow{\mathrm{d}} \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \rightarrow \cdots \rightarrow \Omega^{n-1} \xrightarrow{\mathrm{d}} \Omega^n \xrightarrow{\mathrm{d}} 0$$

De Rham complex (with metric)

$$
\begin{tikzcd}
0 \arrow[r, "\mathrm{d}"] & \Omega^0 \arrow[r, "\mathrm{d}"] \arrow[rrrr, <->, dashed, bend right, "\star"'] & \Omega^1 \arrow[r] \arrow[rr, <->, dashed, bend right, "\star"'] & \cdots \arrow[r] & \Omega^{n-1} \arrow[r, "\mathrm{d}"] & \Omega^n \arrow[r, "\mathrm{d}"] & 0
\end{tikzcd}
$$

De Rham complex (with metric)

$$
\begin{tikzcd}
0 \arrow[r, "\mathrm{d}"] & \Omega^0 \arrow[r, shift left, "\mathrm{d}"] \arrow[rrrr, bend right=30, dashed, <->, "\star"'] & \Omega^1 \arrow[l, shift left, "\delta"] \arrow[r, shift left] \arrow[rr, bend right=30, dashed, <->, "\star"'] & \cdots \arrow[l, shift left] \arrow[r, shift left] & \Omega^{n-1} \arrow[l, shift left] \arrow[r, shift left, "\mathrm{d}"] & \Omega^n \arrow[l, shift left, "\delta"] \arrow[r, "\mathrm{d}"] & 0
\end{tikzcd}
$$

De Rham complex (with metric)

$$0 \xrightarrow{\mathrm{d}} \Omega^0 \underset{\delta}{\overset{\mathrm{d}}{\rightleftharpoons}} \Omega^1 \rightleftharpoons \cdots \rightleftharpoons \Omega^{n-1} \underset{\delta}{\overset{\mathrm{d}}{\rightleftharpoons}} \Omega^n \xrightarrow{\mathrm{d}} 0$$

$$\star \qquad \star$$

- Laplace Beltrami operator: $\Delta = \mathrm{d}\,\delta + \delta\,\mathrm{d}$
- Hodge-Helmholtz decomposition: $\Omega^k = \Omega^k_{\mathrm{d}} \oplus \Omega^k_{\delta} \oplus \Omega^k_h$

Fourier transform

$$\mathcal{F}(f) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)\, e^{i\xi(x)}\, \mathrm{dx}$$

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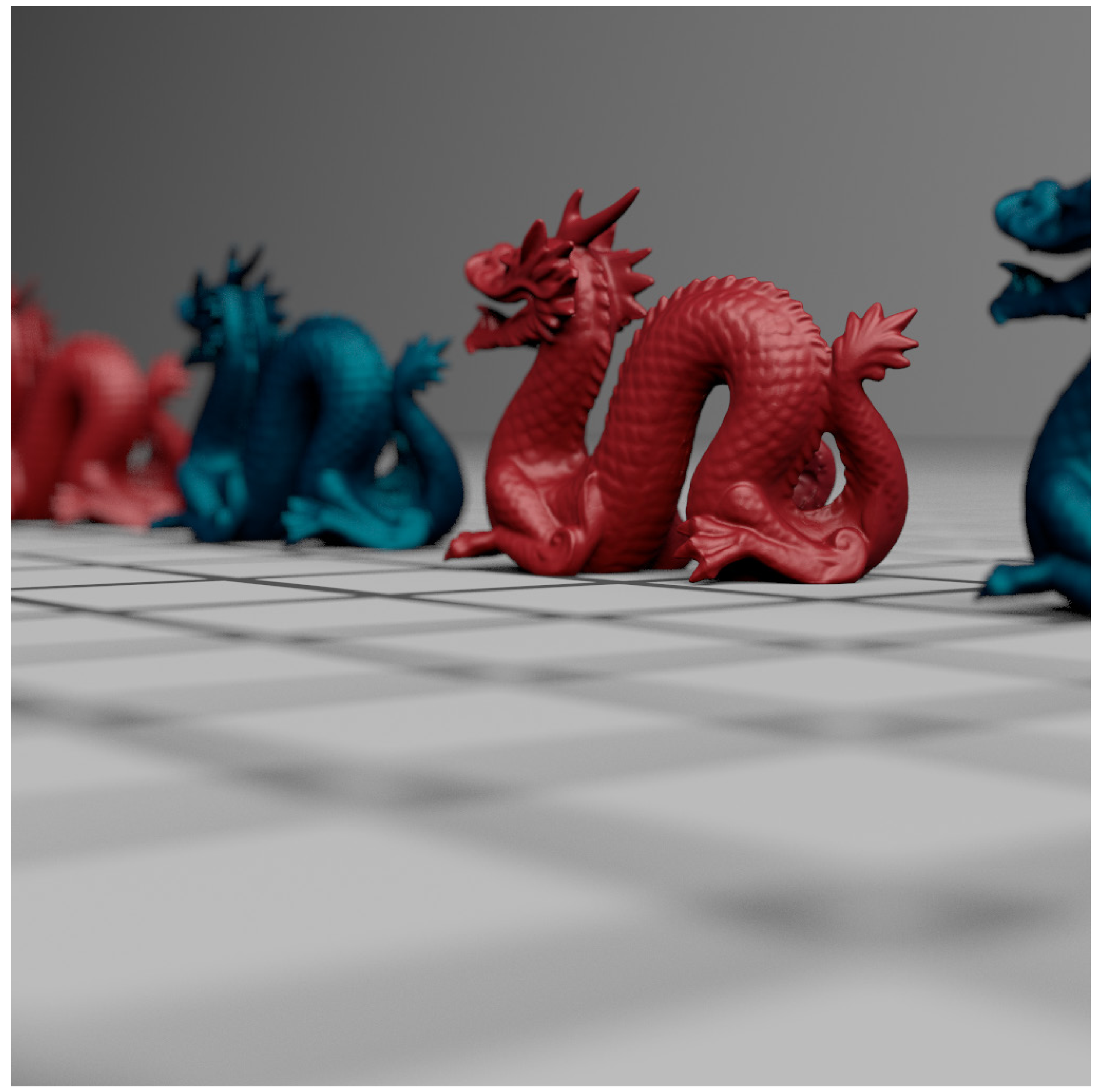

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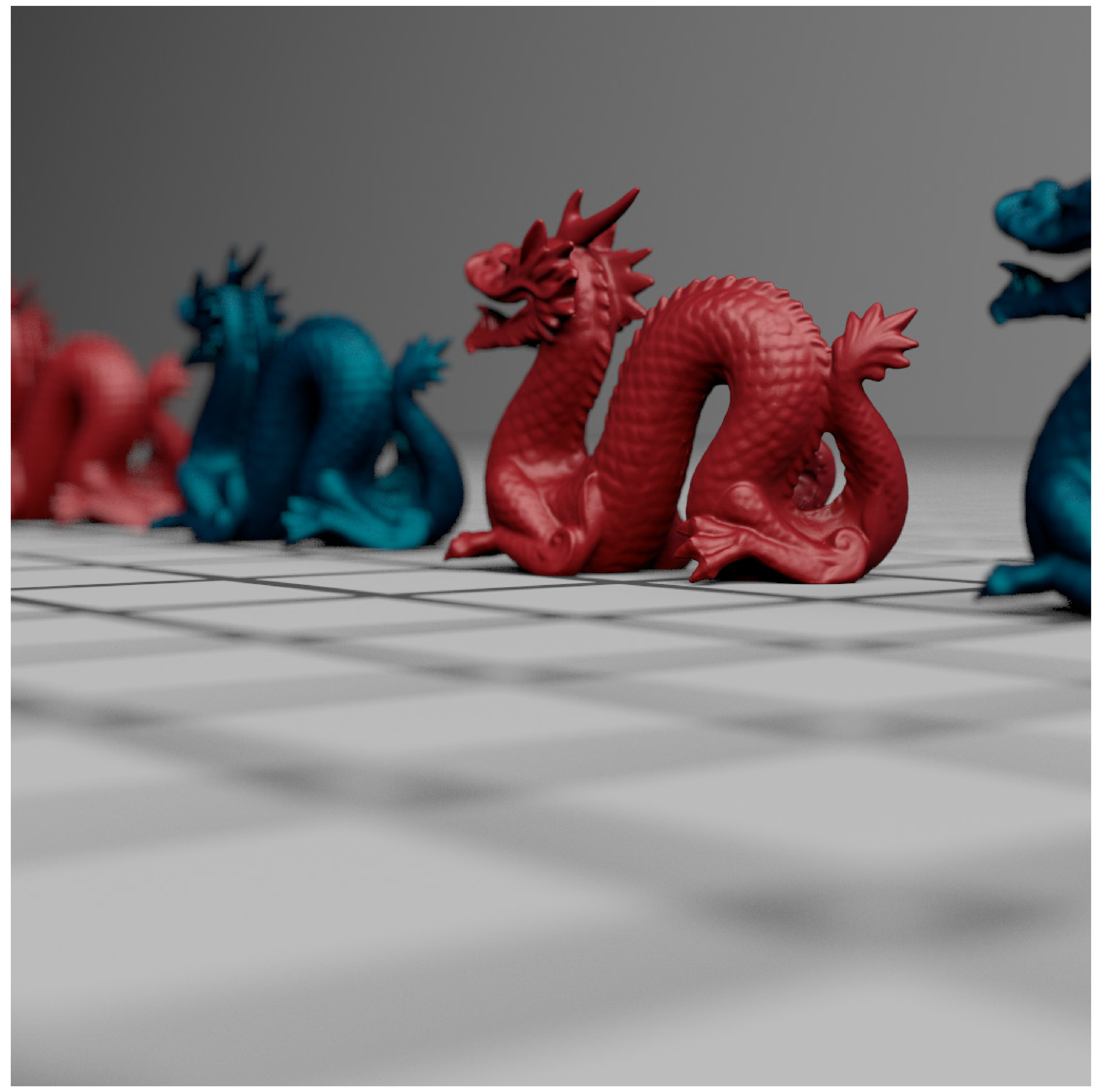
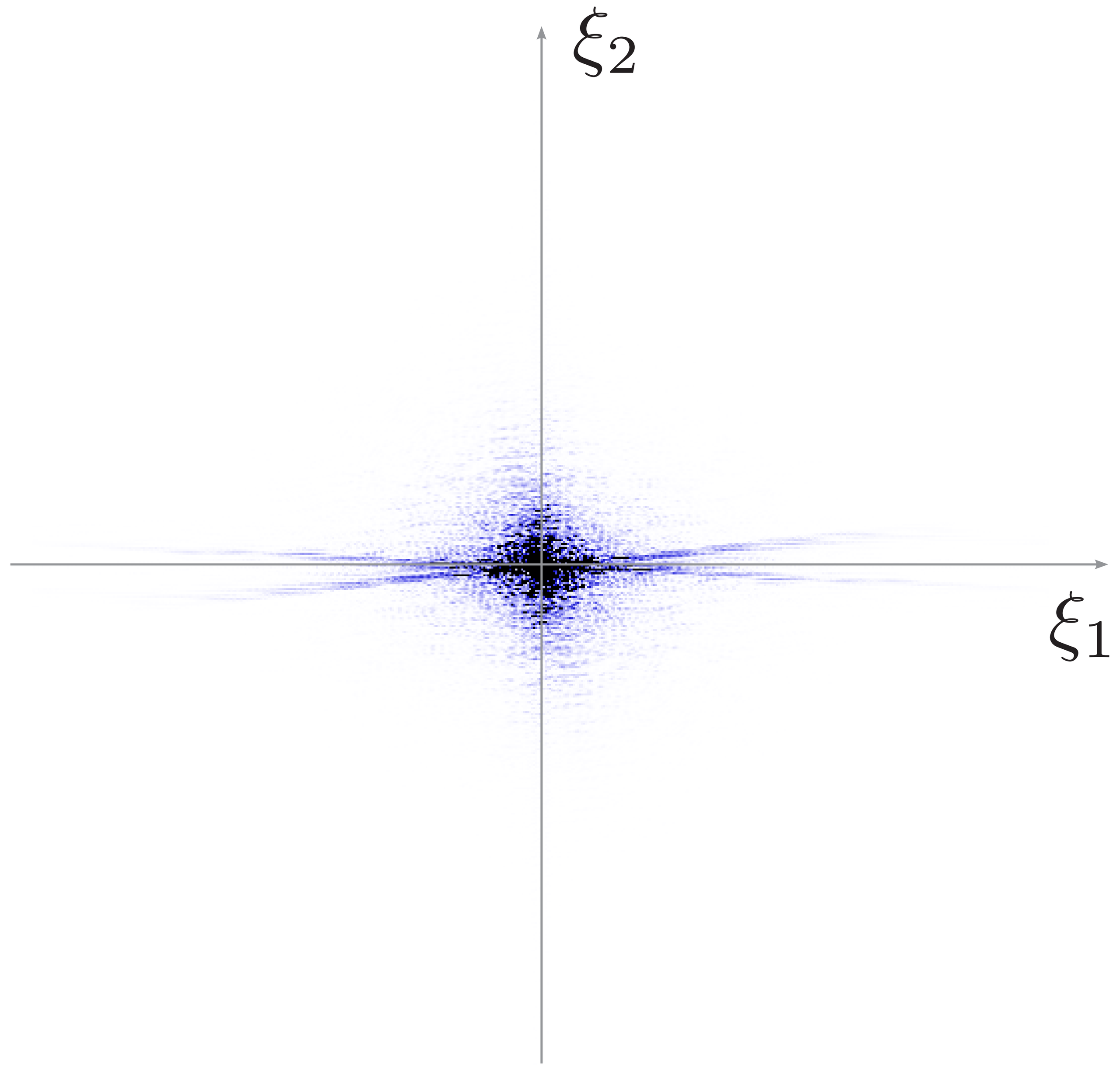


$$\xrightarrow{\quad\mathcal{F}\quad}$$



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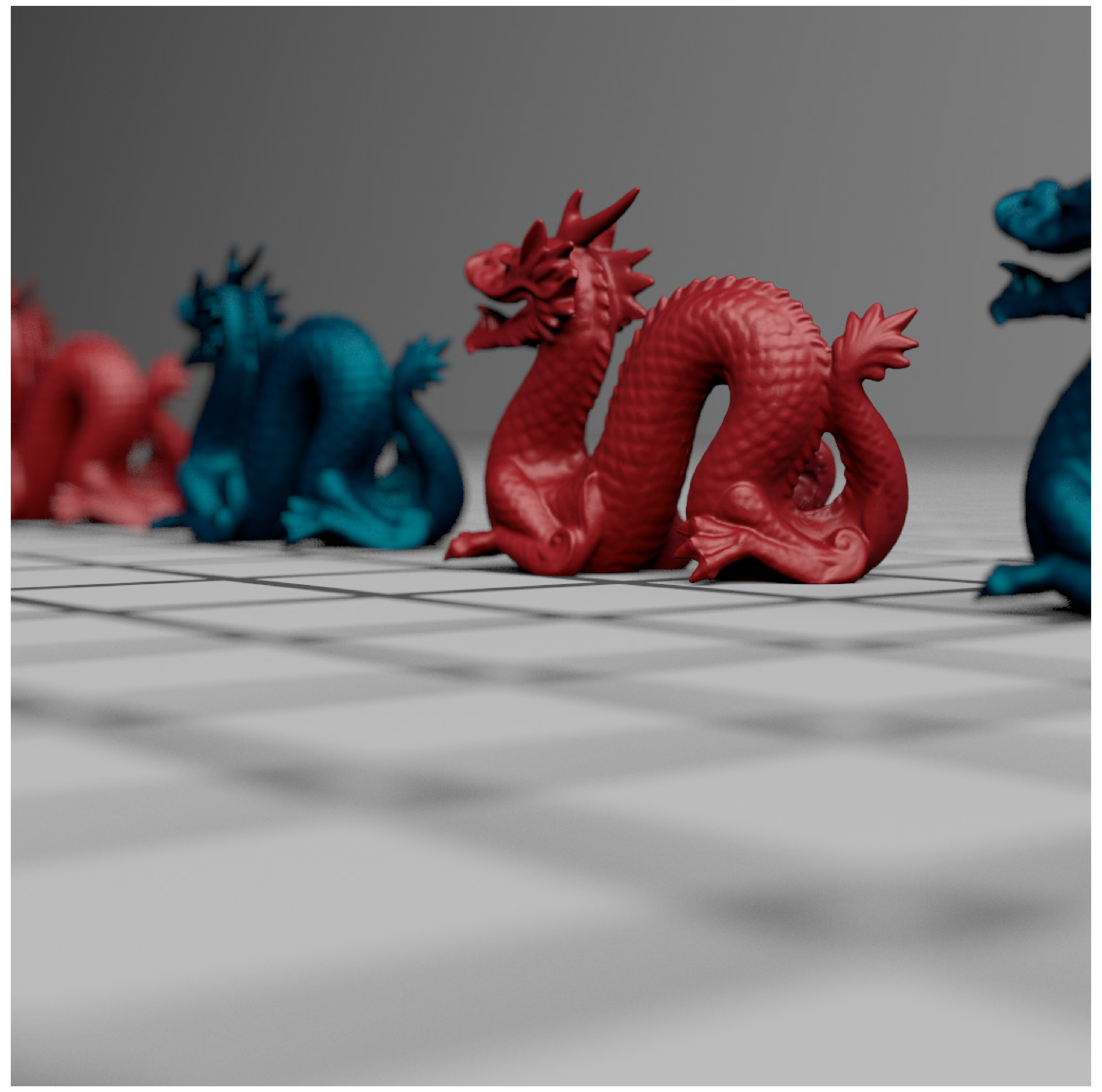
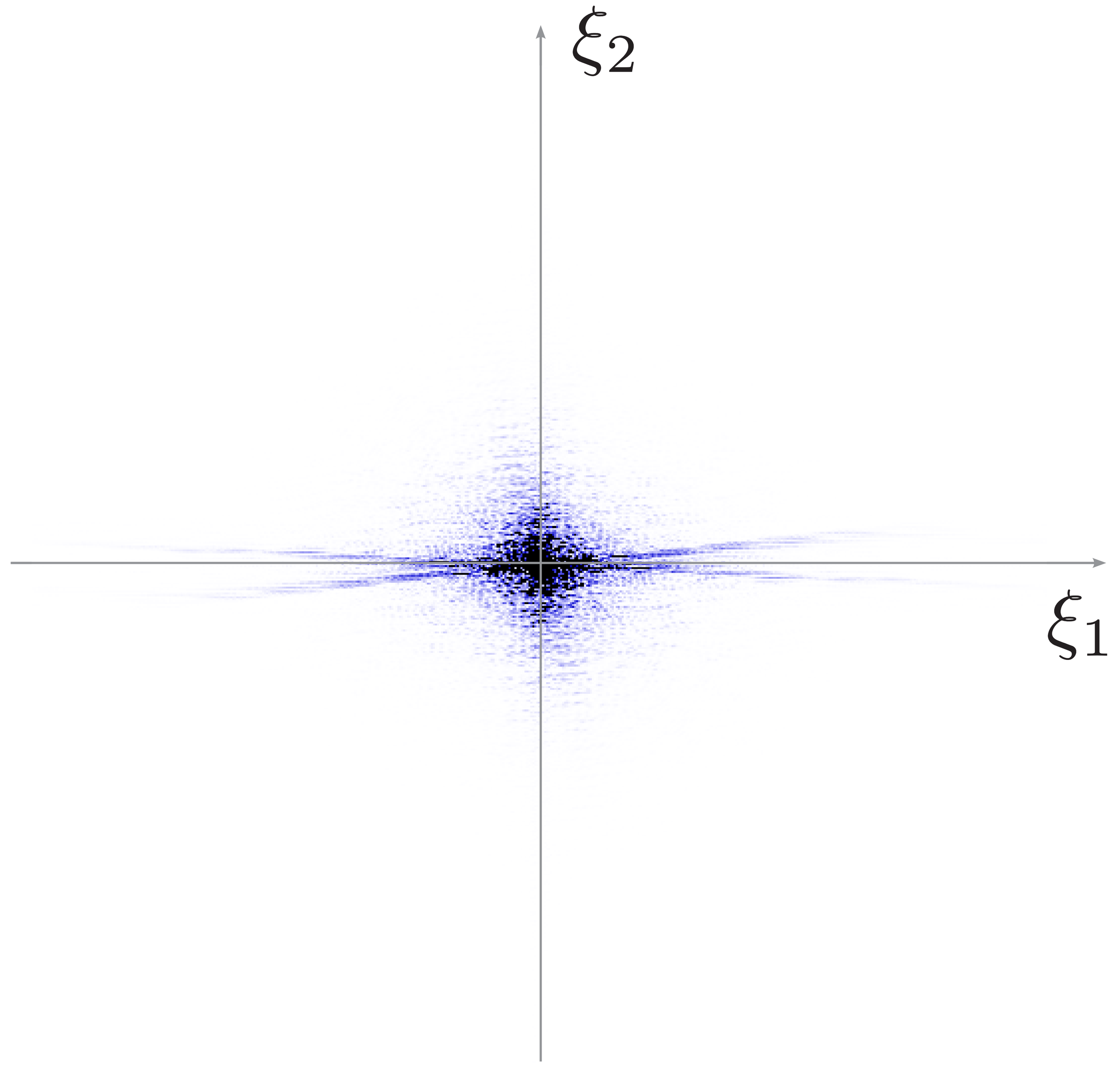


$$\underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}}$$



Exterior calculus and the Fourier transform

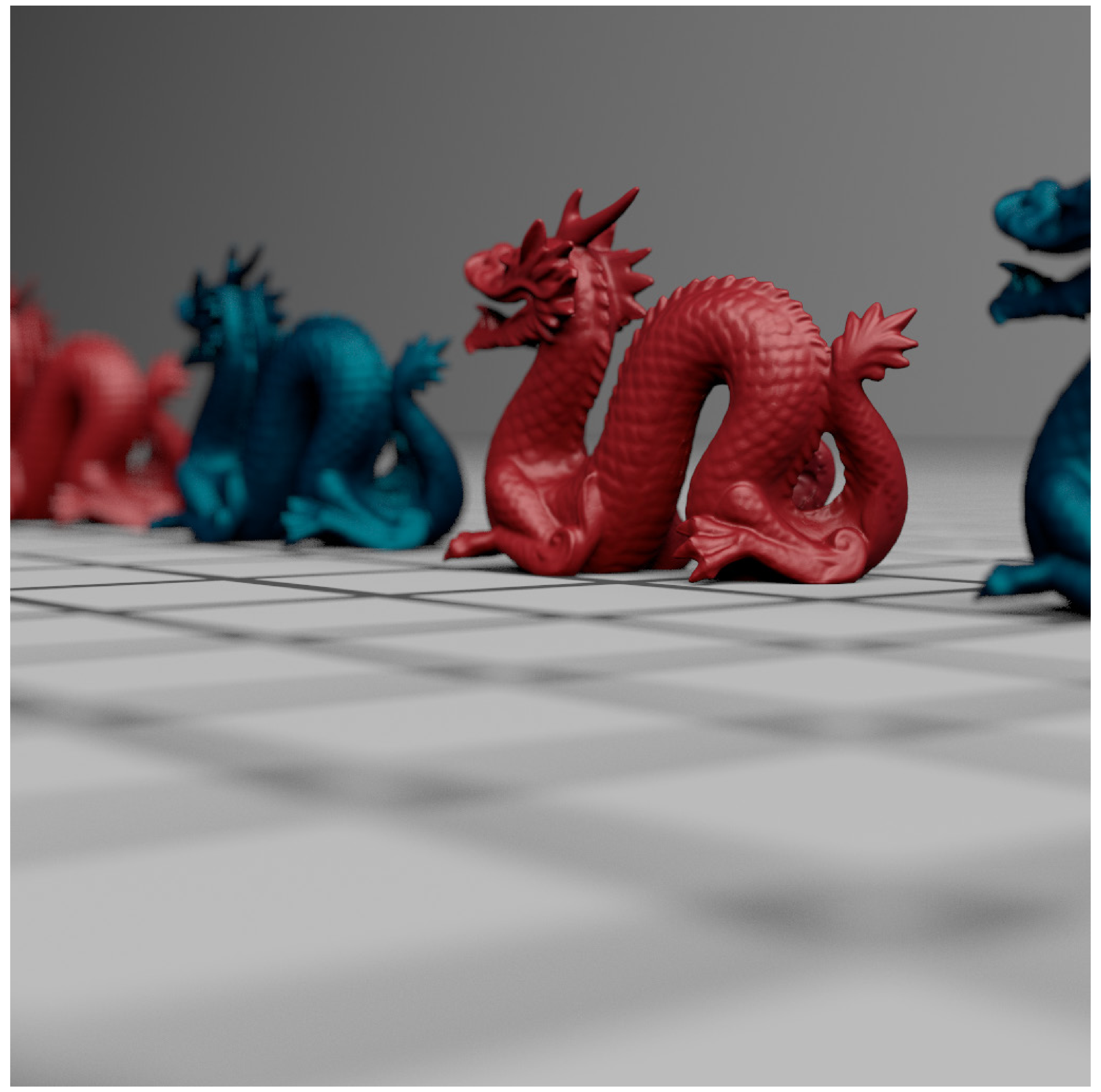
Exterior calculus and the Fourier transform

- Integration:

Exterior calculus and the Fourier transform

- Integration:

$$\int$$



$$\mathrm{d}x$$

Exterior calculus and the Fourier transform

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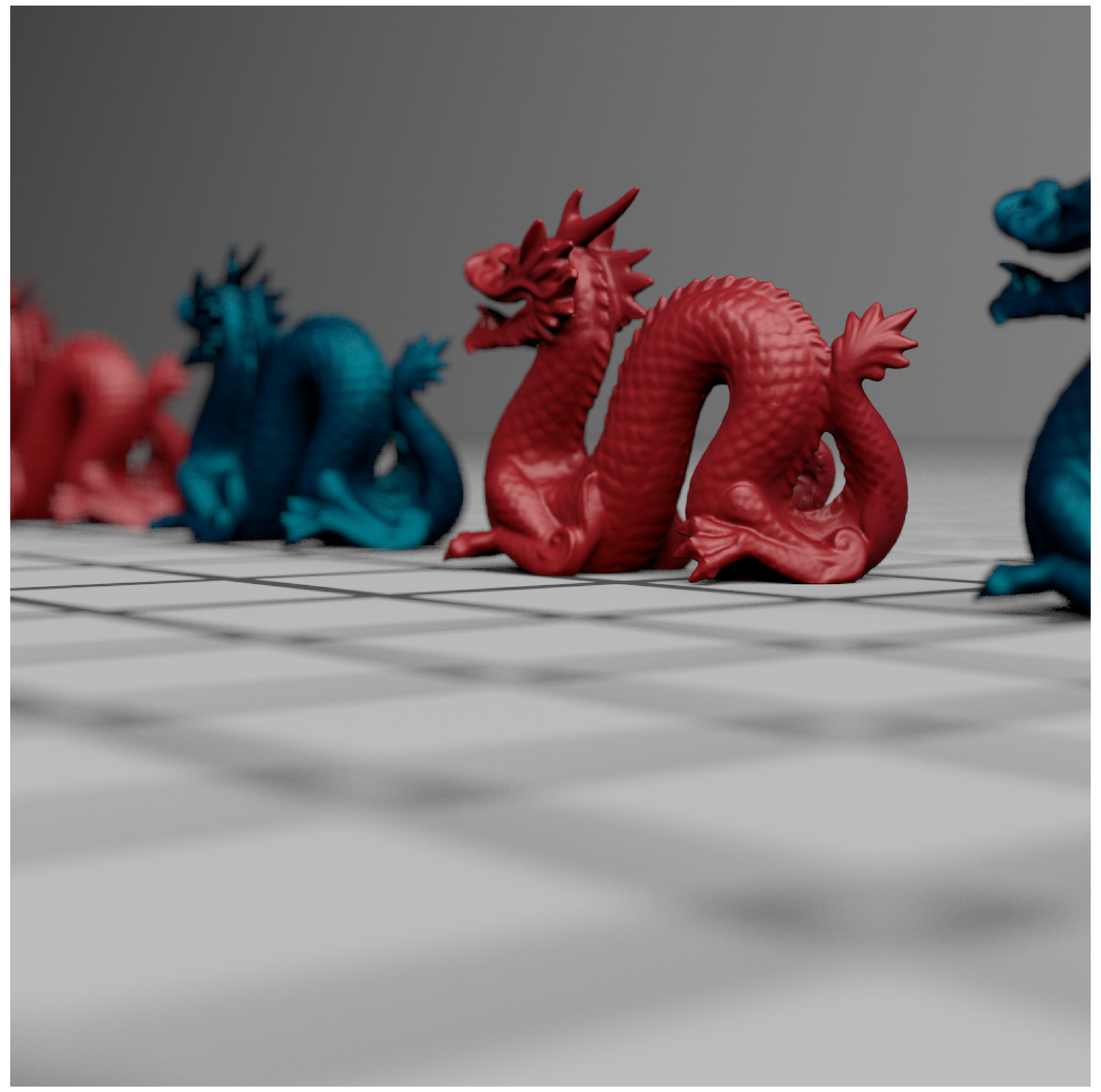
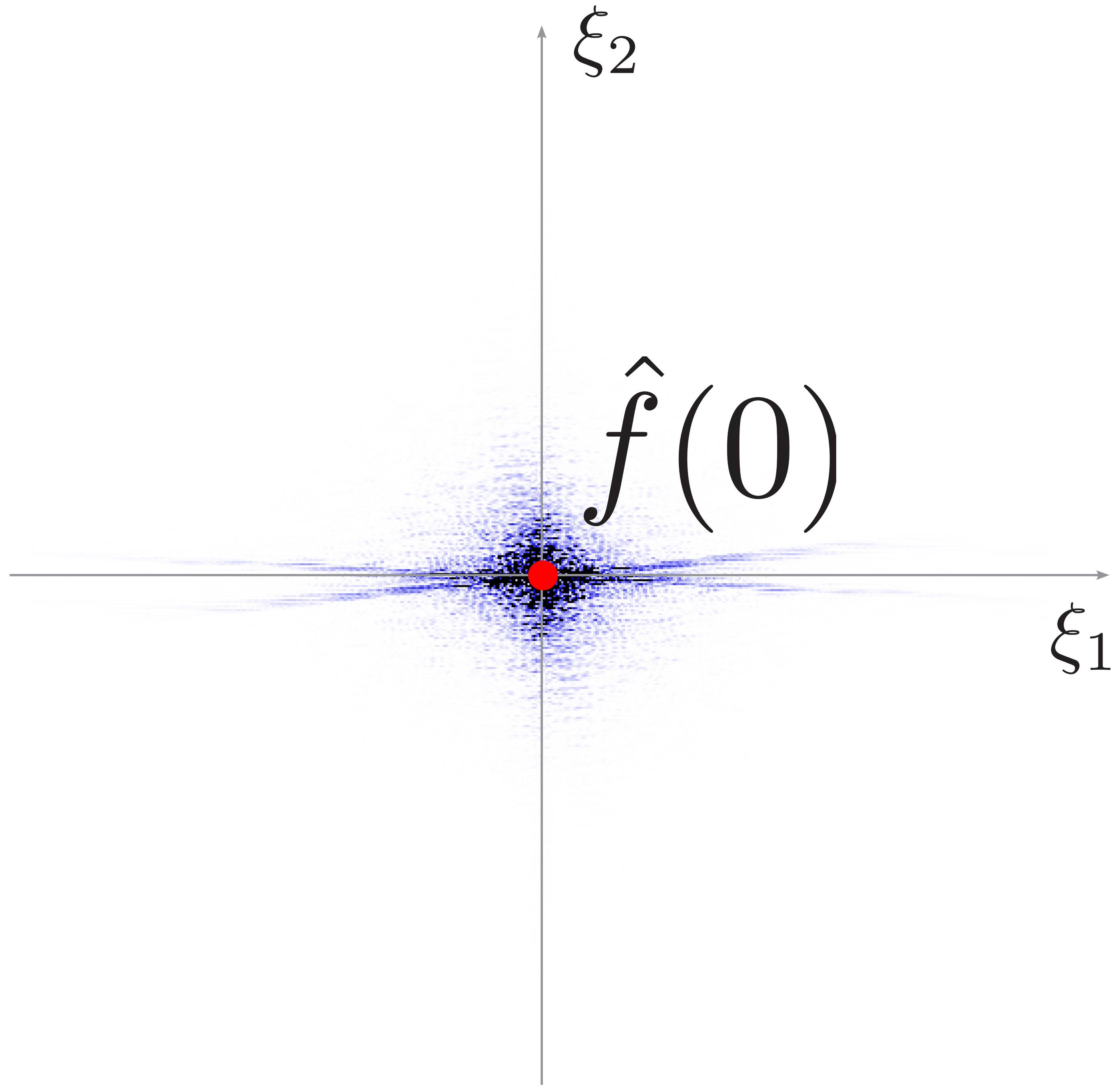
$\int$  $\mathrm{d}x$ $\Longleftrightarrow$ 

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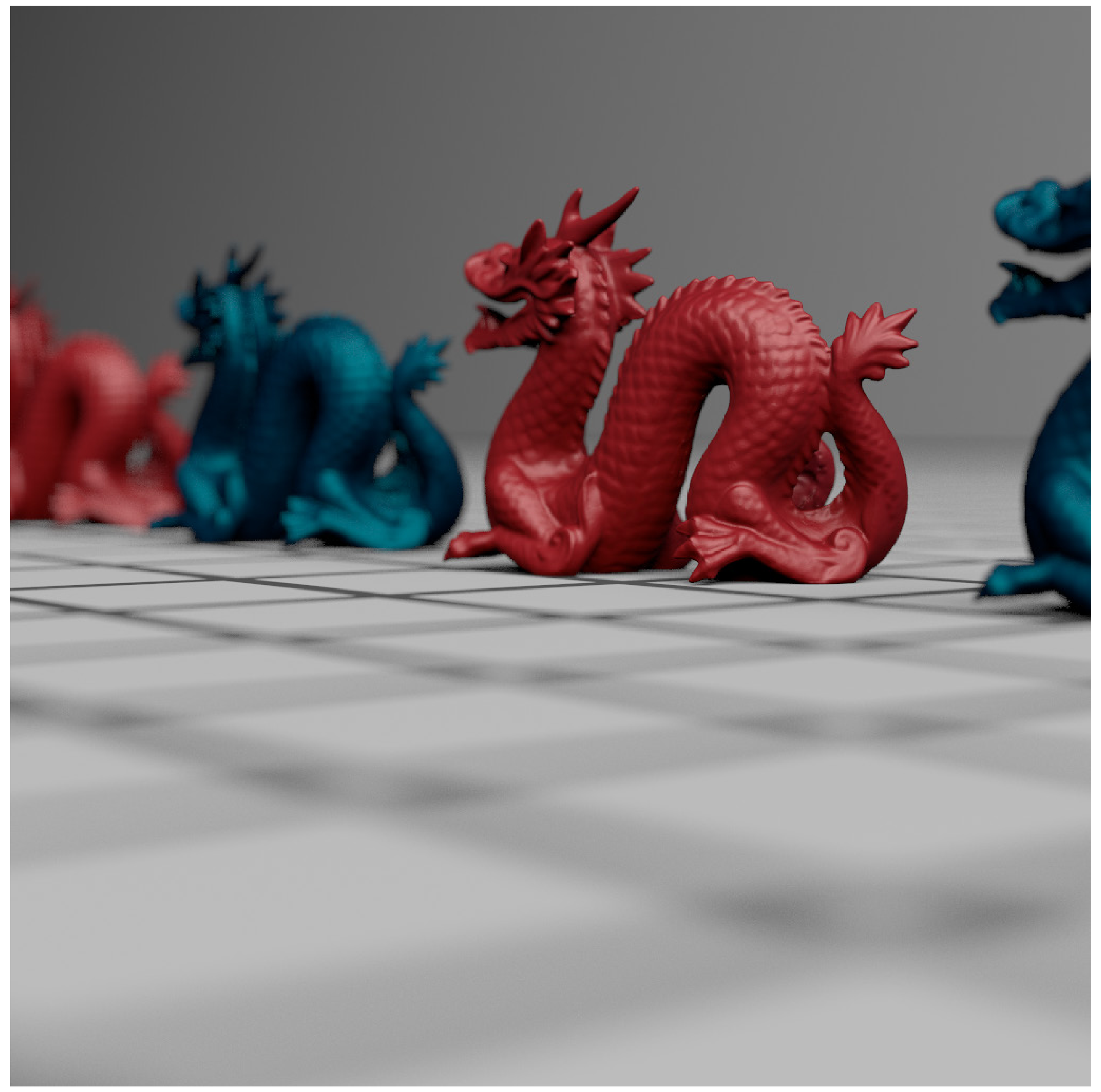
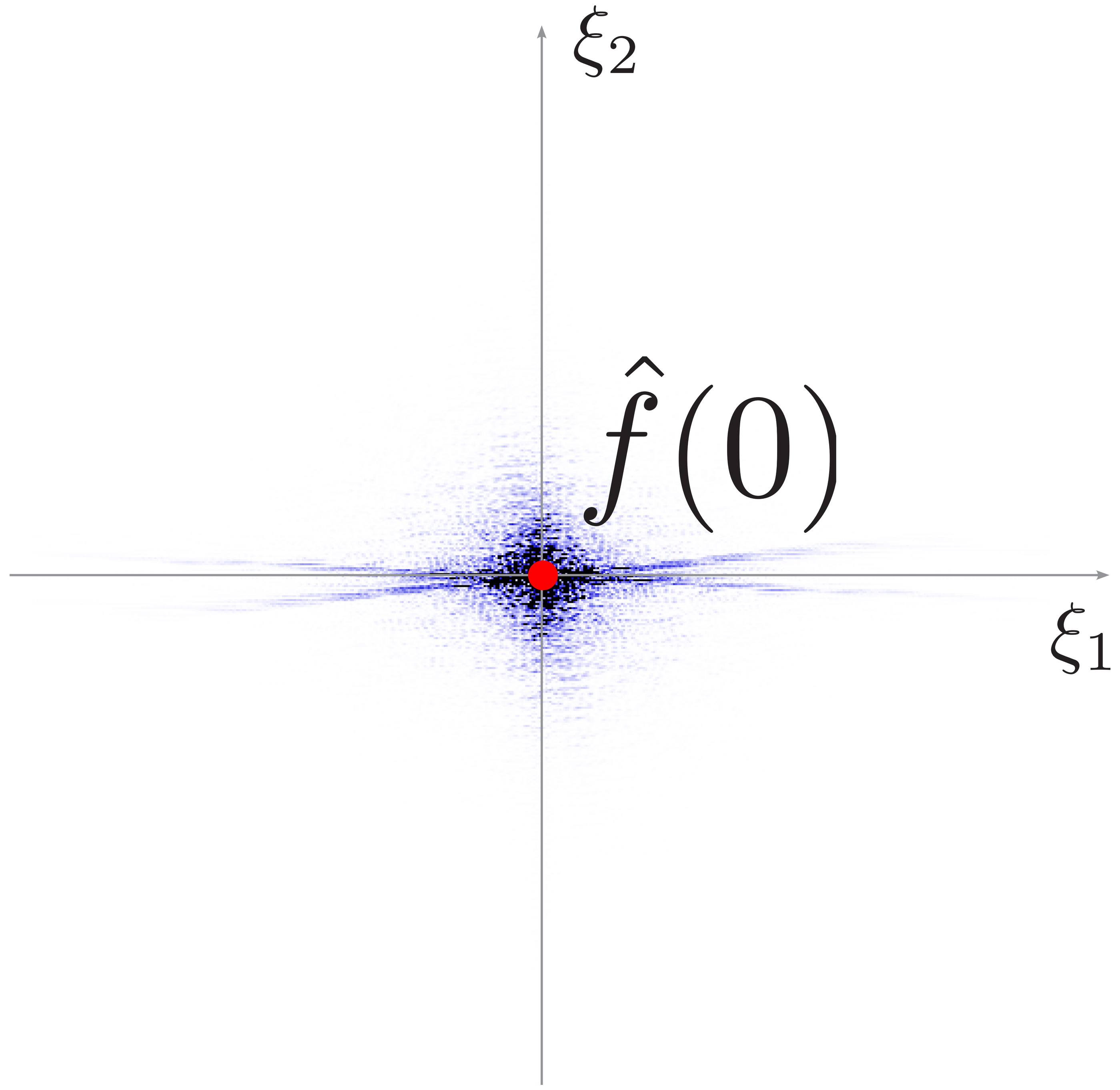


$\mathrm{d}x$

2-form

$\Longleftrightarrow$



Exterior calculus and the Fourier transform

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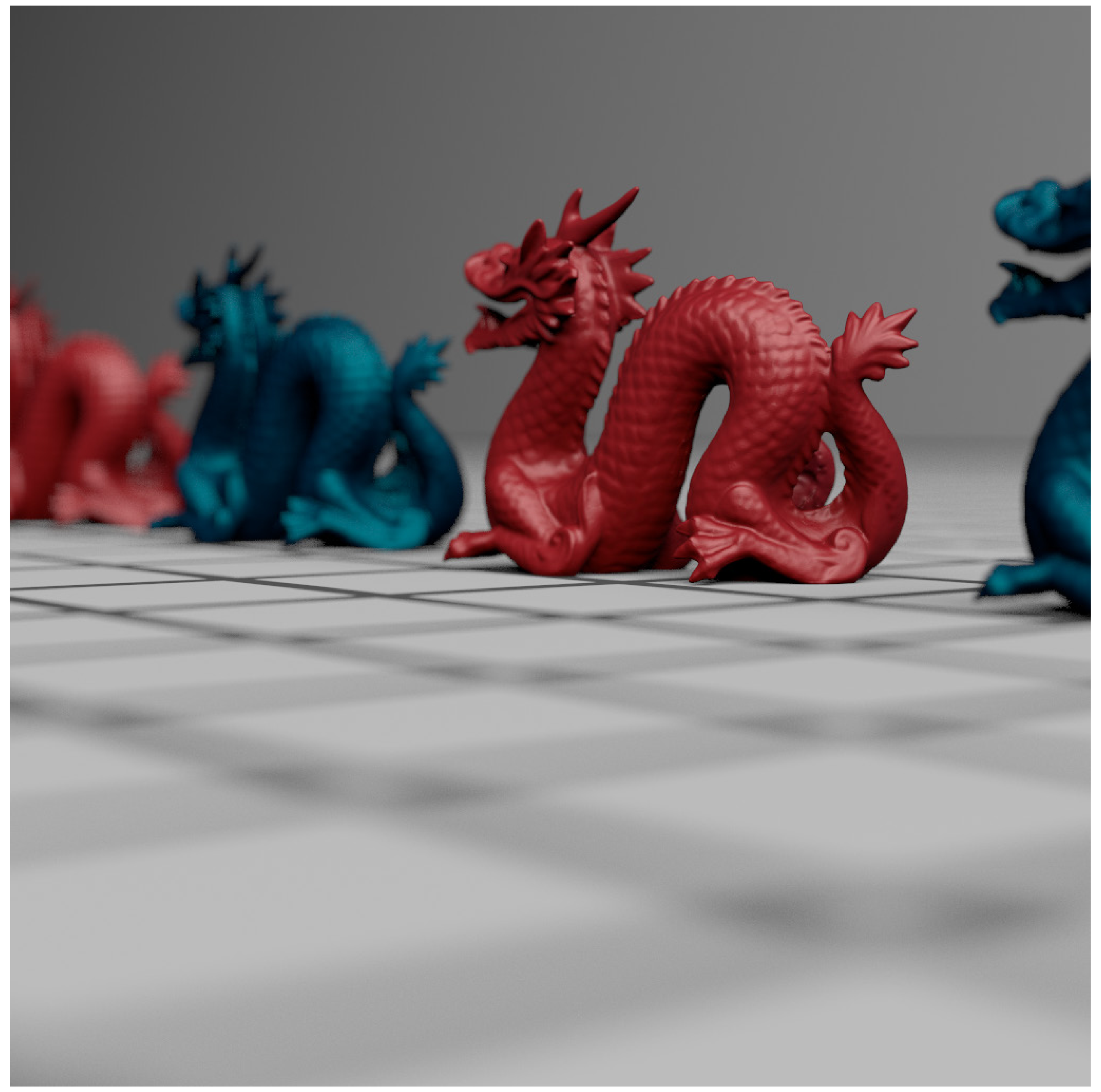
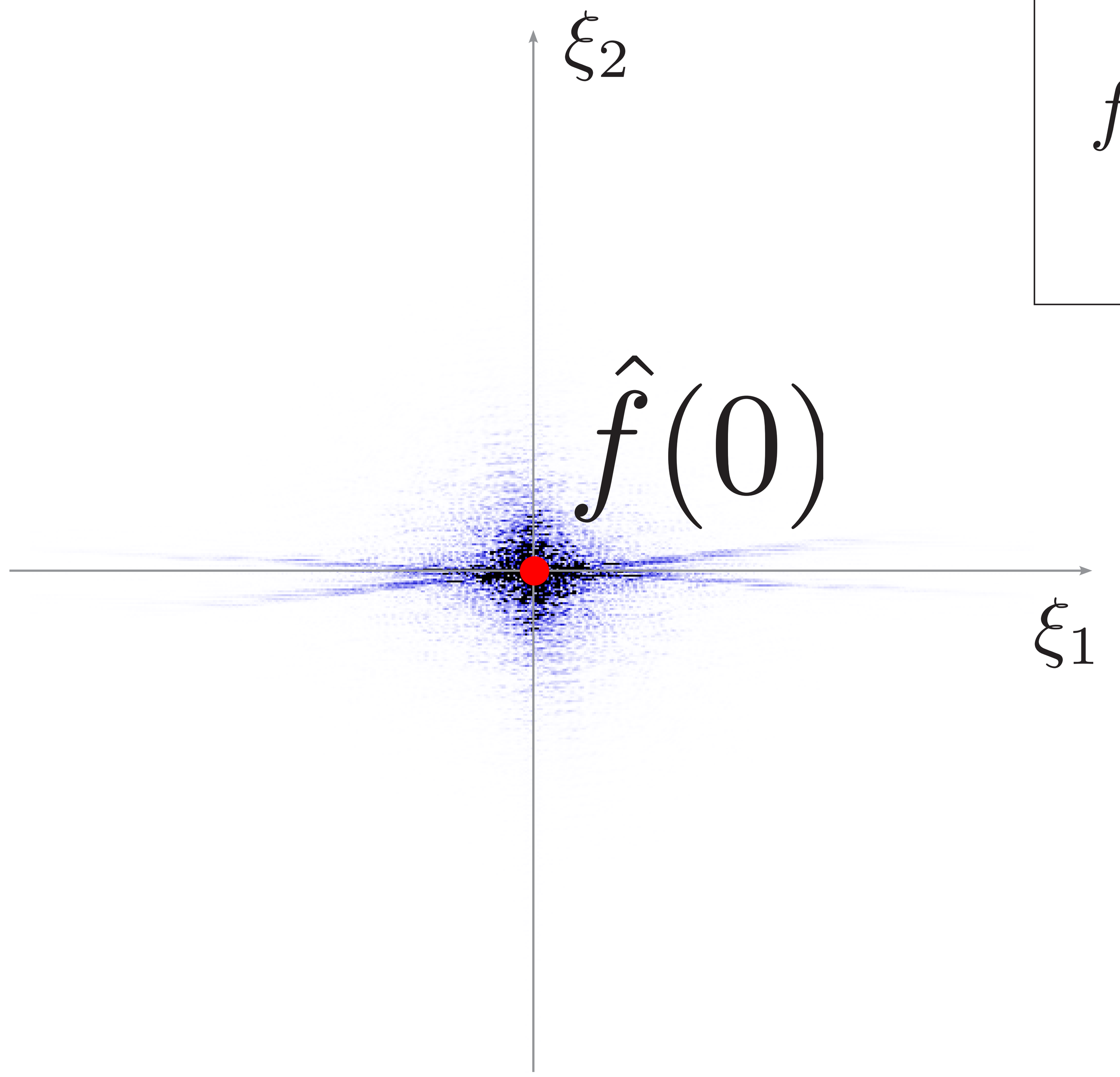
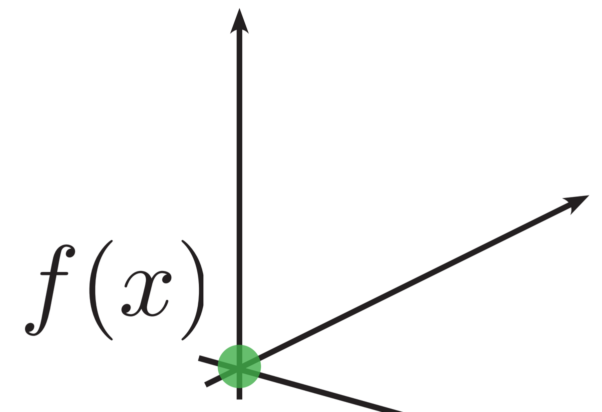
$\int$  $\mathrm{d}x$ $\iff$ 

2-form

Recall:

0-forms



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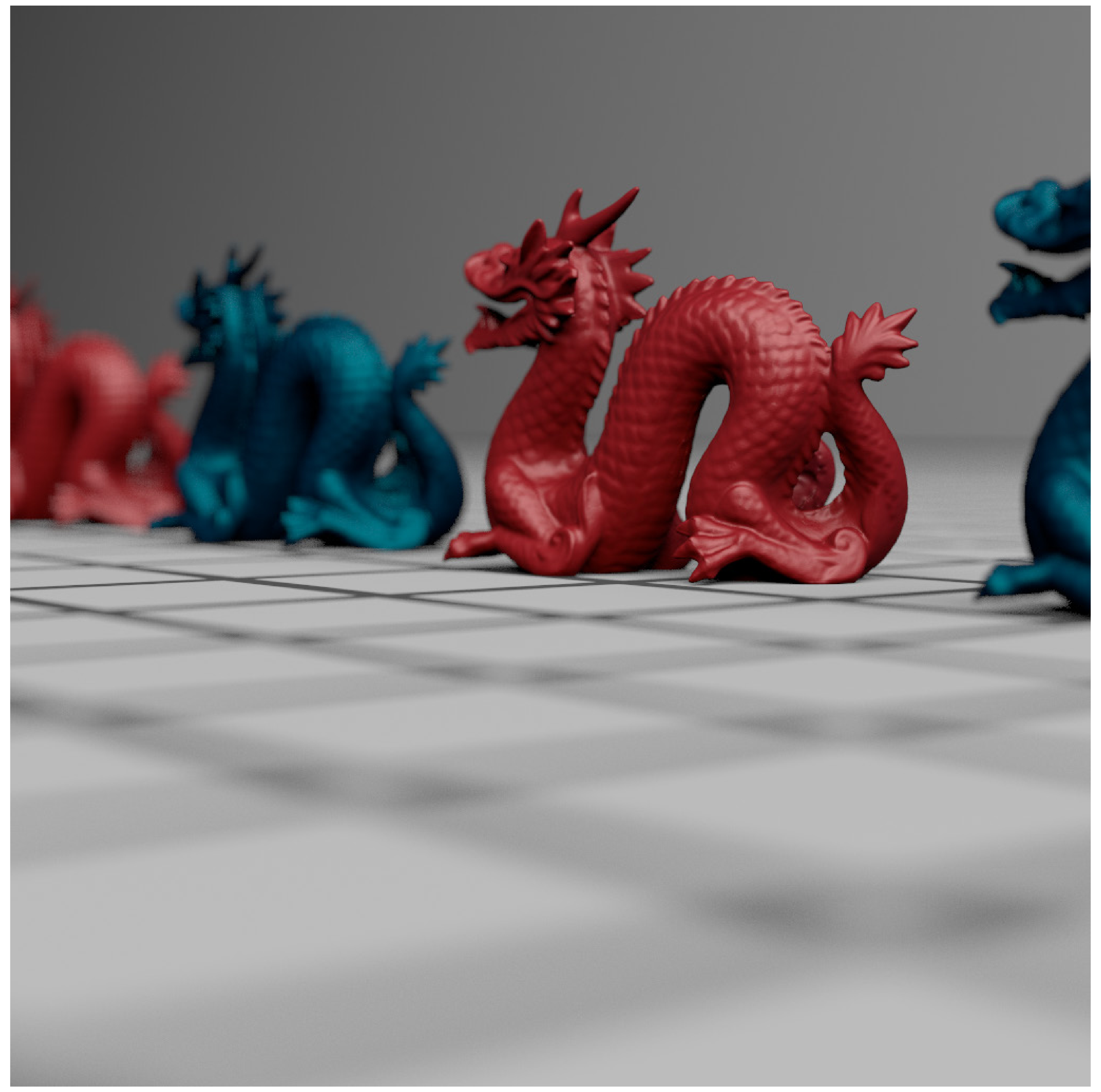
$\int$  $\mathrm{d}x$

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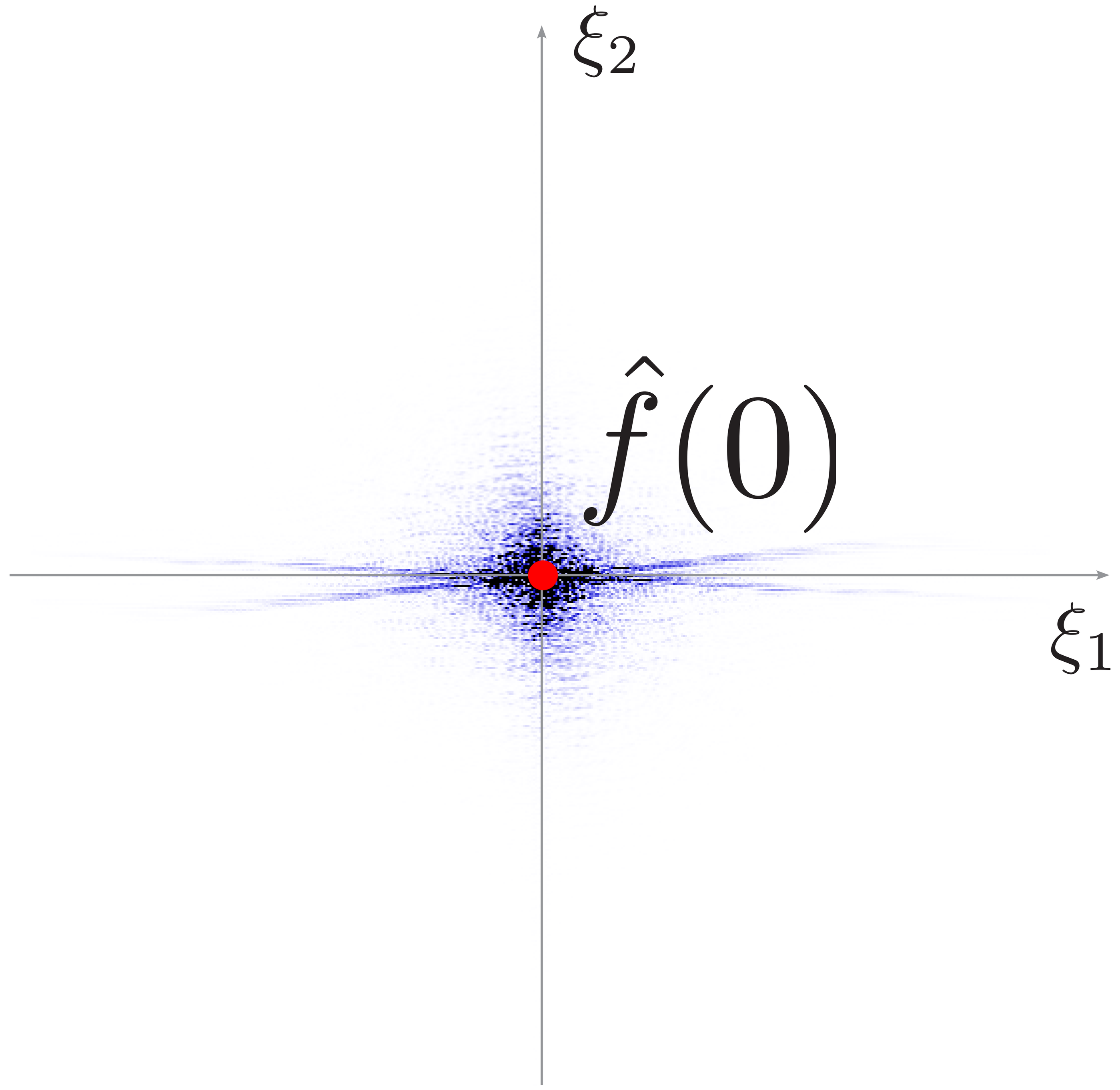

0-form

Exterior calculus and the Fourier transform

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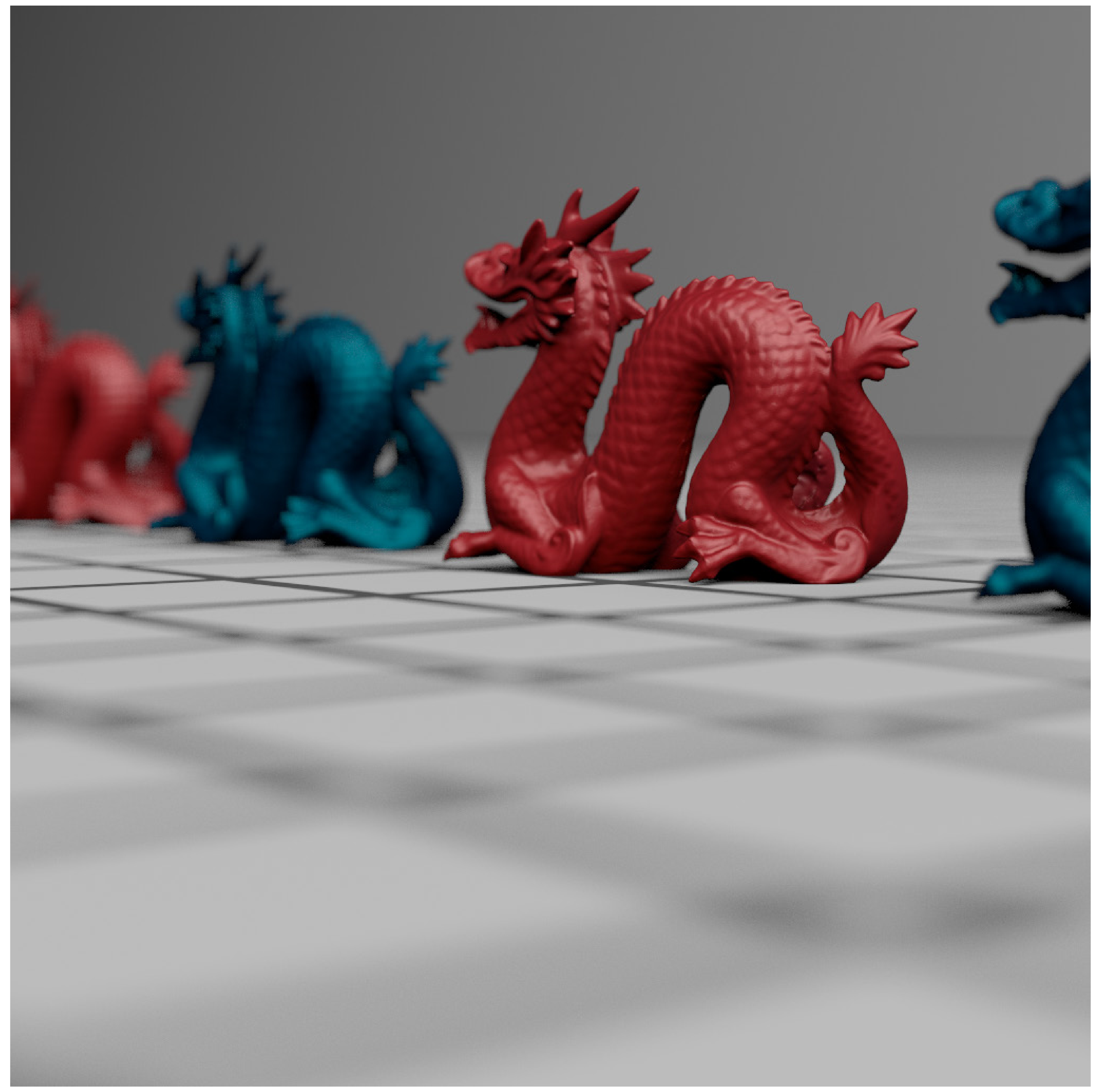
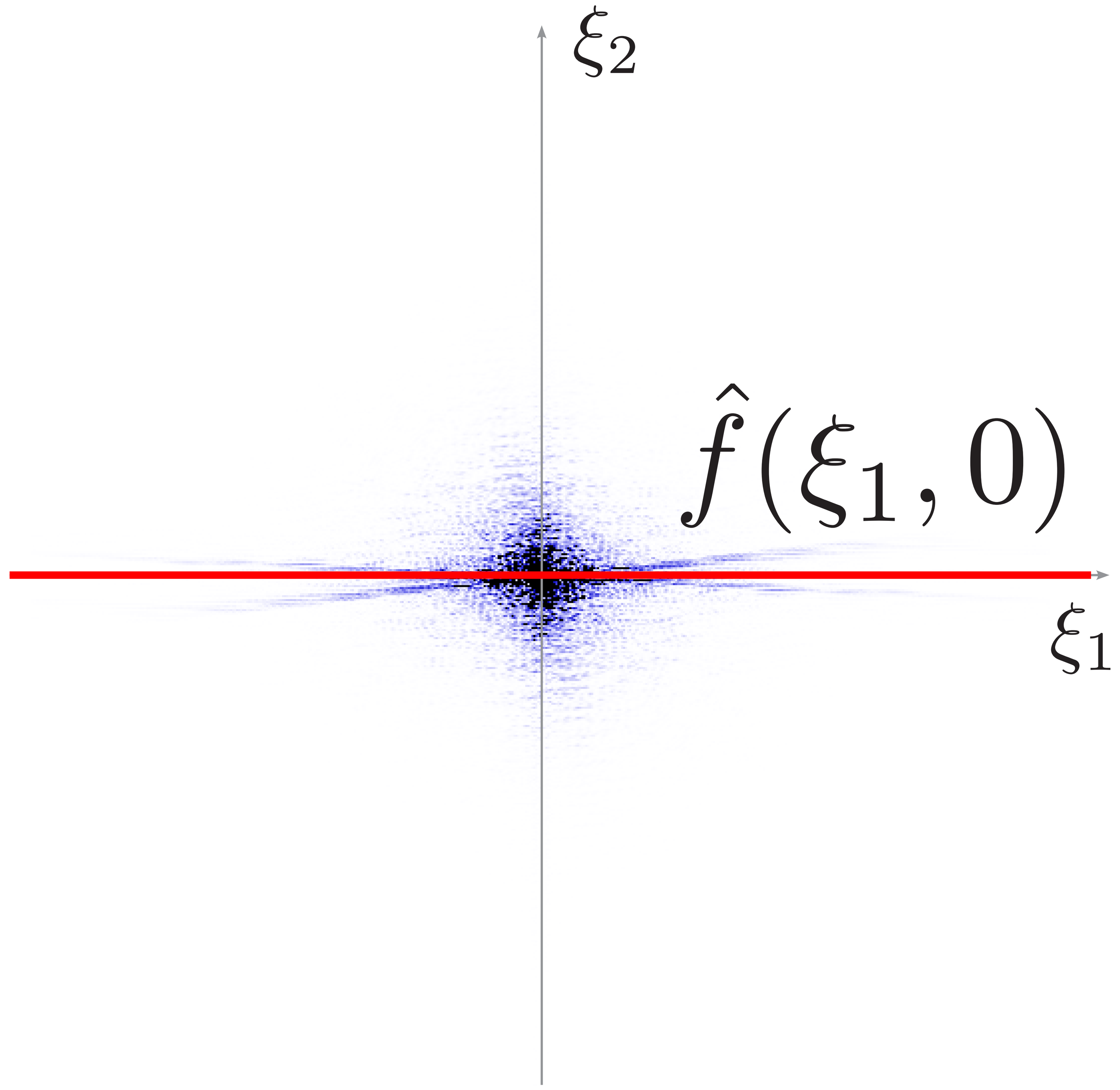


$\mathrm{d}x_2 \quad \Longleftrightarrow$



1-form

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Exterior calculus and the Fourier transform

- Differential operators:

$$\frac{d^k f}{d^k x}$$

Exterior calculus and the Fourier transform

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$$\frac{d^k f}{d^k x} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n_\xi} \left((i\xi)^k \hat{f}(\xi) \right) e^{-i\xi(x)} \, \mathrm{d}\xi$$

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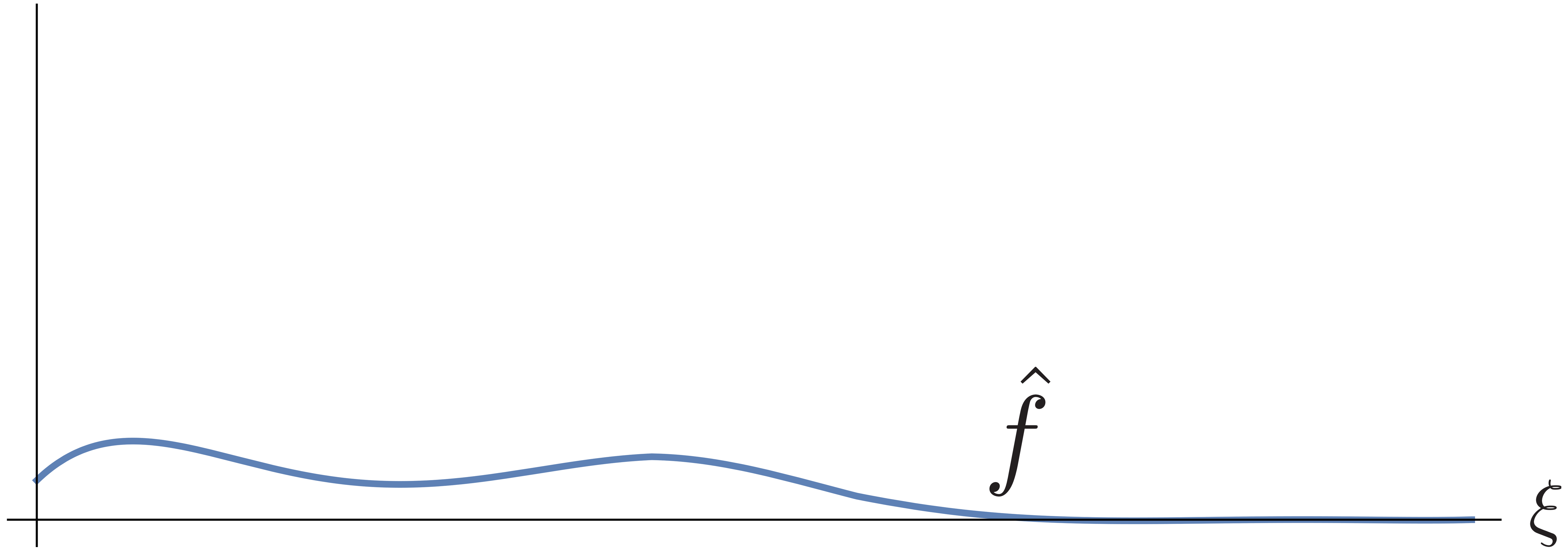

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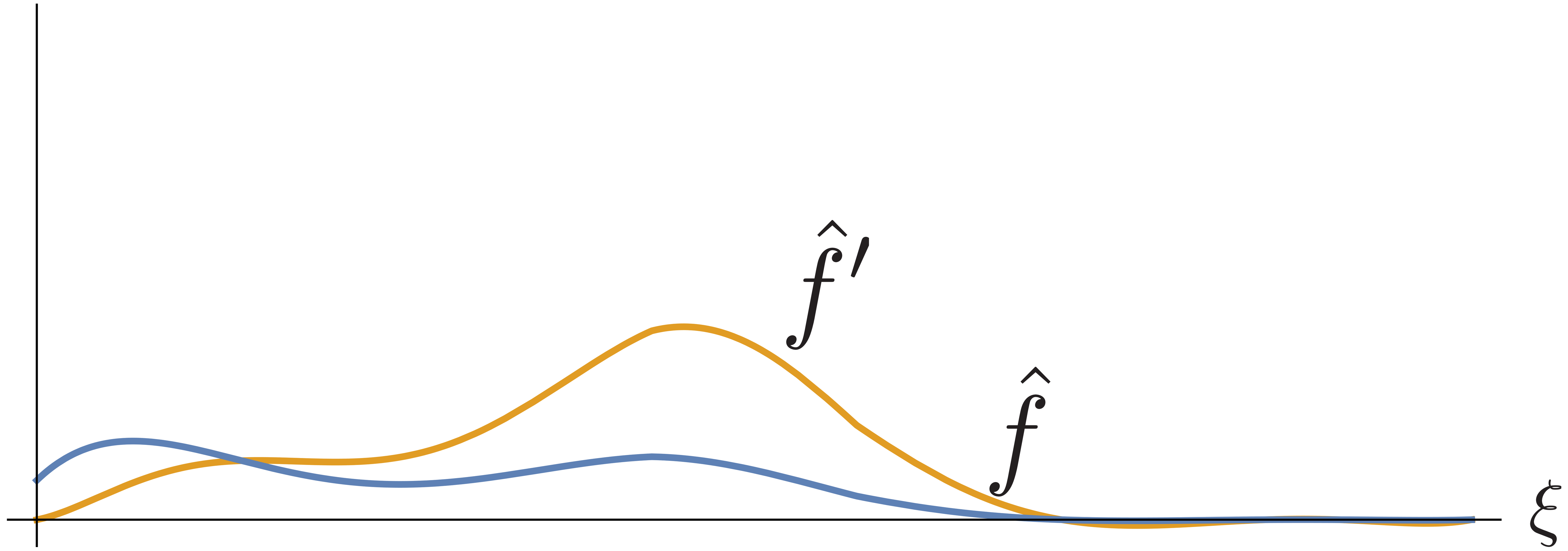

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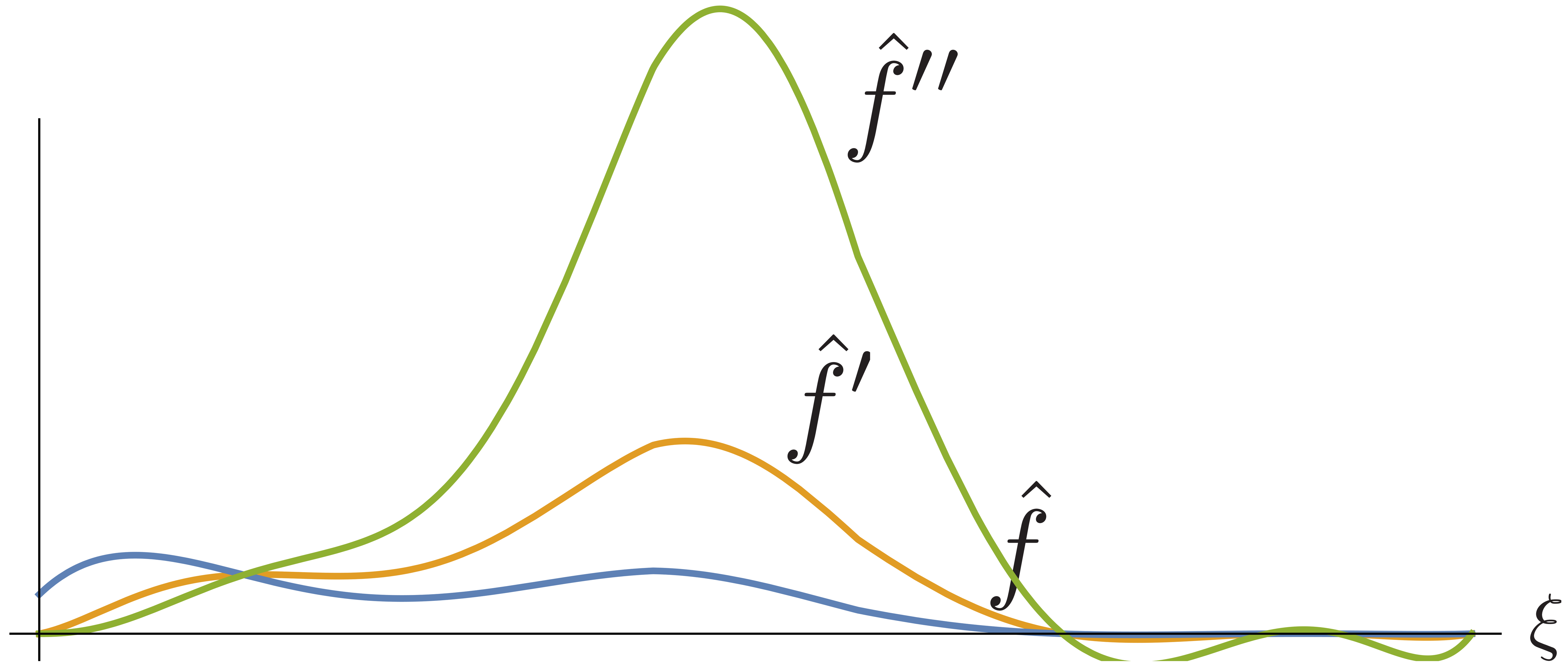

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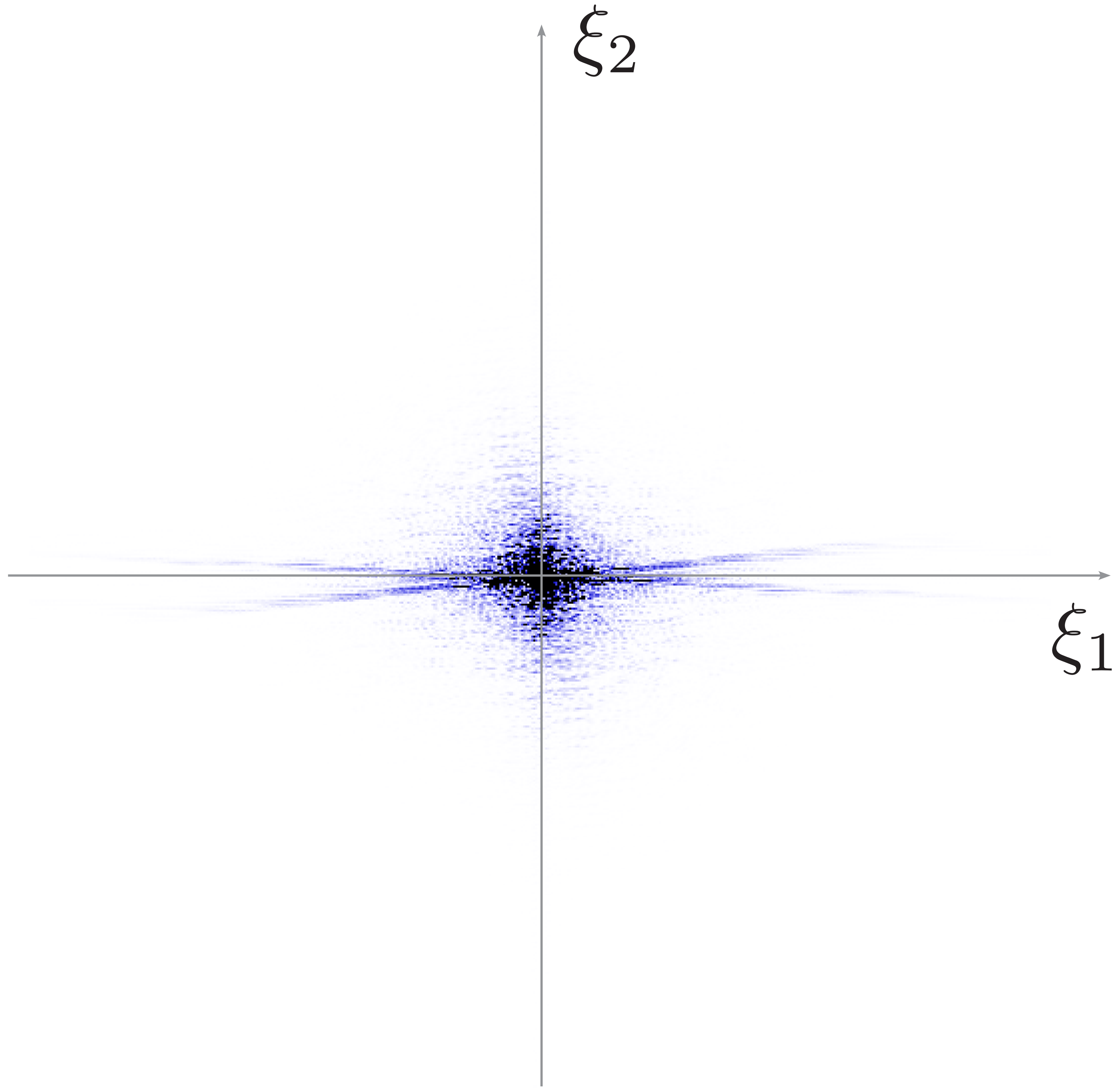

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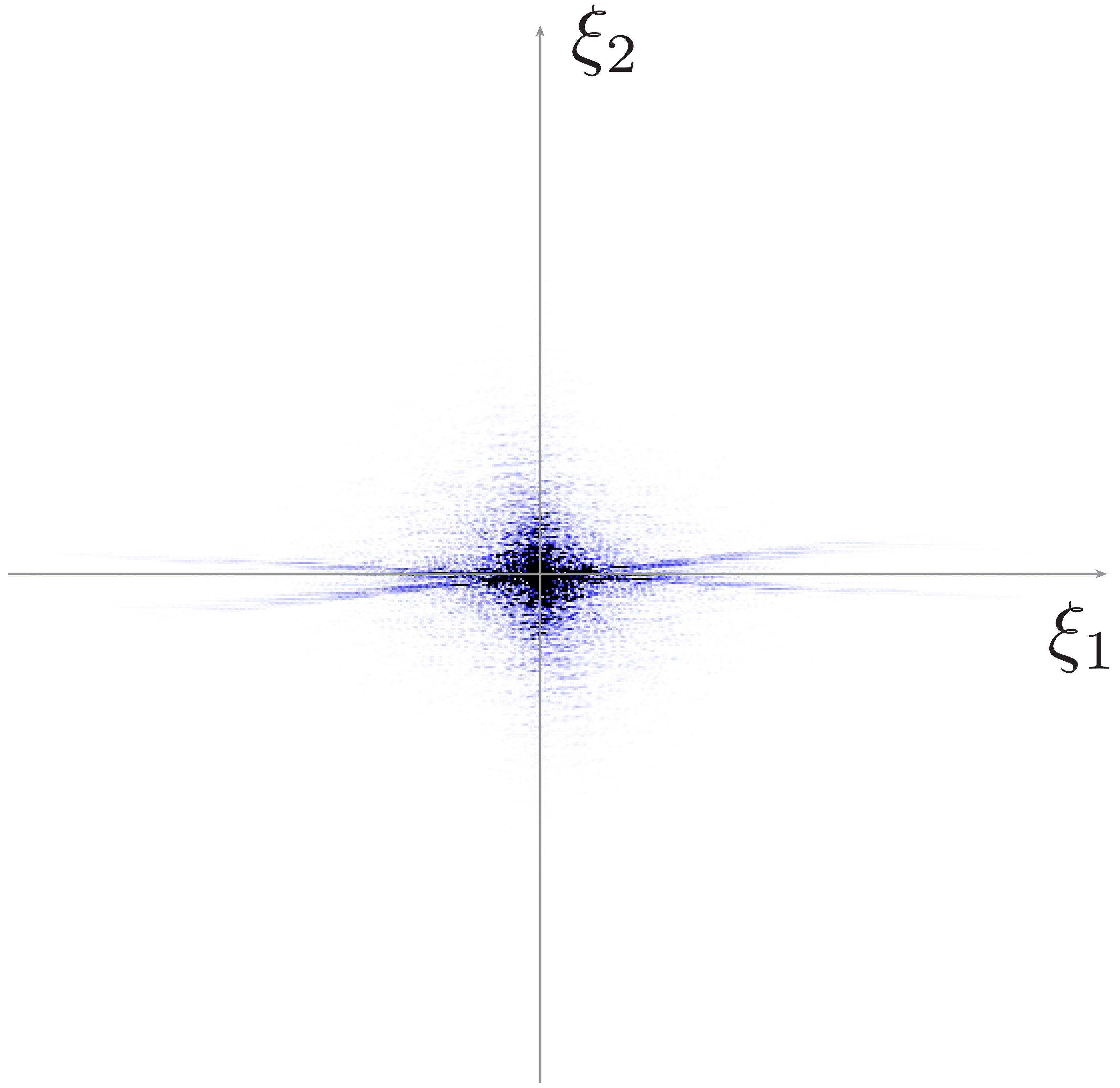
$$\begin{aligned} \widehat{\nabla f} &= (i\underbrace{\vec{\xi}}_{})\hat{f} \\ &= \bar{\xi}\,|\vec{\xi}| \end{aligned}$$



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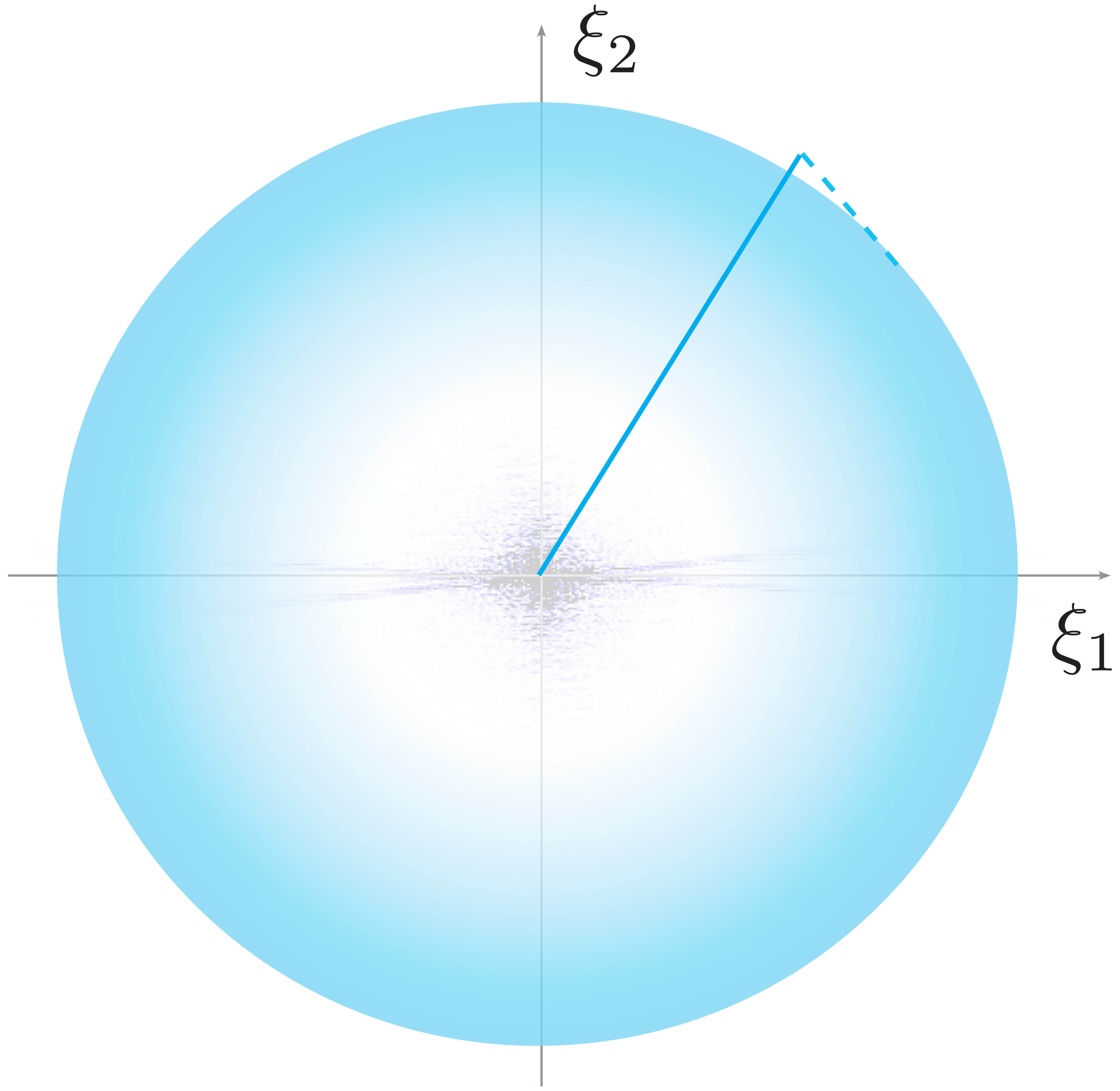

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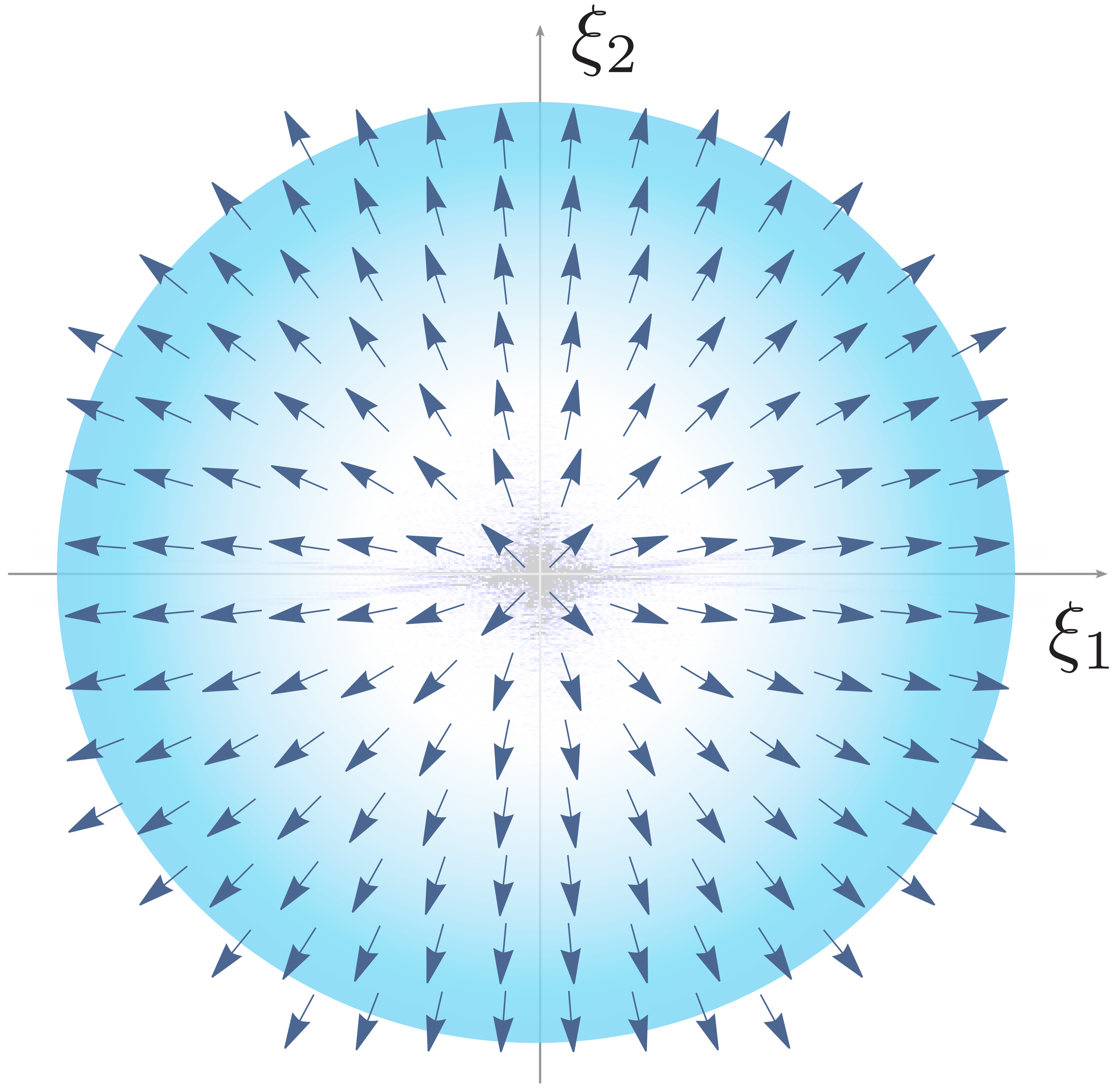

Exterior calculus and the Fourier transform

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$$\widehat{\vec{Y}} = \hat{Y}_\theta \vec{e}_\theta + \hat{Y}_r \vec{e}_r$$

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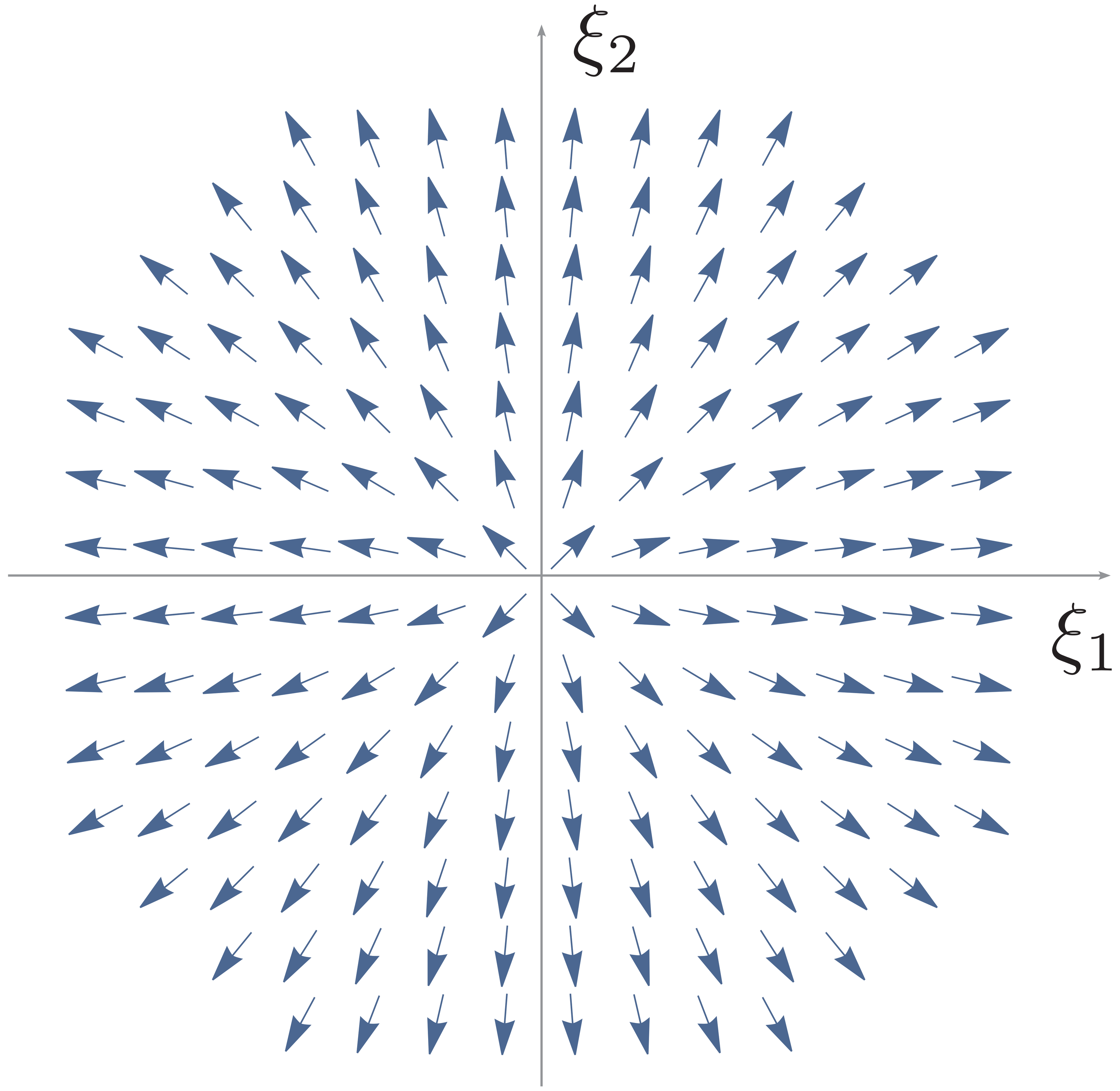

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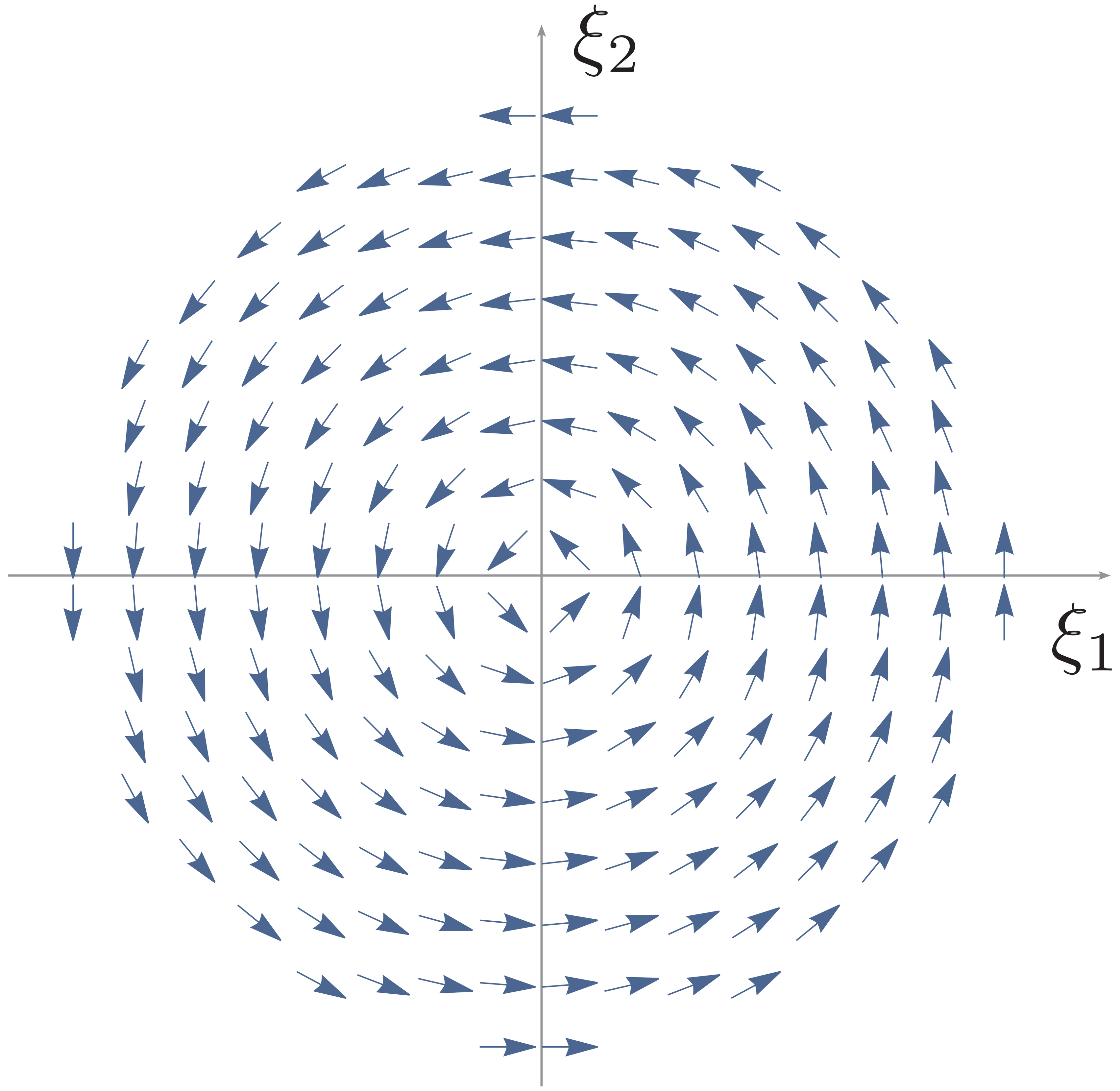

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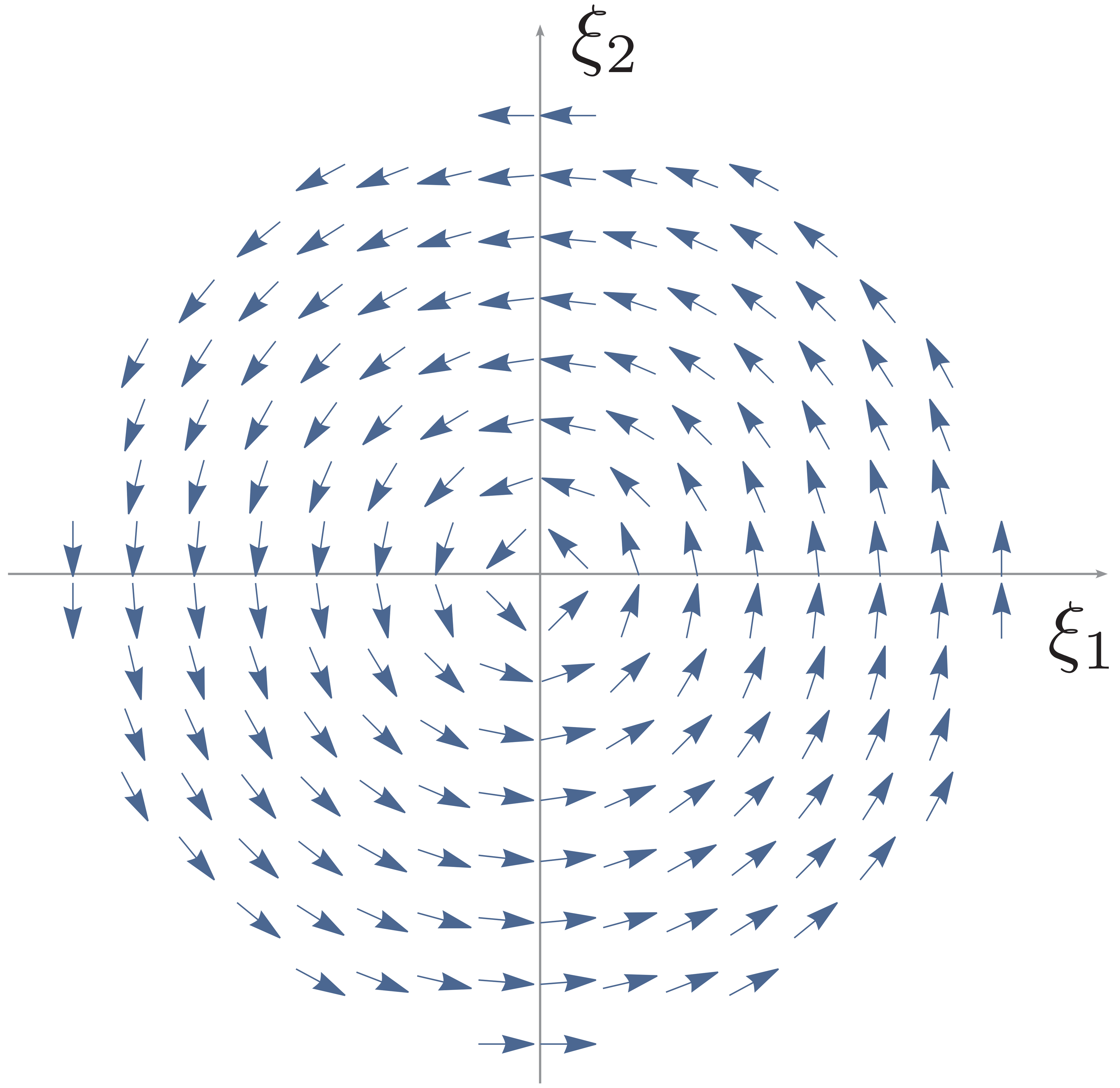

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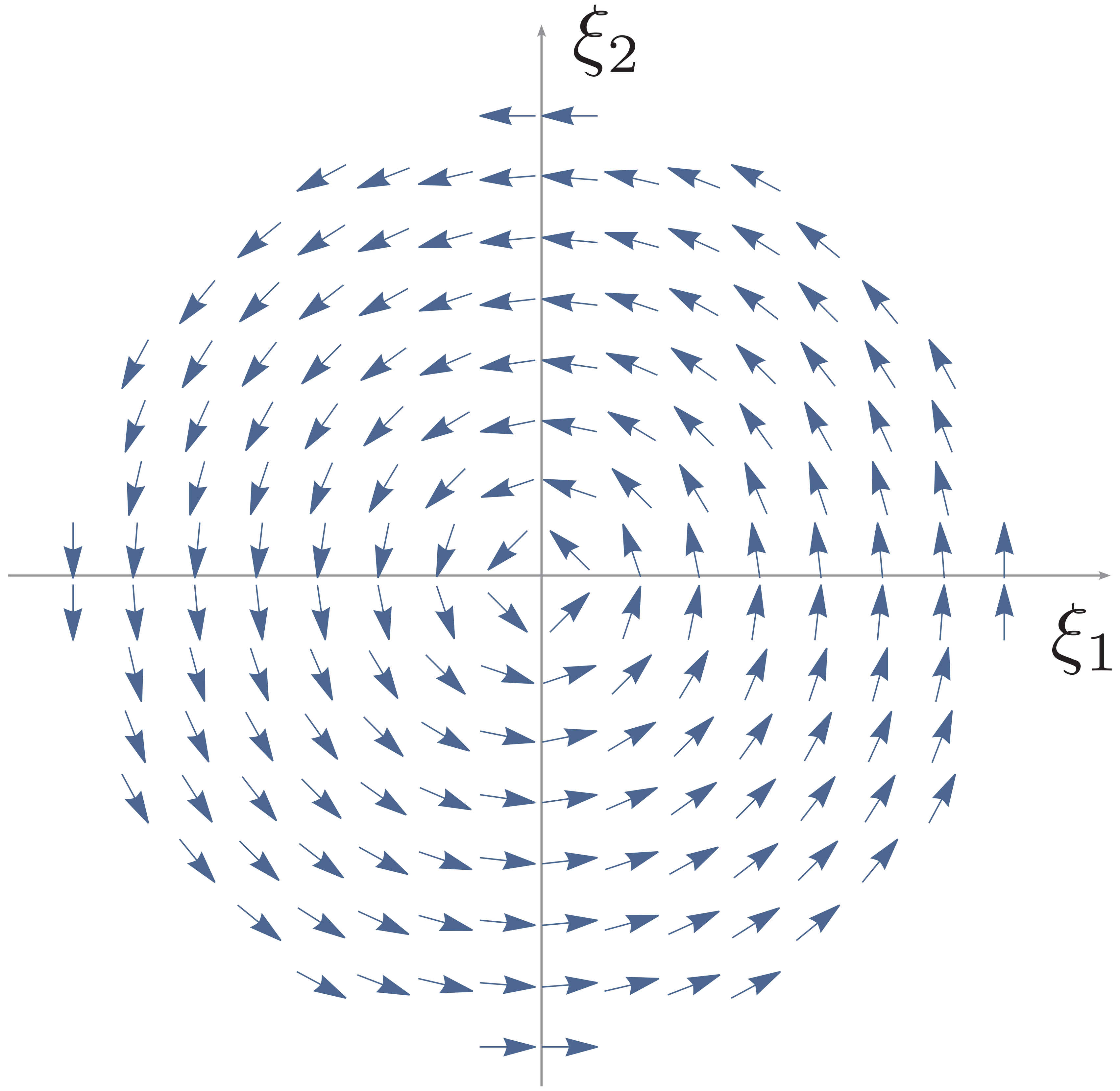

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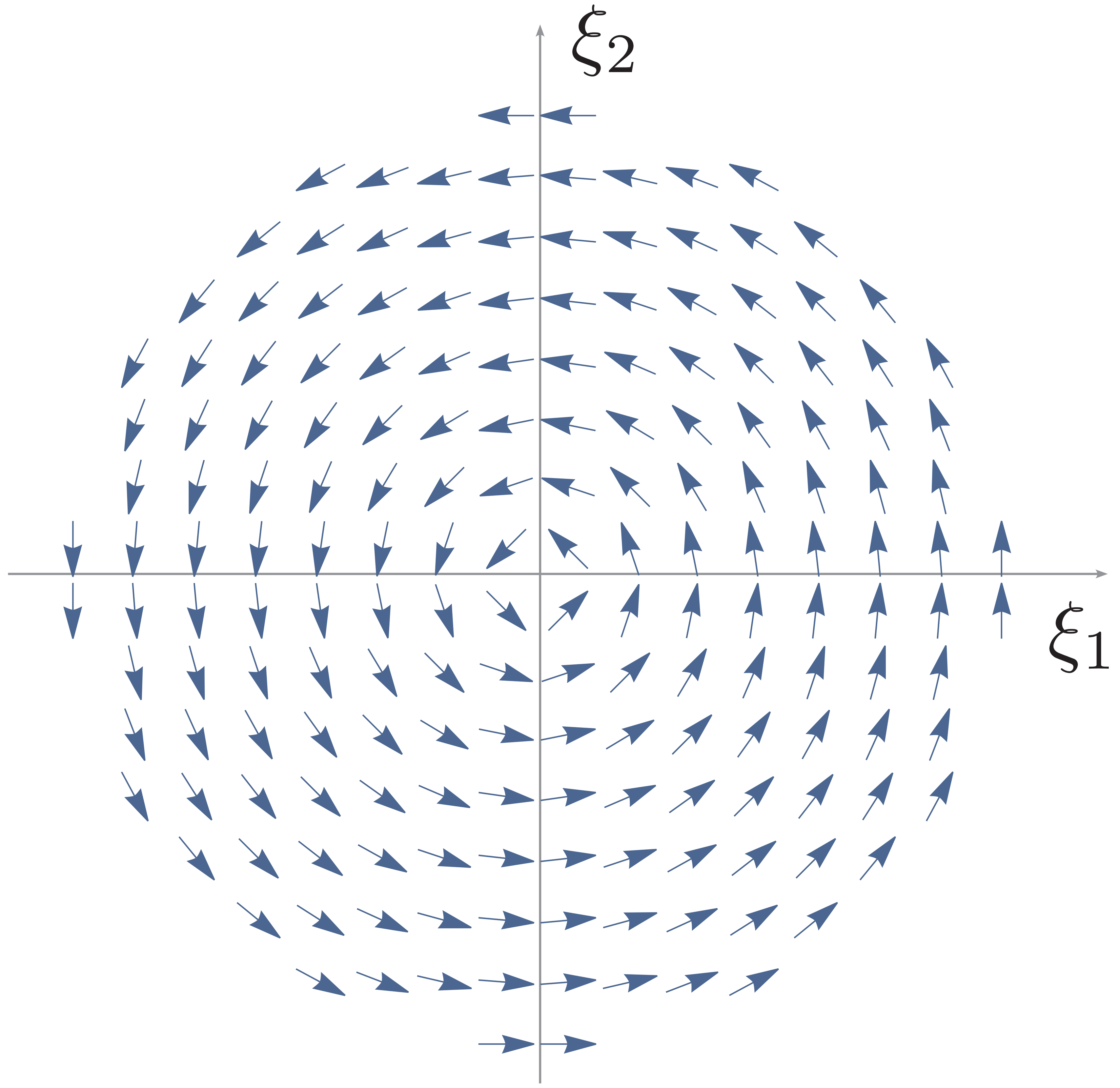

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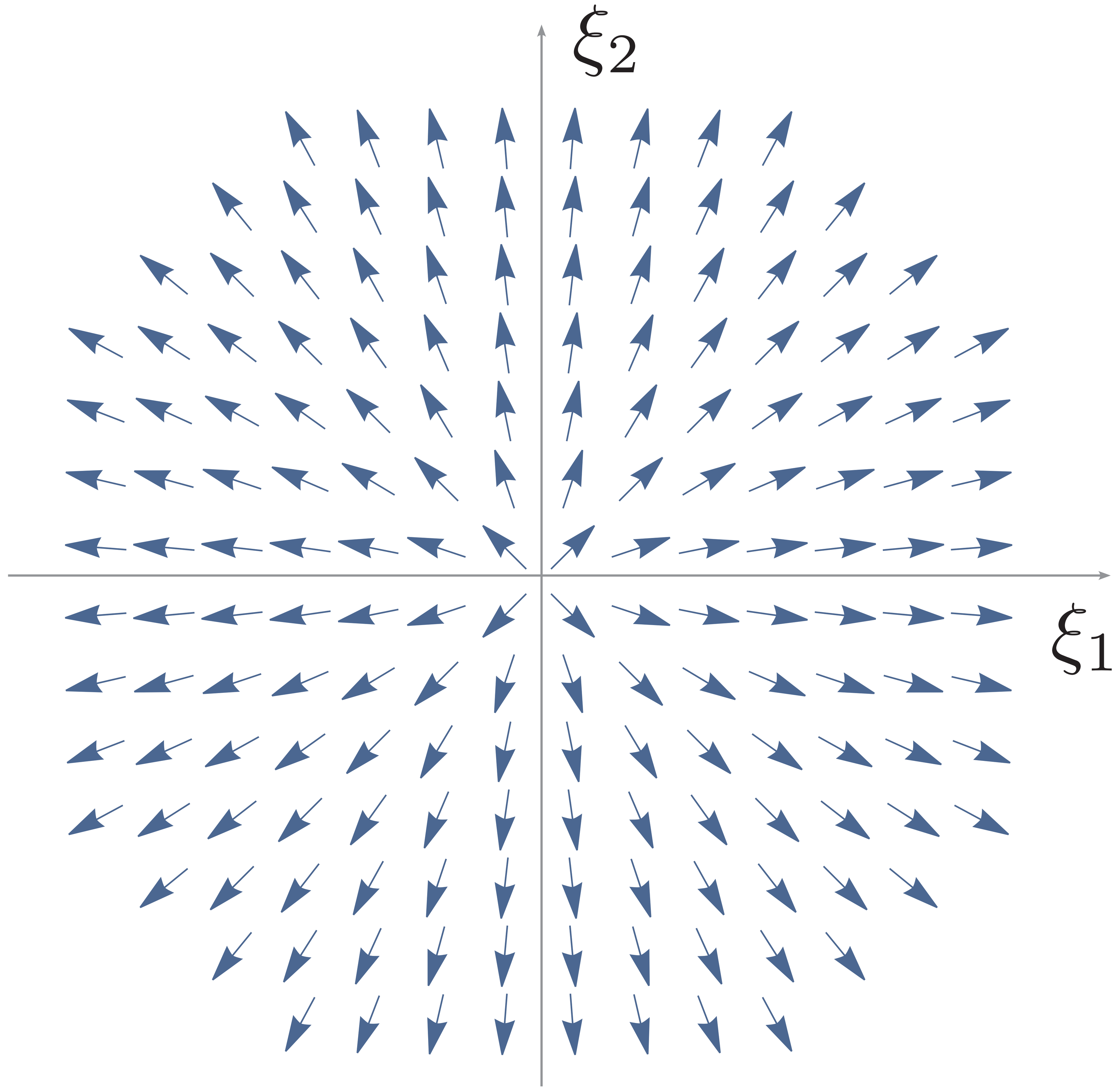

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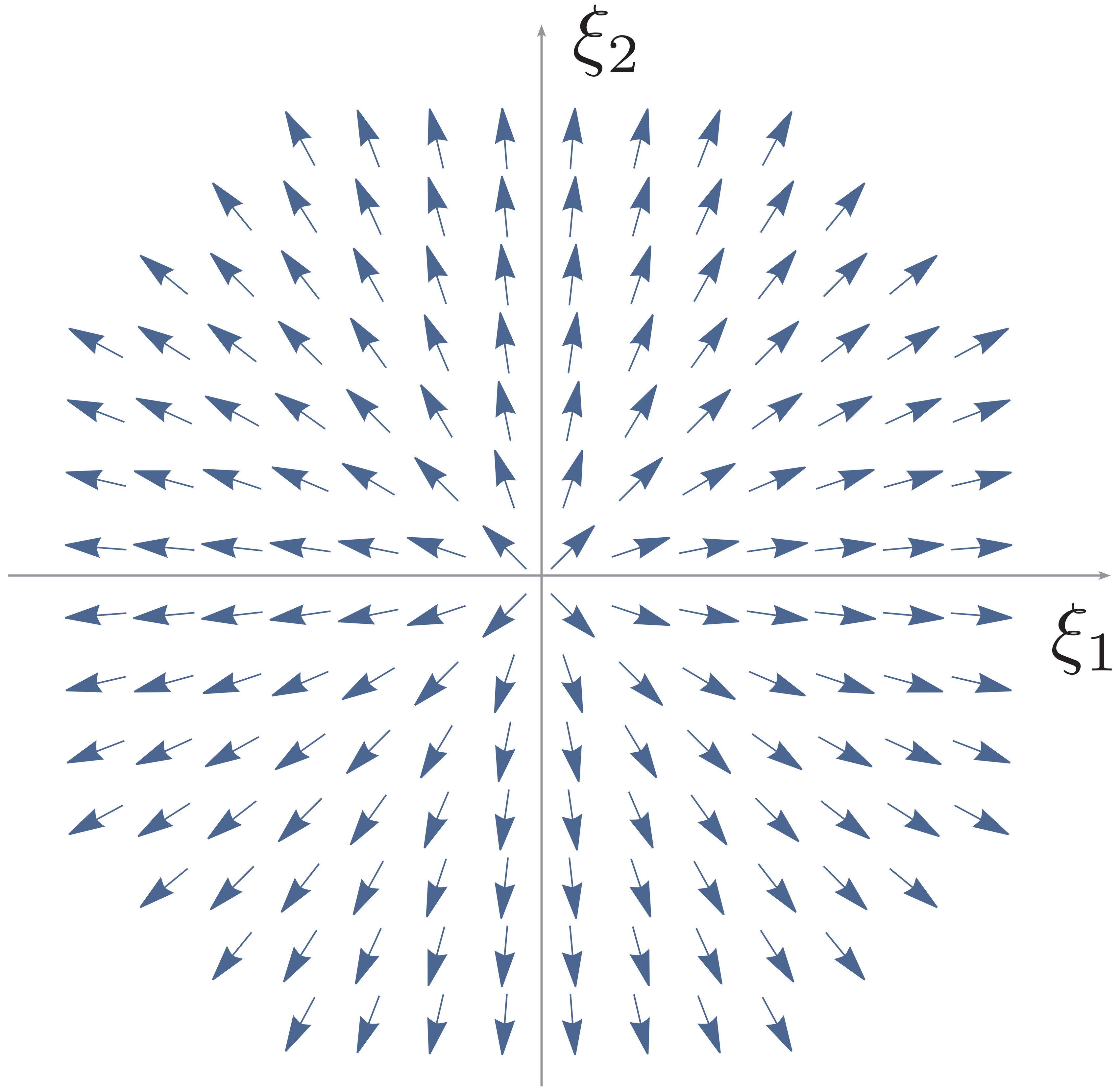

Exterior calculus and the Fourier transform

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$$\widehat{\nabla f} = (i\vec{\xi})\hat{f}$$

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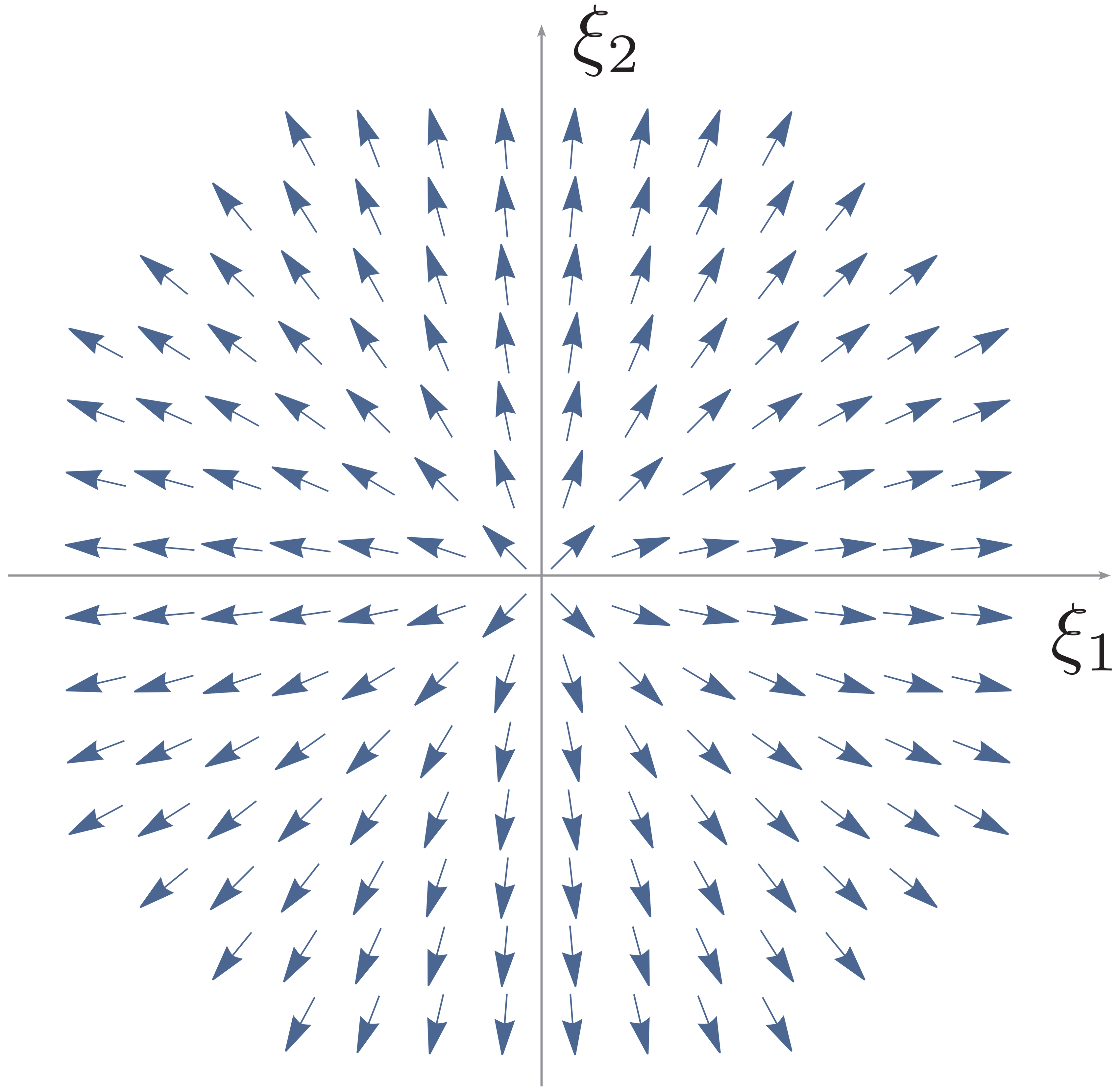

Exterior calculus and the Fourier transform

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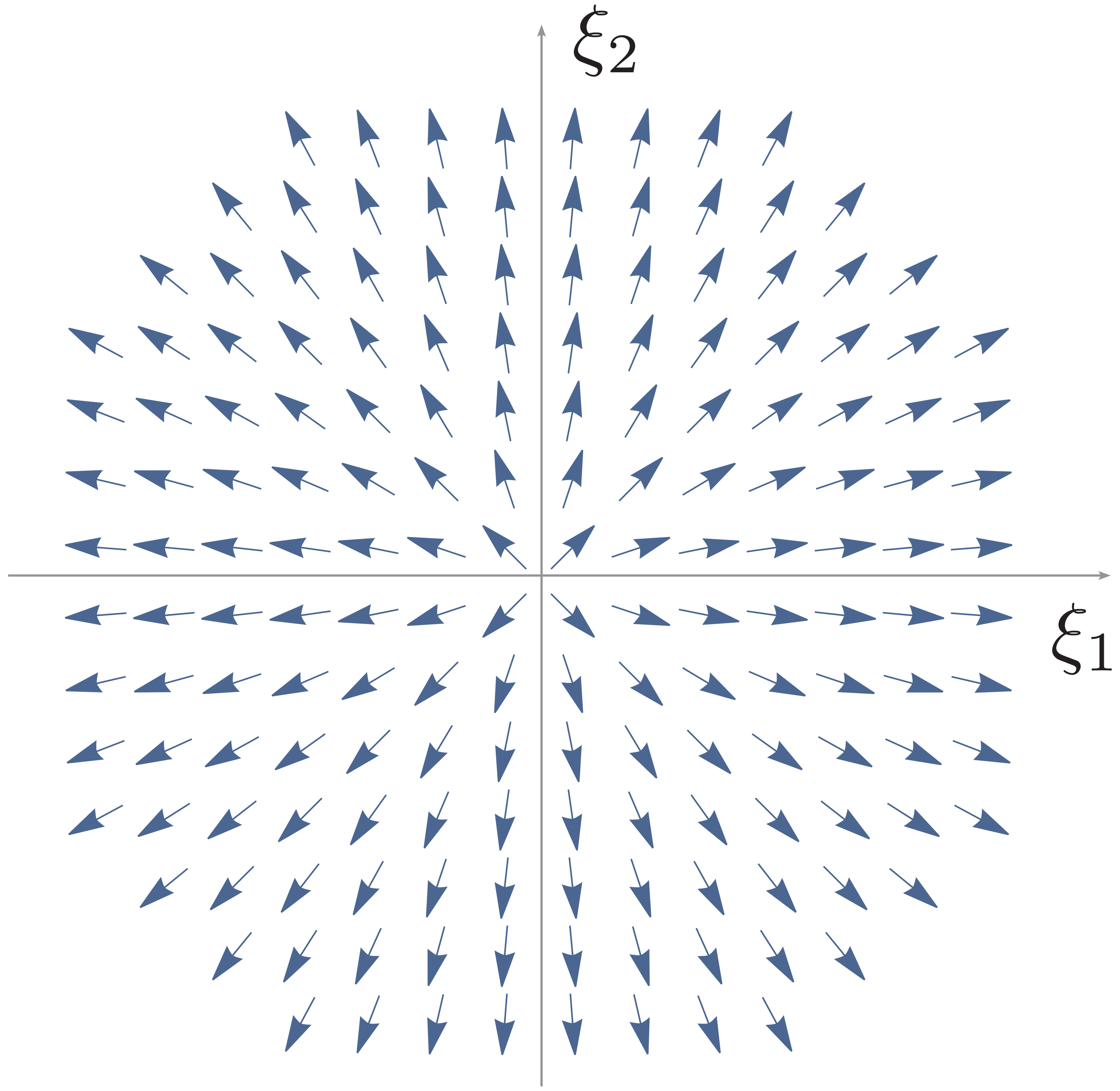
$$\widehat{\nabla f} = (i\vec{\xi})\hat{f} \qquad \cong \mathrm{d}f$$

$$\widehat{\nabla \times \vec{Y}} = i\vec{\xi} \times \hat{\vec{Y}} \quad \cong \mathrm{d}\alpha$$

$$\widehat{\nabla \cdot \vec{Y}} = i\vec{\xi} \cdot \hat{\vec{Y}} \quad \cong \mathrm{d} \star \beta$$



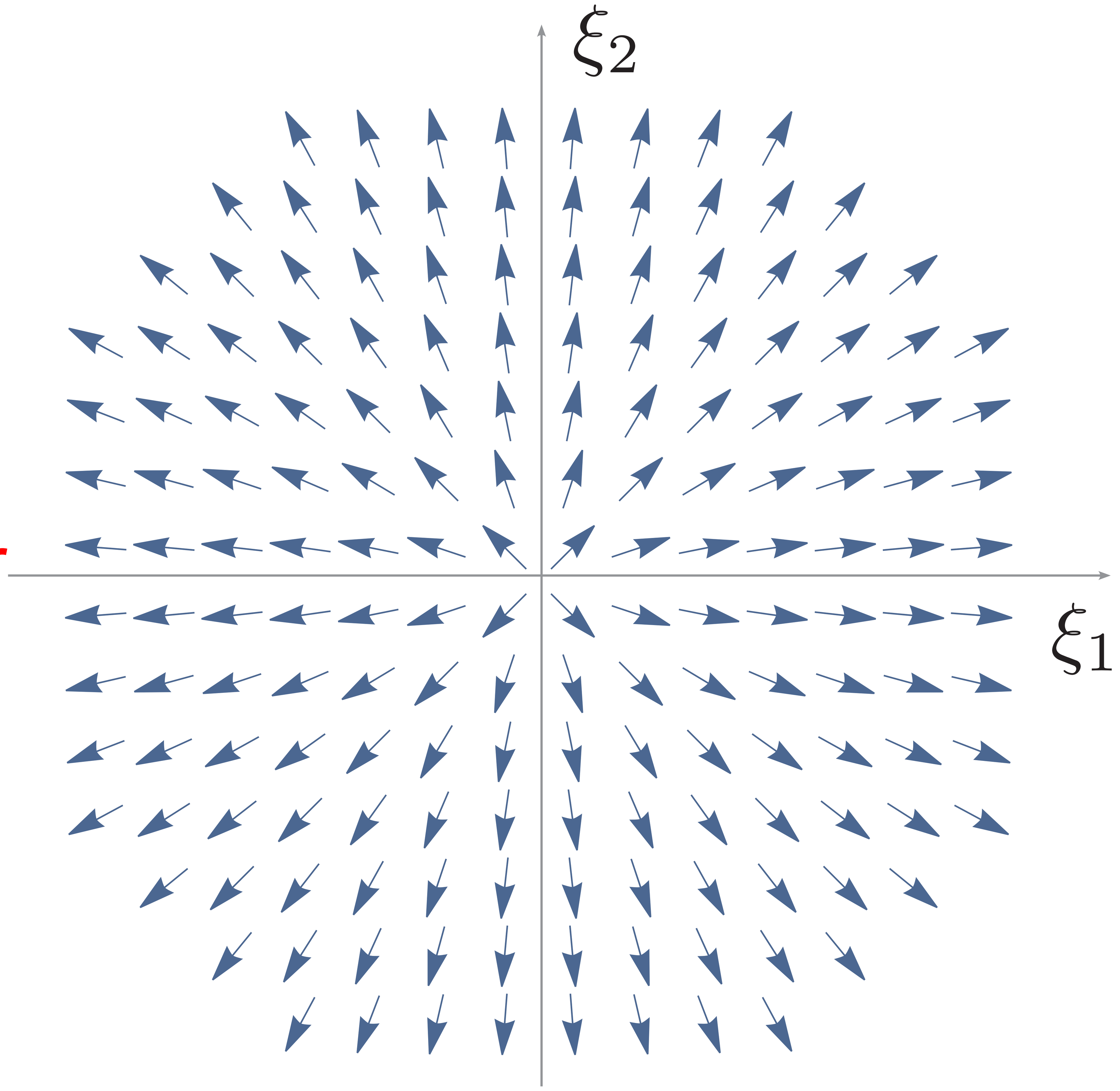
Exterior calculus and the Fourier transform

- Differential operators:

$$\widehat{\nabla f} = (i\vec{\xi})\hat{f} \quad \cong \mathrm{d}f \qquad \text{radial}$$

$$\widehat{\nabla \times \vec{Y}} = i\vec{\xi} \times \hat{\vec{Y}} \quad \cong \mathrm{d}\alpha \qquad \text{angular}$$

$$\widehat{\nabla \cdot \vec{Y}} = i\vec{\xi} \cdot \hat{\vec{Y}} \quad \cong \mathrm{d} \star \beta \qquad \text{radial}$$



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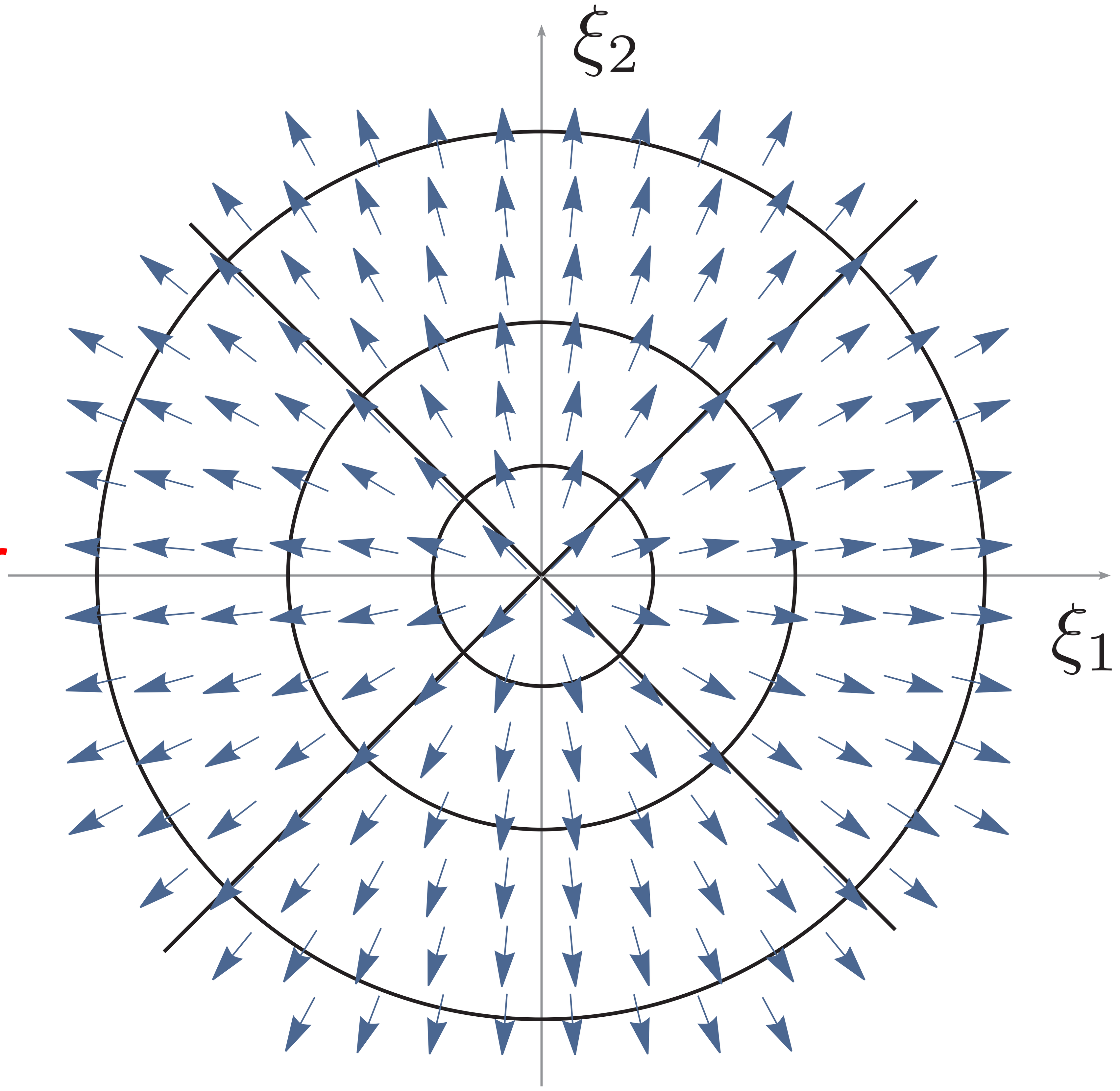
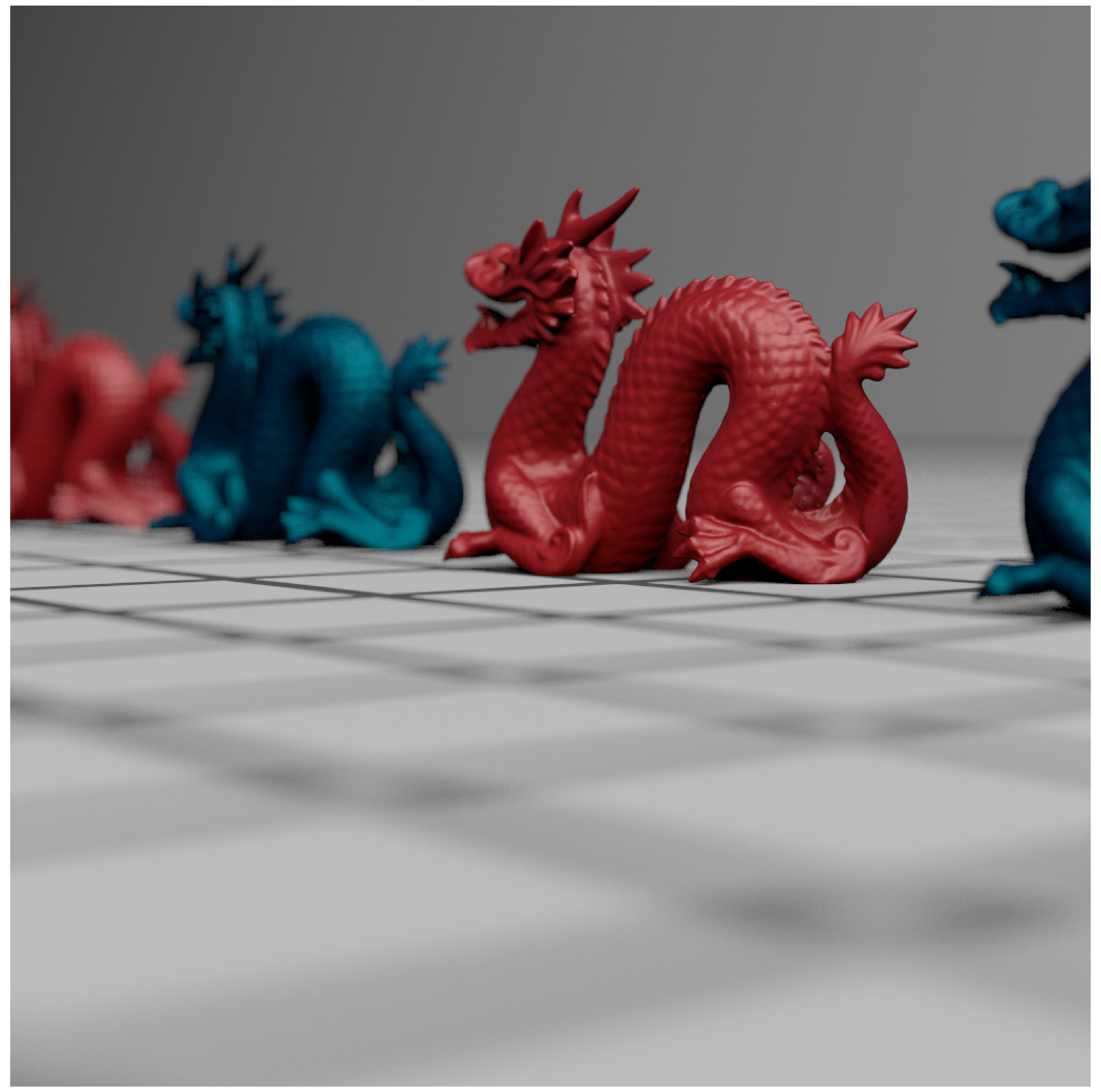
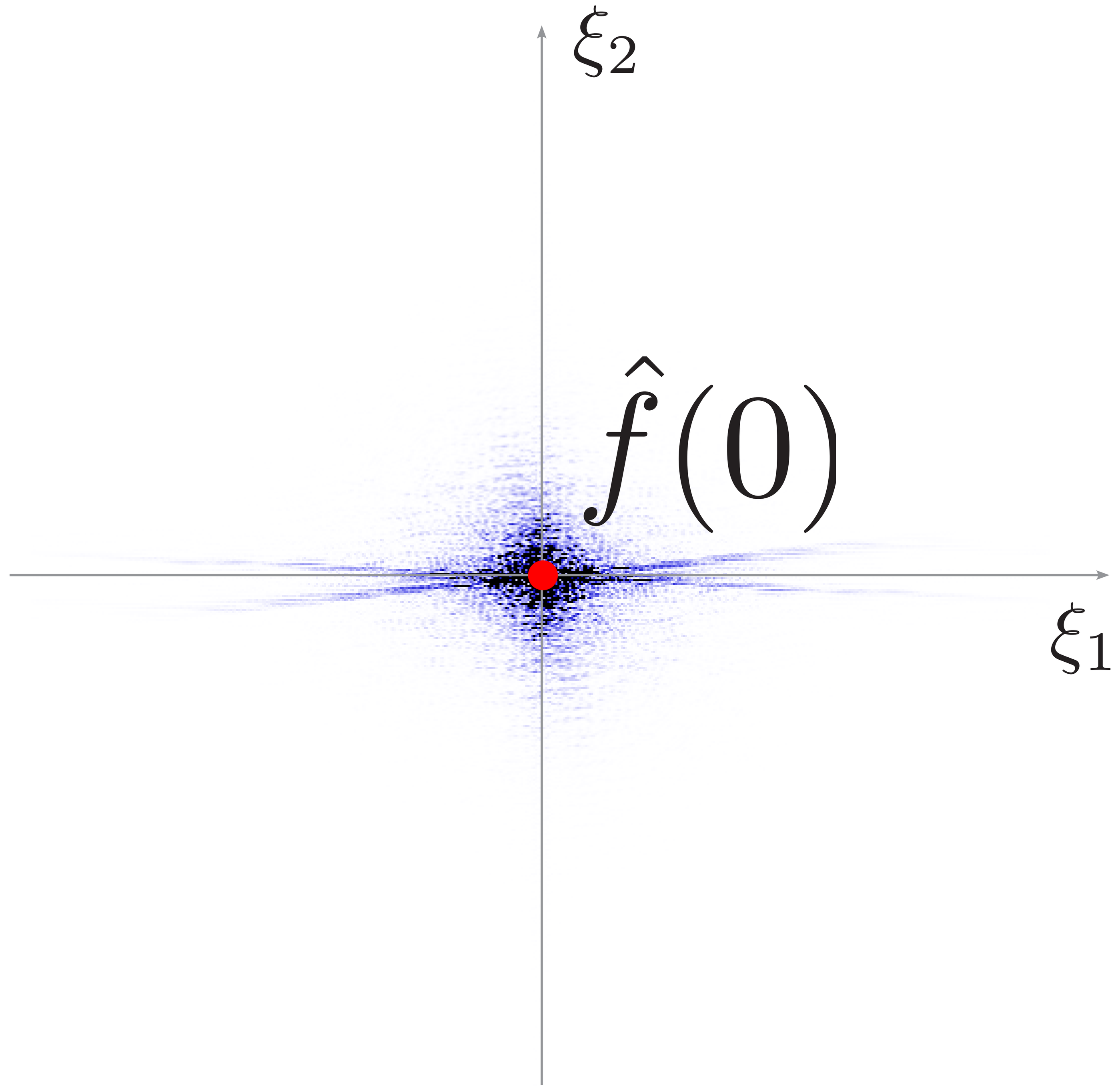


Exterior calculus and the Fourier transform

- Integration:

$\int$  $\mathrm{d}x \iff$ 

2-form

0-form

Exterior calculus and the Fourier transform

$$\mathrm{d}x^1 \wedge \mathrm{d}x^2 \qquad \longrightarrow \qquad 1_\xi$$

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Exterior calculus and the Fourier transform

- Fourier transform:

$$\mathcal{F}(f) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)\, e^{i\boxed{\xi(x)}}\, \mathrm{dx}$$

for covariance ξ has to be a co-vector

Exterior calculus and the Fourier transform

$$1_x \quad \longrightarrow \quad \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2}$$

$$\mathrm{d}x^1 \wedge \mathrm{d}x^2 \quad \longrightarrow \quad 1_\xi$$

Exterior calculus and the Fourier transform

$$
\begin{aligned}
1_x \quad &\longrightarrow \quad \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \\
\mathrm{d}x^1 \quad &\longrightarrow \quad \frac{\partial}{\partial \xi^2} \\
\mathrm{d}x^2 \quad &\longrightarrow \quad -\frac{\partial}{\partial \xi^1} \\
\mathrm{d}x^1 \wedge \mathrm{d}x^2 \quad &\longrightarrow \quad 1_\xi
\end{aligned}
$$

Exterior calculus and the Fourier transform

- Fourier transform of differential forms:

$$\mathfrak{F}(\alpha)(\xi) = \frac{1}{(2\pi)^{n/2}} \sum_{j_1 < \cdots < j_r} \int_{\mathbb{R}^n} \alpha_{j_1, \cdots, j_r}(x) \, \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_r} \wedge e^{-i\, x^p \xi_p} \wedge e^{i\, \mathrm{d}x^q \frac{\partial}{\partial \xi^q}}$$

Exterior calculus and the Fourier transform

- Fourier transform of differential forms:

$$\mathfrak{F}(\alpha)(\xi) = \frac{1}{(2\pi)^{n/2}} \sum_{j_1 < \cdots < j_r} \int_{\mathbb{R}^n} \underbrace{\boxed{\alpha_{j_1,\cdots,j_r}(x)\, \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_r} \wedge e^{-i\, x^p \xi_p}}}_{\text{regular Fourier transform of coordinate functions}} \wedge e^{i\, \mathrm{d}x^q \frac{\partial}{\partial \xi^q}}$$

Exterior calculus and the Fourier transform

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Exterior calculus and the Fourier transform

- Fourier transform of differential forms:

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$$e^{\mathrm{d}x^q \frac{\partial}{\partial \xi^q}} = \sum_{a=0}^{n} \frac{\left(\mathrm{d}x^q \otimes \frac{\partial}{\partial \xi^q}\right)^a}{a!}$$

Exterior calculus and the Fourier transform

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$$\mathfrak{F}(\alpha)(\xi) = \frac{1}{(2\pi)^{n/2}} \sum_{j_1 < \cdots < j_r} \int_{\mathbb{R}^n} \alpha_{j_1, \cdots, j_r}(x) \, \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_r} \wedge e^{-i\, x^p \xi_p} \wedge e^{i\, \mathrm{d}x^q \frac{\partial}{\partial \xi^q}}$$

$$e^{\mathrm{d}x^q \frac{\partial}{\partial \xi^q}} = \sum_{a=0}^{n} \frac{\left(\mathrm{d}x^q \otimes \frac{\partial}{\partial \xi^q}\right)^a}{a!}$$

has to respect $\mathbb{Z}^2$ grading
to ensure proper anti-symmetry

Exterior calculus and the Fourier transform

- Exponential form factor in $\mathbb{R}^3$:

$$
\begin{aligned}
e^{\mathrm{d}x^q \frac{\partial}{\partial \xi^q}} = 1 &+ \left(\mathrm{d}x^1 \frac{\partial}{\partial \xi^1} + \mathrm{d}x^2 \frac{\partial}{\partial \xi^2} + \mathrm{d}x^3 \frac{\partial}{\partial \xi^3} \right) \\
&- \left(\mathrm{d}x^2 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3} + \mathrm{d}x^1 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^3} + \mathrm{d}x^1 \wedge \mathrm{d}x^2 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \right) \\
&- \left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3} \right)
\end{aligned}
$$

Exterior calculus and the Fourier transform

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\begin{aligned}
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&- \left(\mathrm{d}x^2 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3} + \mathrm{d}x^1 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^3} + \mathrm{d}x^1 \wedge \mathrm{d}x^2 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \right) \\
&- \left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3} \right)
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Exterior calculus and the Fourier transform

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- \left(\mathrm{d}x^2 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3} + \mathrm{d}x^1 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^3} + \mathrm{d}x^1 \wedge \mathrm{d}x^2 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \right) \\
- \left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3} \right)
\end{aligned}
$$

Exterior calculus and the Fourier transform

- Exponential form factor in $\mathbb{R}^3$:

$$e^{\mathrm{d}x^q \frac{\partial}{\partial \xi^q}} = 1 + \left(\mathrm{d}x^1 \frac{\partial}{\partial \xi^1} + \mathrm{d}x^2 \frac{\partial}{\partial \xi^2} + \mathrm{d}x^3 \frac{\partial}{\partial \xi^3} \right)$$

$$- \left(\mathrm{d}x^2 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3} + \mathrm{d}x^1 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^3} + \mathrm{d}x^1 \wedge \mathrm{d}x^2 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \right)$$

$$- \left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3} \right)$$

Exterior calculus and the Fourier transform

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&- \left(\mathrm{d}x^2 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3} + \mathrm{d}x^1 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^3} + \mathrm{d}x^1 \wedge \mathrm{d}x^2 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \right) \\
&- \left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \, \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3} \right)
\end{aligned}
$$

Exterior calculus and the Fourier transform

$$
\begin{aligned}
1_x \quad &\longrightarrow \quad \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \\
\mathrm{d}x^1, \mathrm{d}x^2 \quad &\longrightarrow \quad \frac{\partial}{\partial \xi^2}, -\frac{\partial}{\partial \xi^1} \\
\mathrm{d}x^1 \wedge \mathrm{d}x^2 \quad &\longrightarrow \quad 1_\xi
\end{aligned}
$$

Exterior calculus and the Fourier transform

$$
\mathrm{d} \Bigg\downarrow \quad
\begin{array}{rcl}
1_x & \longrightarrow & \dfrac{\partial}{\partial \xi^1} \wedge \dfrac{\partial}{\partial \xi^2} \\[2ex]
\mathrm{d}x^1, \mathrm{d}x^2 & \longrightarrow & \dfrac{\partial}{\partial \xi^2}, -\dfrac{\partial}{\partial \xi^1} \\[2ex]
\mathrm{d}x^1 \wedge \mathrm{d}x^2 & \longrightarrow & 1_\xi
\end{array}
$$

Exterior calculus and the Fourier transform

- Fourier transform of exterior derivative:

$$\mathfrak{F}\big(\mathrm{d}\alpha\big)(\xi) = \big(\widehat{\mathrm{d}}\,\widehat{\alpha}\big)(\xi) = \mathrm{i}_{i\xi}\widehat{\alpha}(\xi)$$

Exterior calculus and the Fourier transform

- Fourier transform of exterior derivative:

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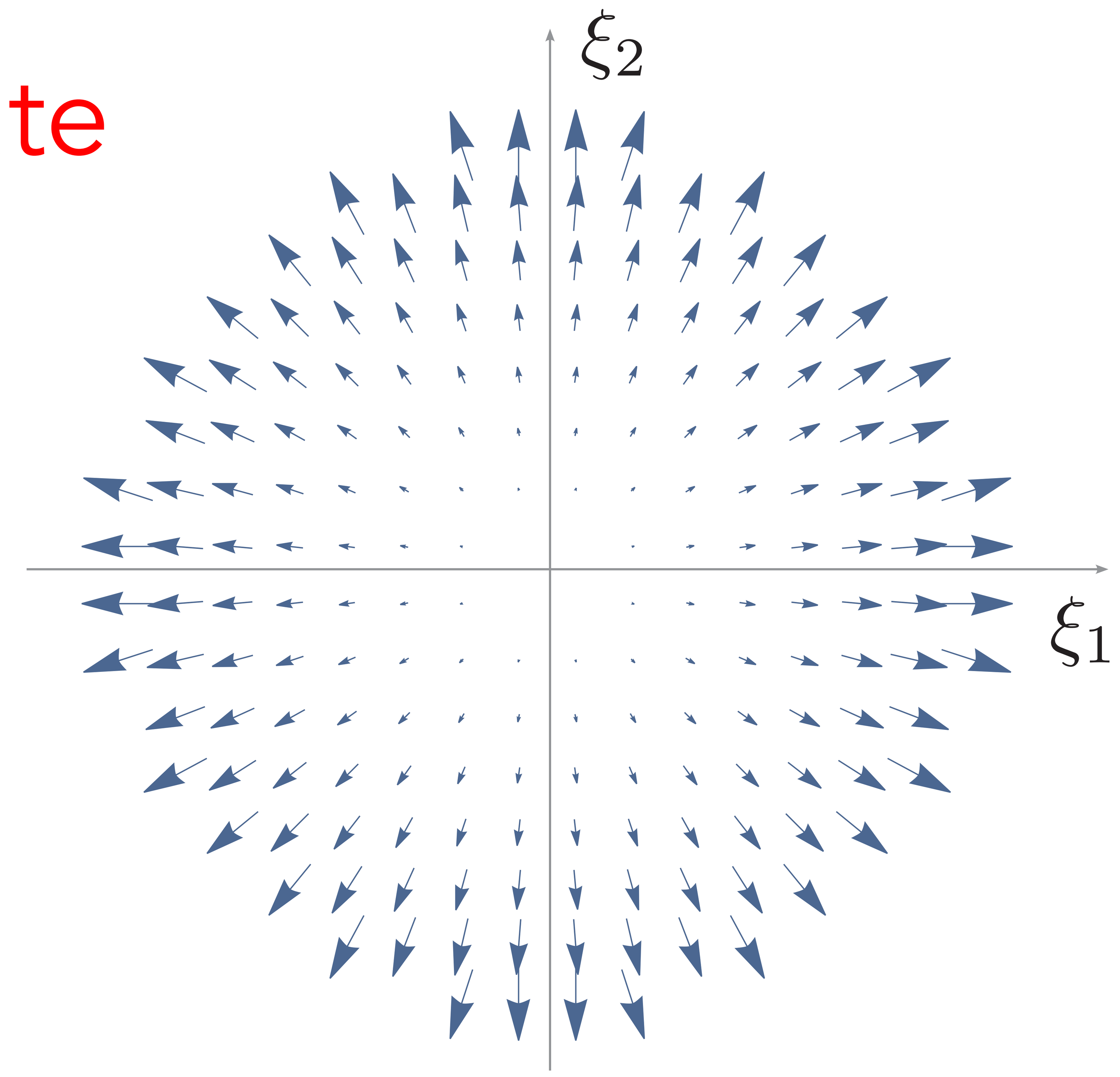
radial coordinate
vector

Exterior calculus and the Fourier transform

- Fourier transform of exterior derivative:

$$\mathfrak{F}(\mathrm{d}\alpha)(\xi) = \big(\widehat{\mathrm{d}}\,\widehat{\alpha}\big)(\xi) = \mathrm{i}\, i_{\boxed{\xi}}\widehat{\alpha}(\xi)$$

radial coordinate vector



Exterior calculus and the Fourier transform

- Fourier transform of exterior derivative:

$$\mathfrak{F}\big(\mathrm{d}\alpha\big)(\xi) = \big(\widehat{\mathrm{d}}\,\widehat{\alpha}\big)(\xi) = \underbrace{\mathrm{i}_{i\xi}\widehat{\alpha}(\xi)}_{\text{Koszul differential that also appears in FEEC}}$$

Exterior calculus and the Fourier transform

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$$\mathfrak{F}\big(\mathrm{d}\alpha\big)(\xi) = \big(\widehat{\mathrm{d}}\,\widehat{\alpha}\big)(\xi) = \underbrace{\mathrm{i}_{i\xi}\widehat{\alpha}(\xi)}$$

Koszul differential that also appears in FEEC

Alternative in the literature:

$$\widehat{\mathrm{d}\alpha} = i\xi \wedge \widehat{\alpha}$$

Exterior calculus and the Fourier transform

- Hodge star:

$$\mathfrak{F}(\alpha) = s_n(\sigma) \star \widehat{\alpha}$$

- Plancherel theorem:

$$\langle\!\langle \alpha, \beta \rangle\!\rangle = \langle\!\langle \widehat{\alpha}, \widehat{\beta} \rangle\!\rangle$$

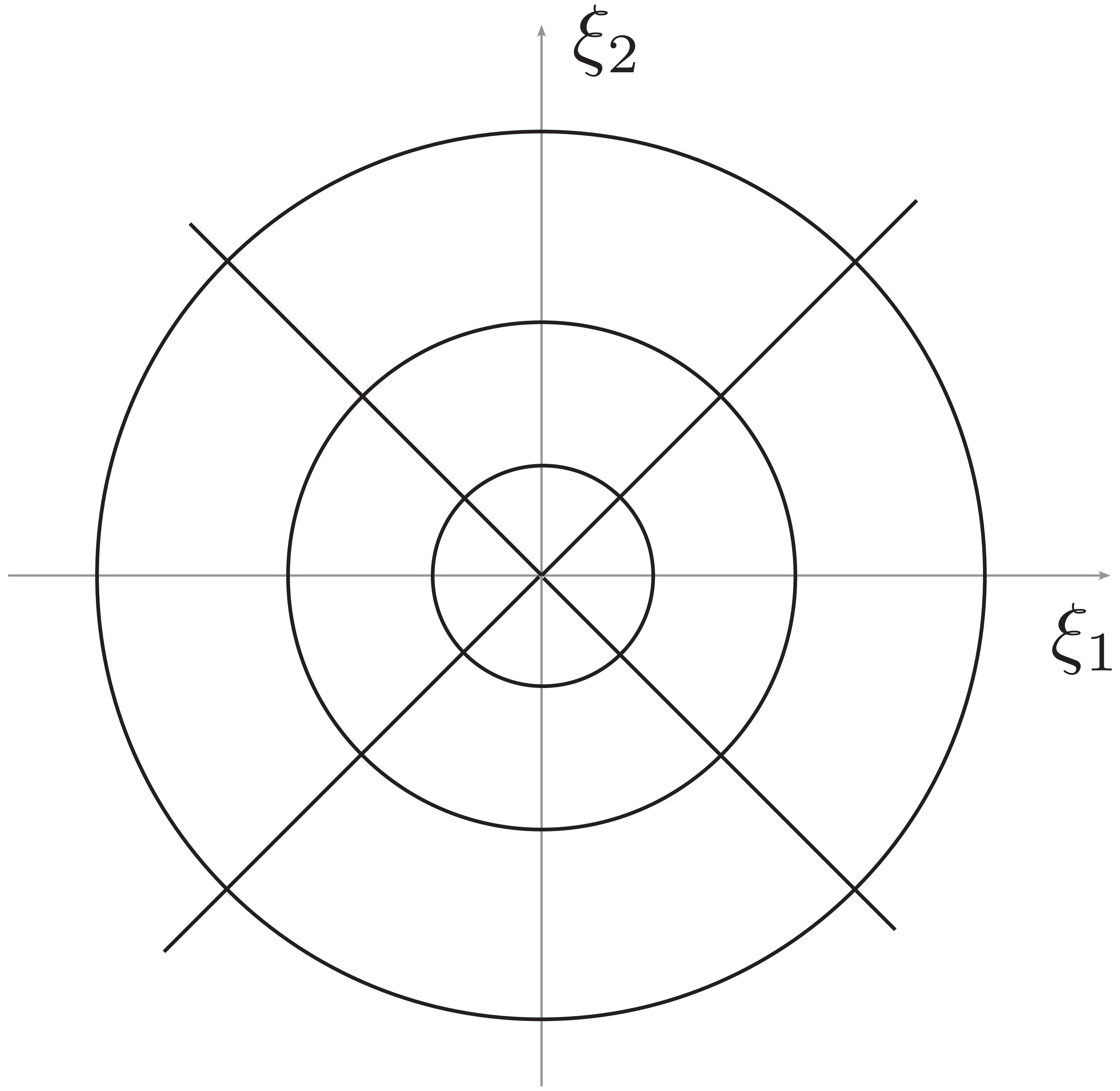
- Stokes theorem

Ψec: A local spectral exterior calculus

- Motto: use simple structure of exterior calculus in polar coordinates to construct a discretization of it

Ψec: A local spectral exterior calculus

- Differential form basis functions:



Ψec: A local spectral exterior calculus

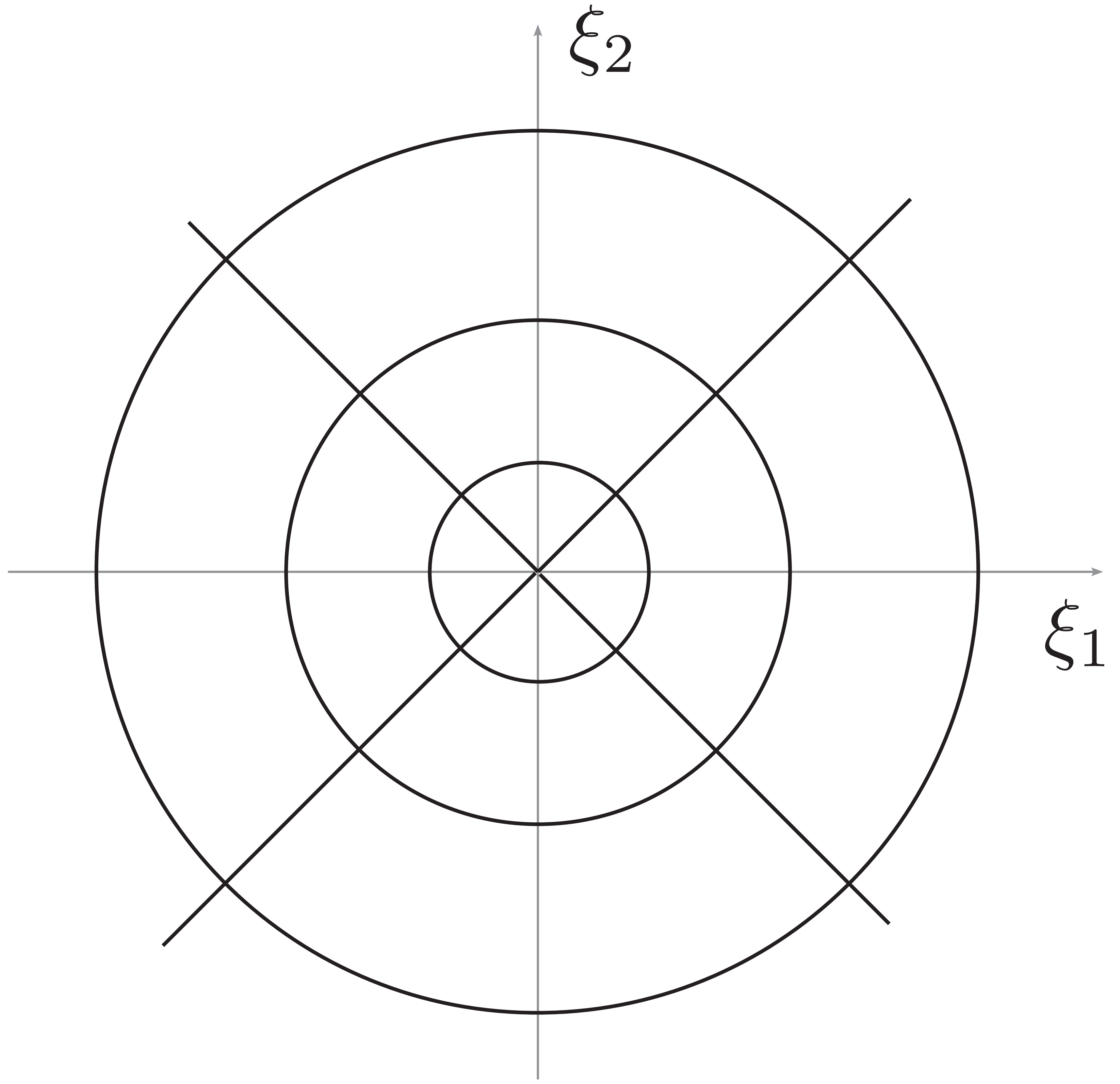
- Differential form basis functions:

$$\frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2}$$

$$\frac{\partial}{\partial \xi^2}$$

$$\frac{\partial}{\partial \xi^1}$$

$$1_\xi$$



Ψec: A local spectral exterior calculus

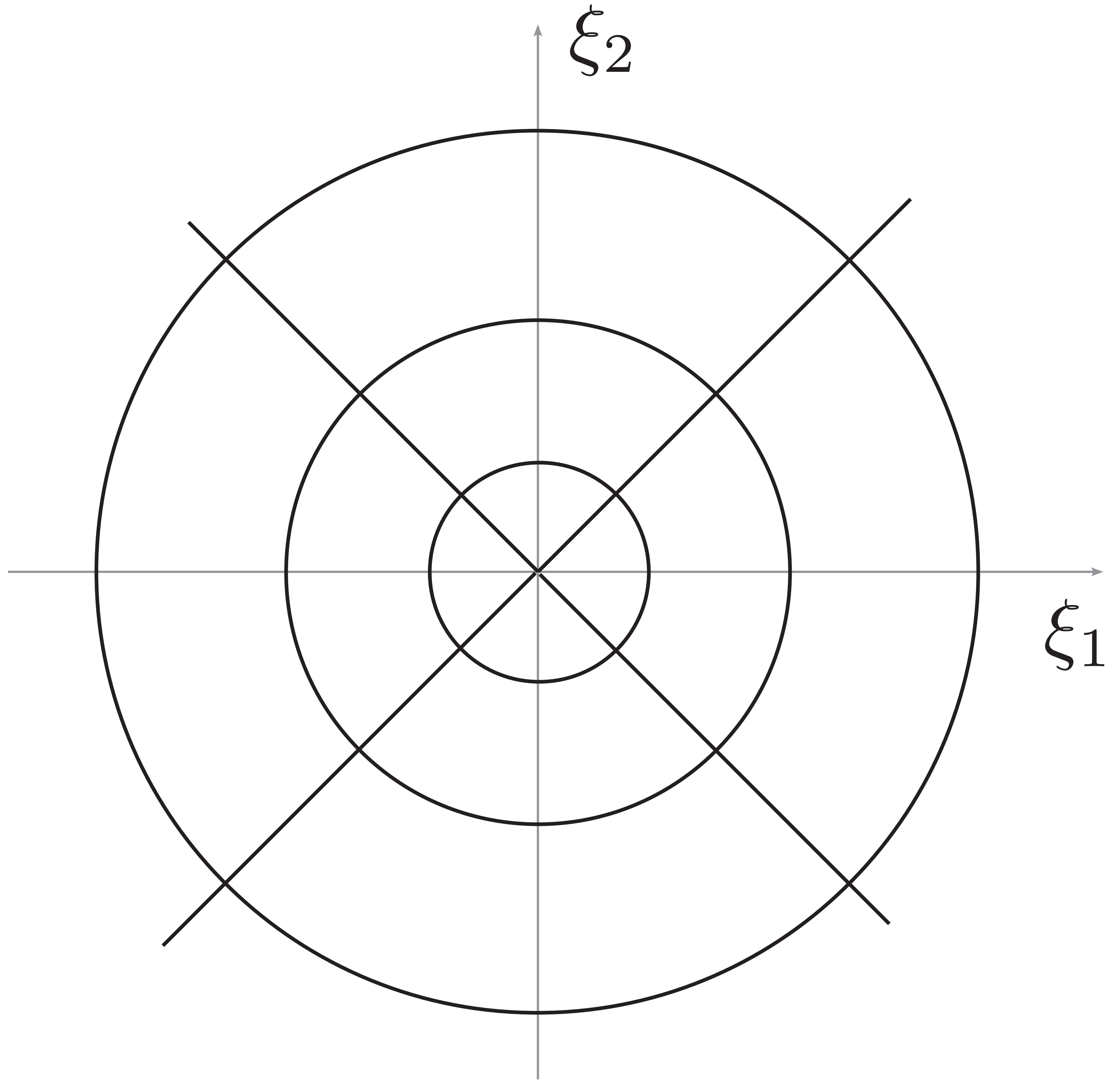
- Differential form basis functions:

$$\frac{\partial}{\partial \xi^{\theta}} \wedge \frac{\partial}{\partial \xi^{r}}$$

$$\frac{\partial}{\partial \xi^{\theta}}$$

$$\frac{\partial}{\partial \xi^{r}}$$

$$1_{\xi}$$



Ψec: A local spectral exterior calculus

- Differential form basis functions:

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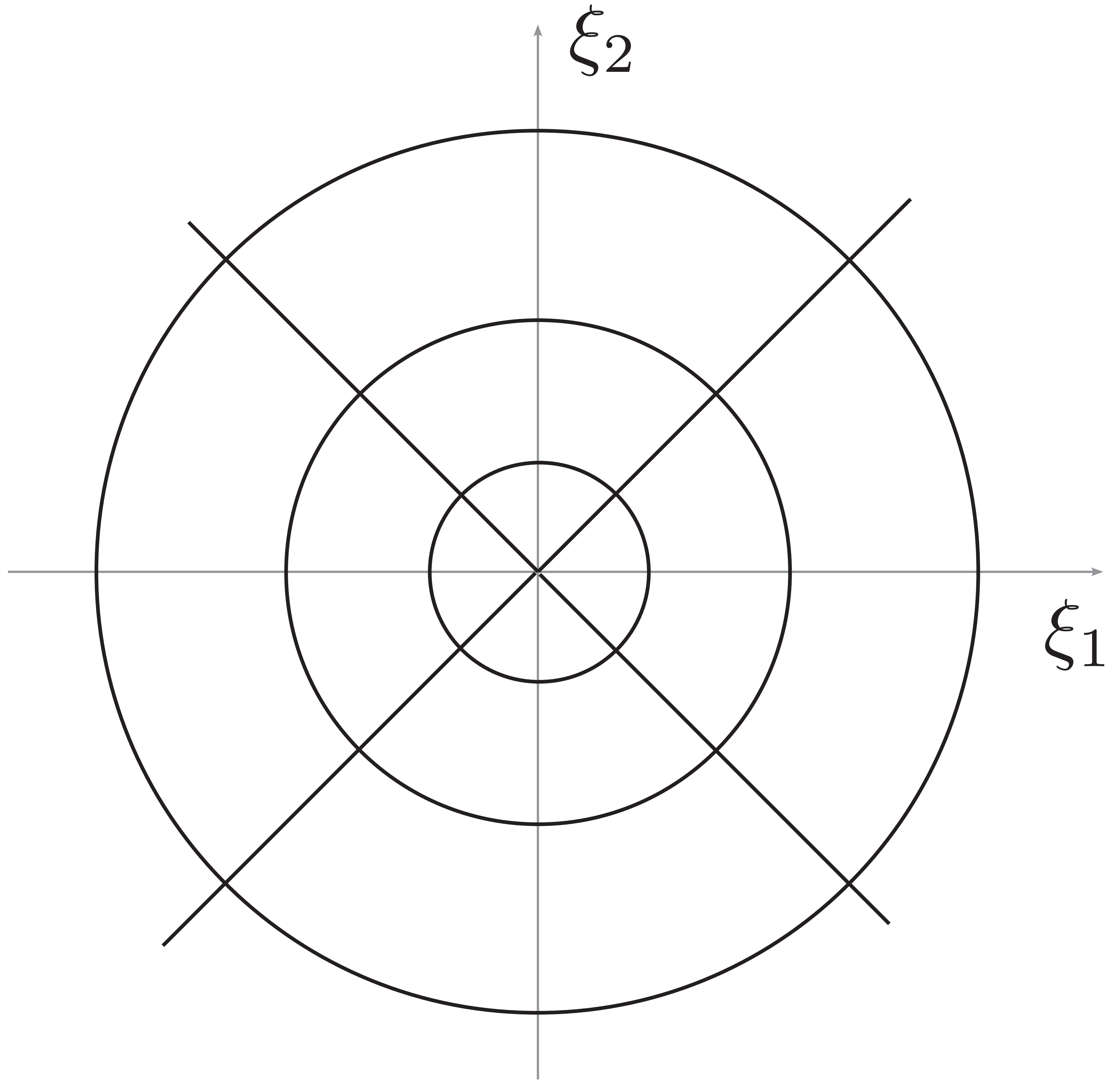

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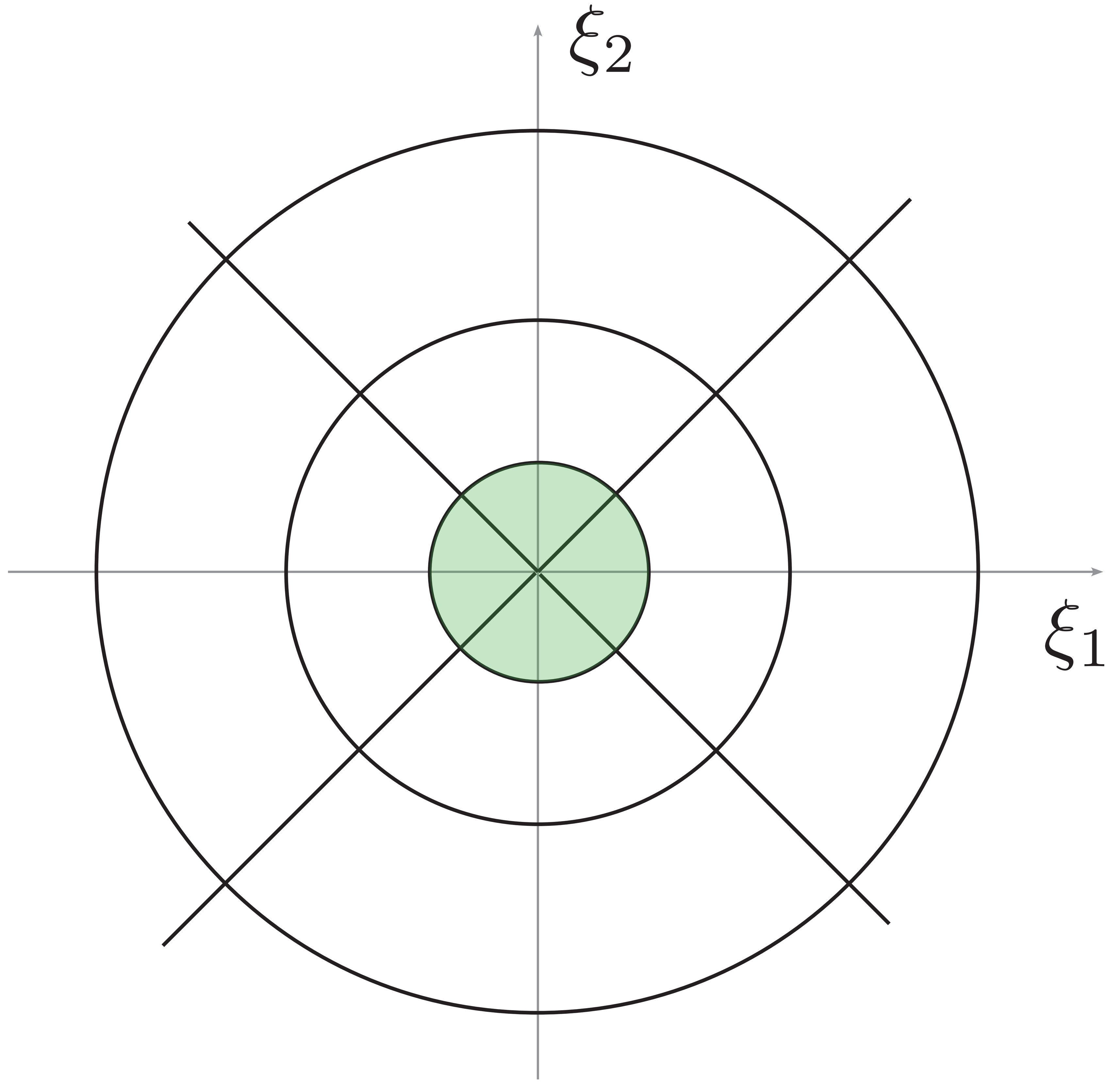

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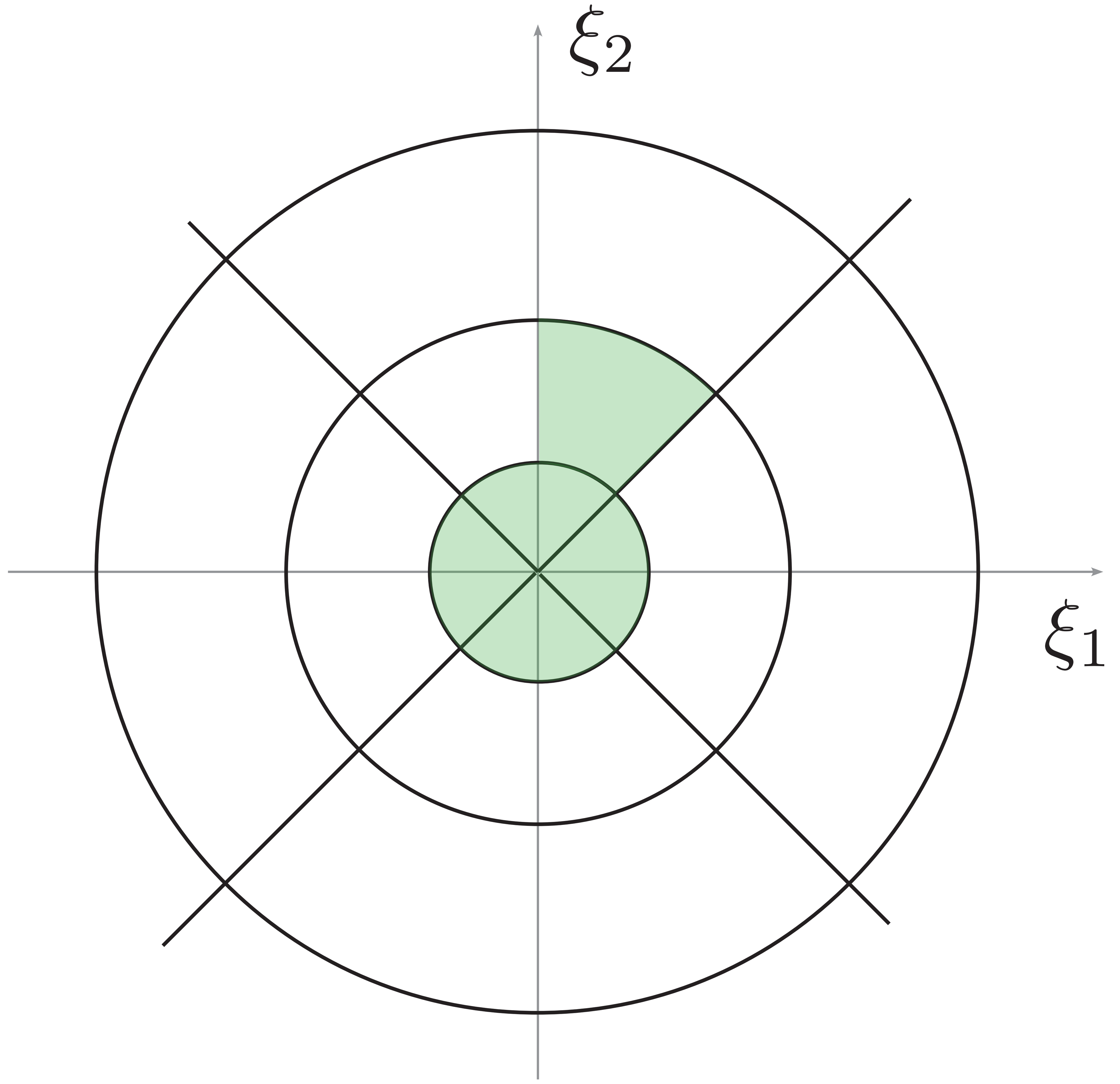

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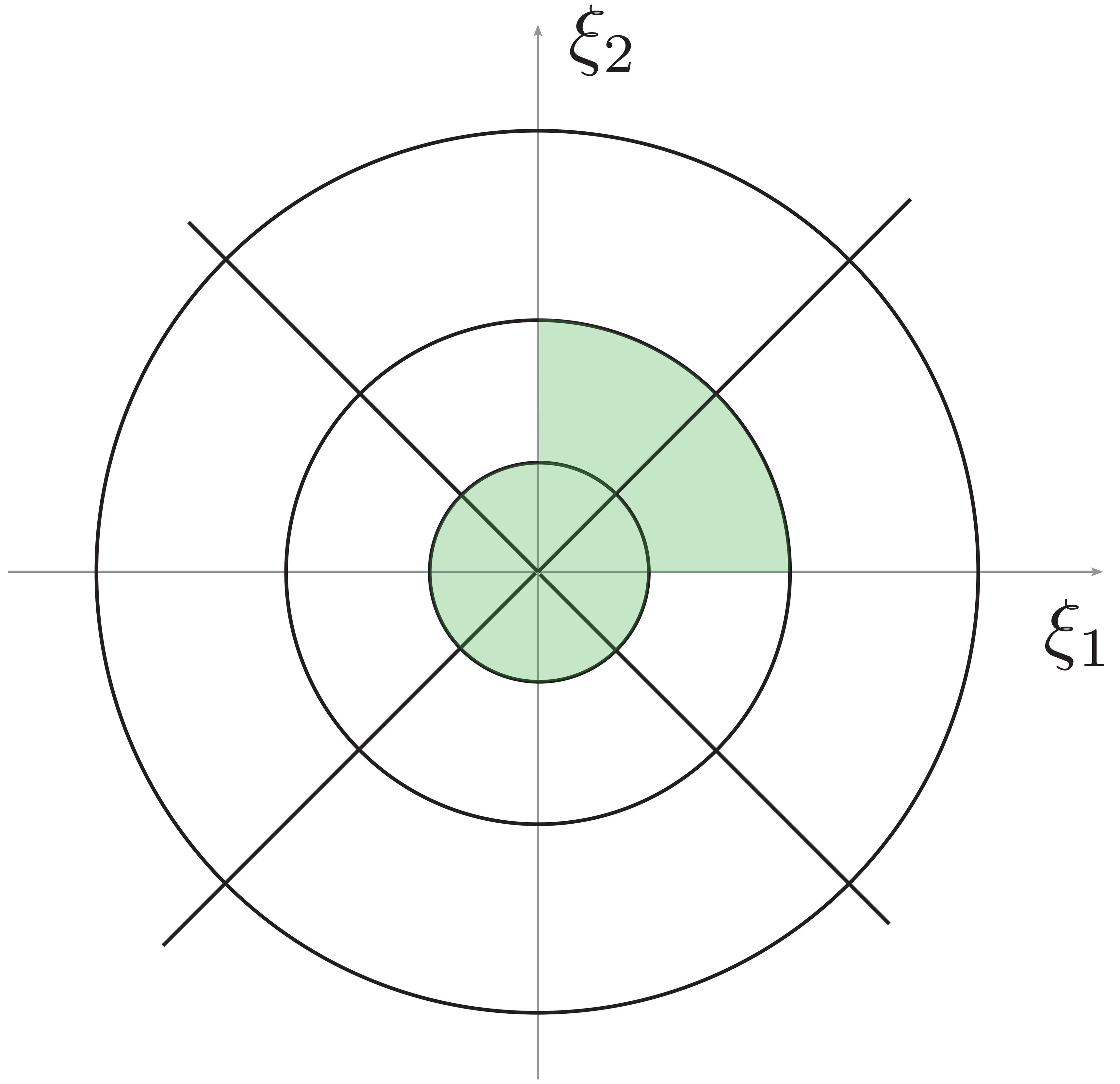

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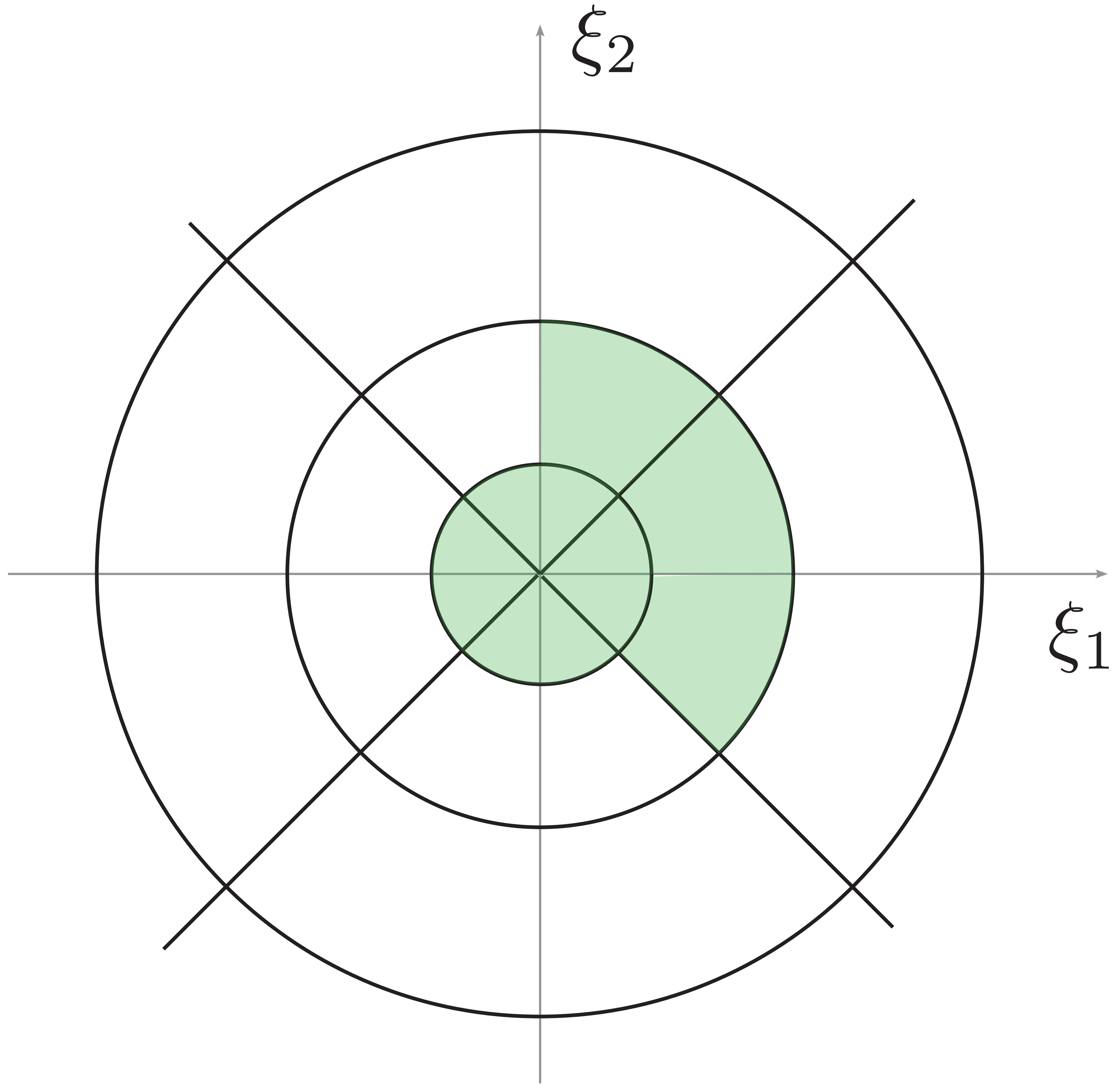

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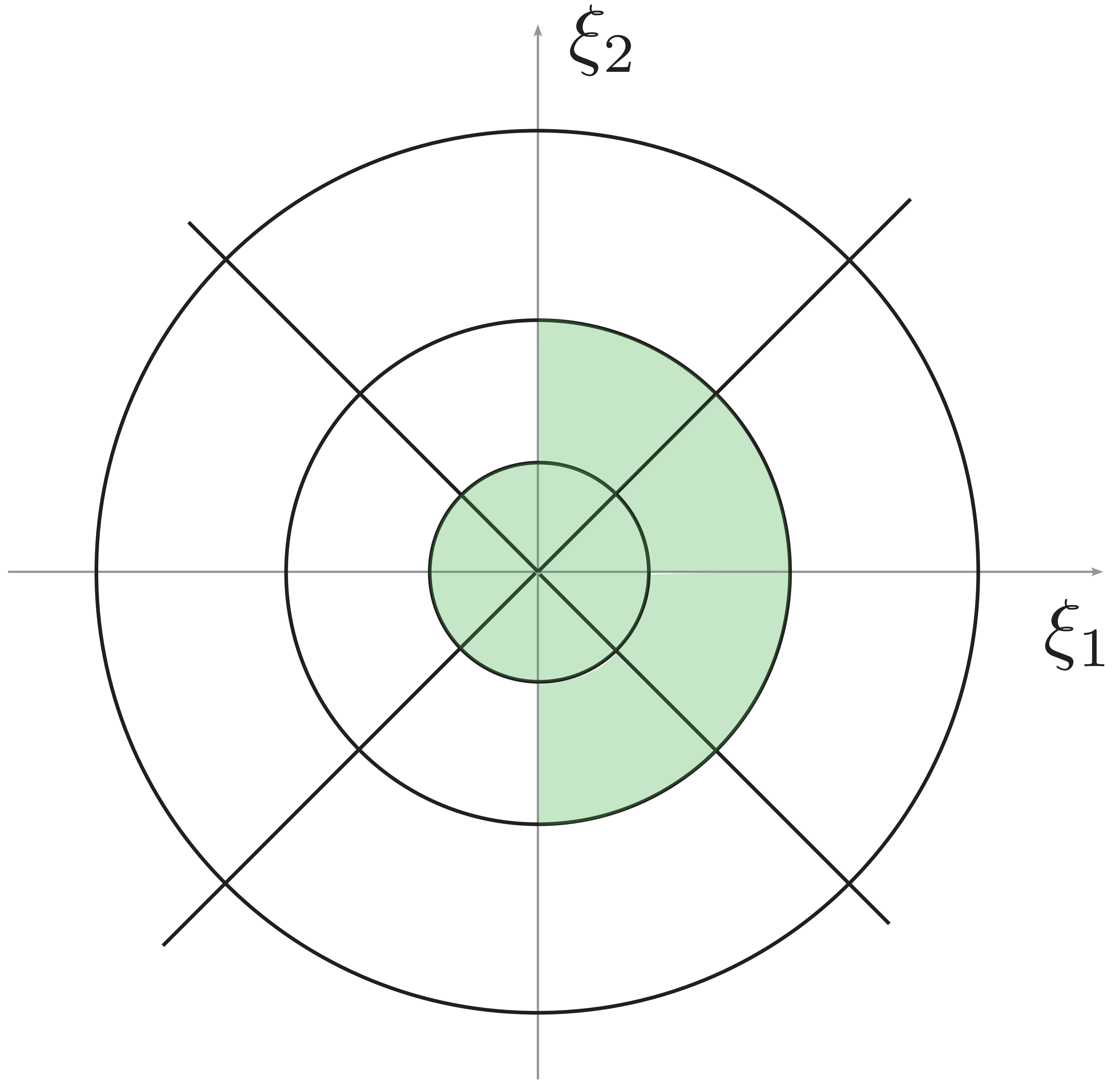

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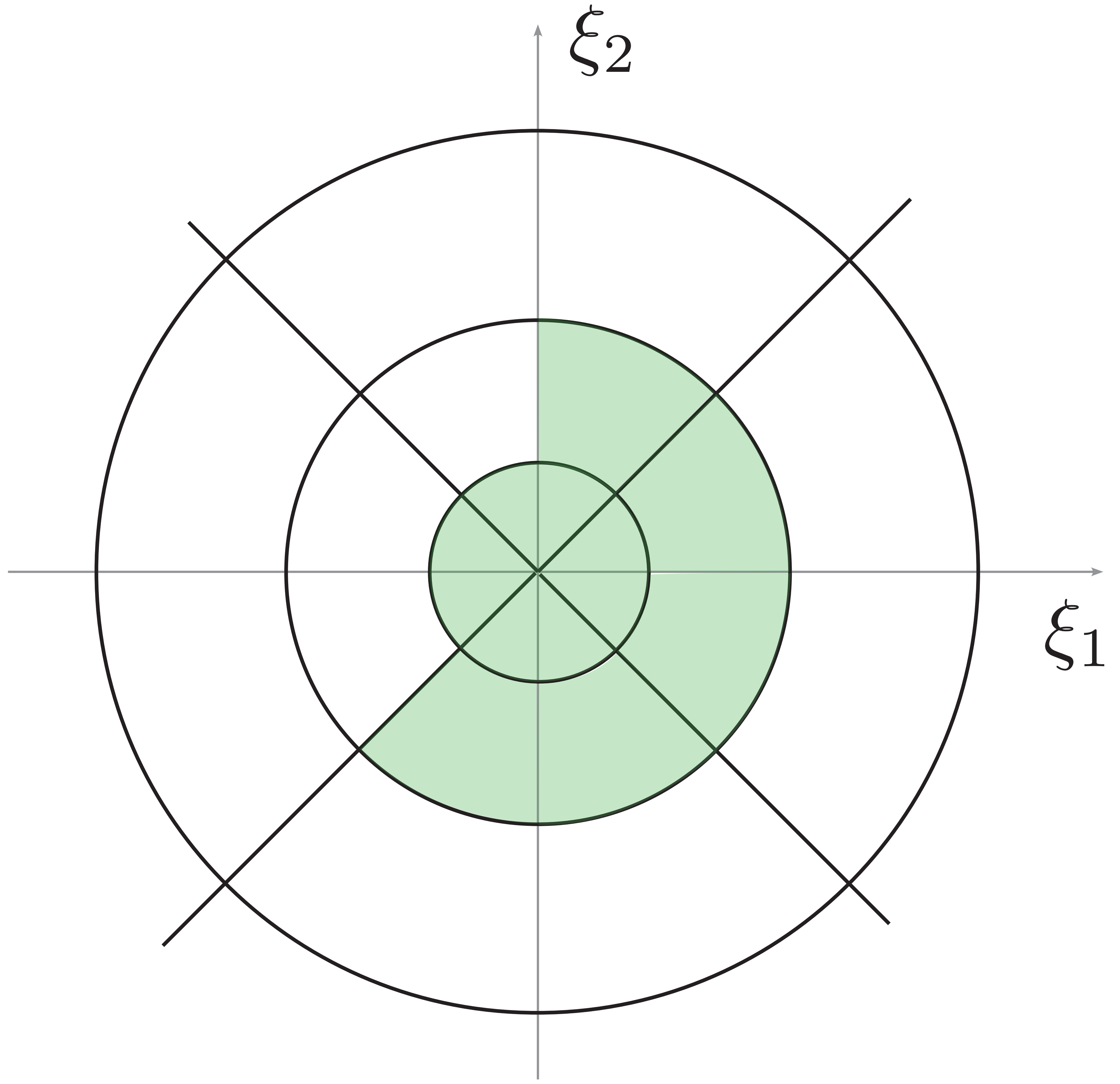

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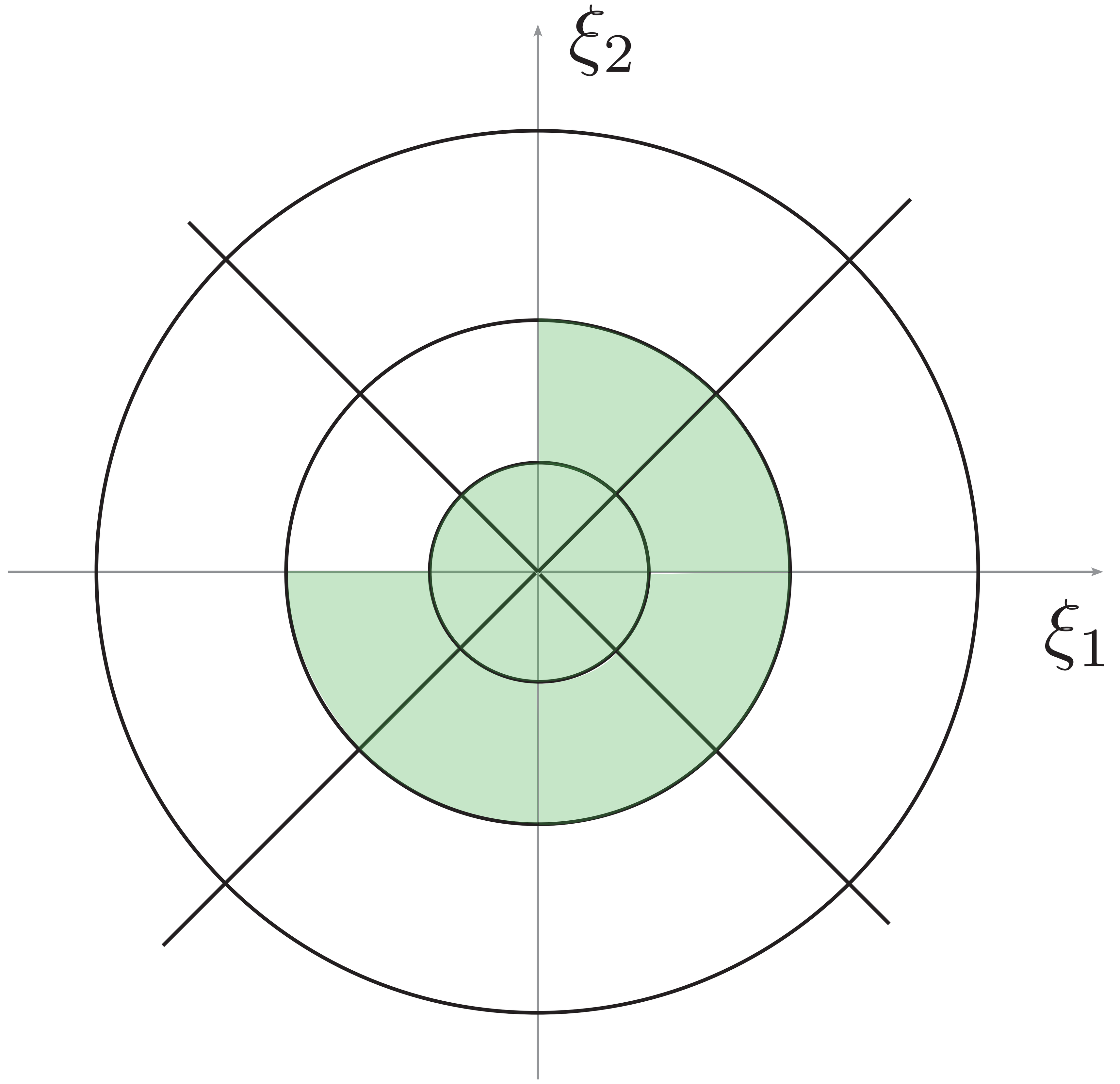

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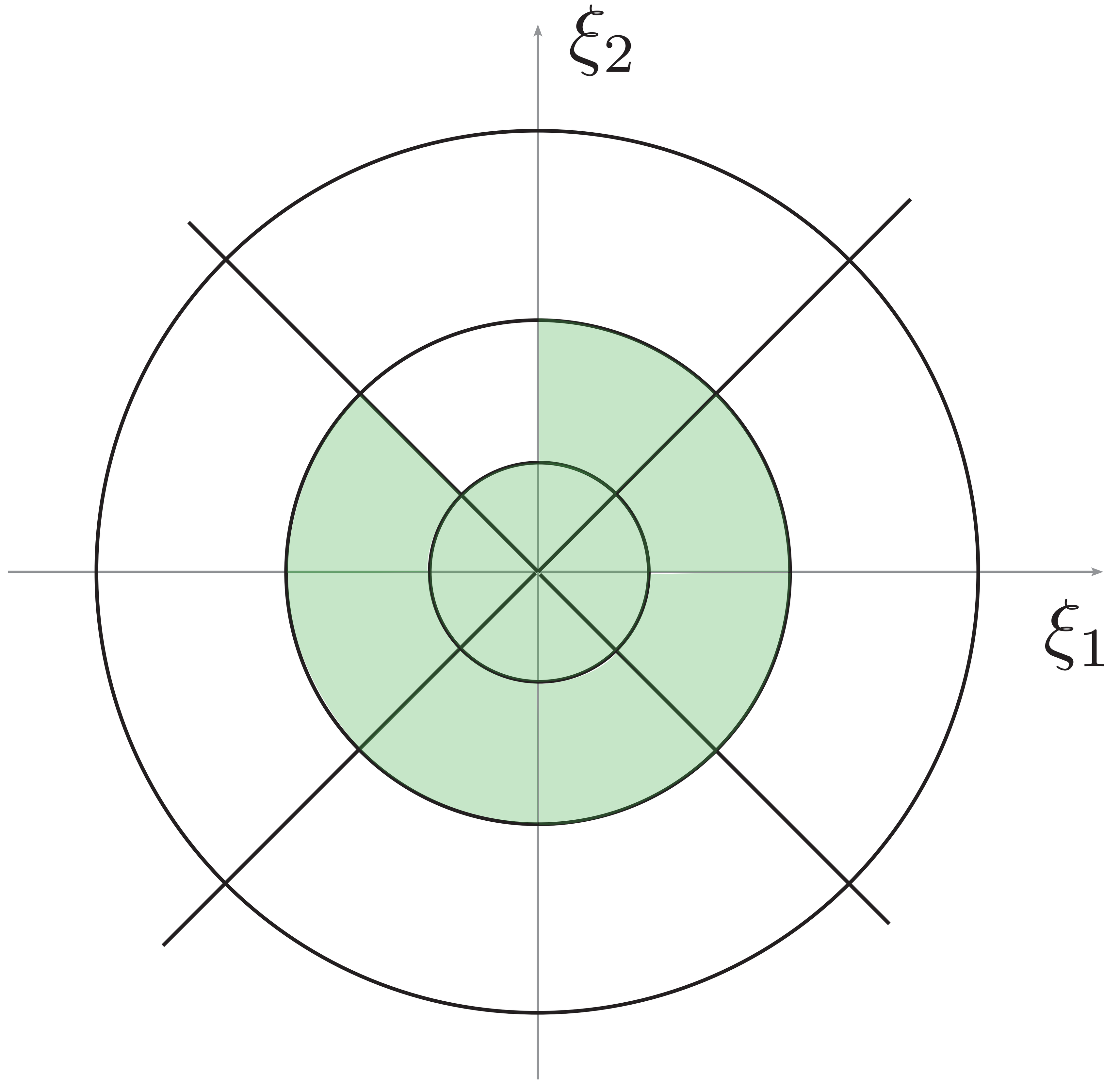

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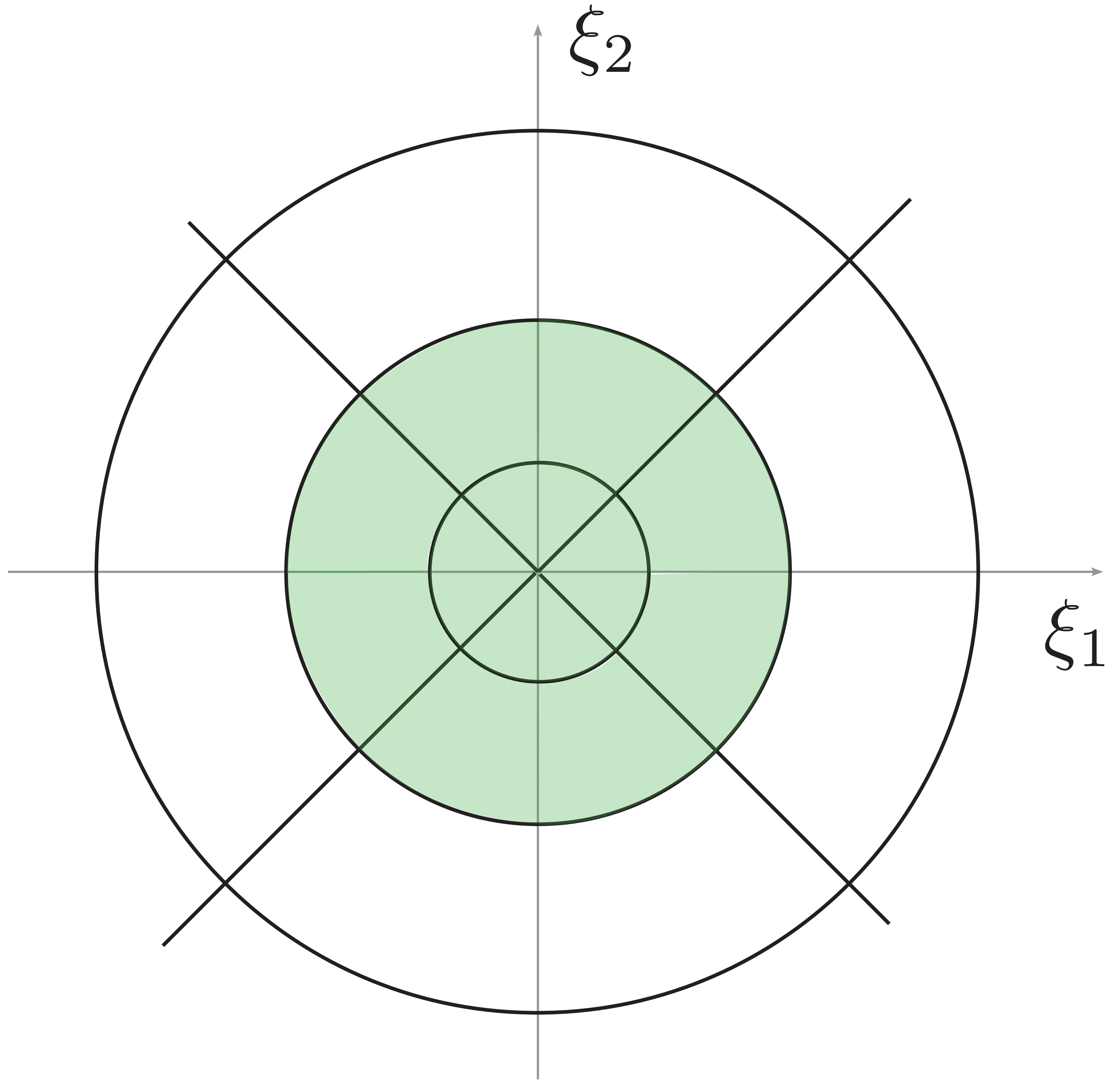

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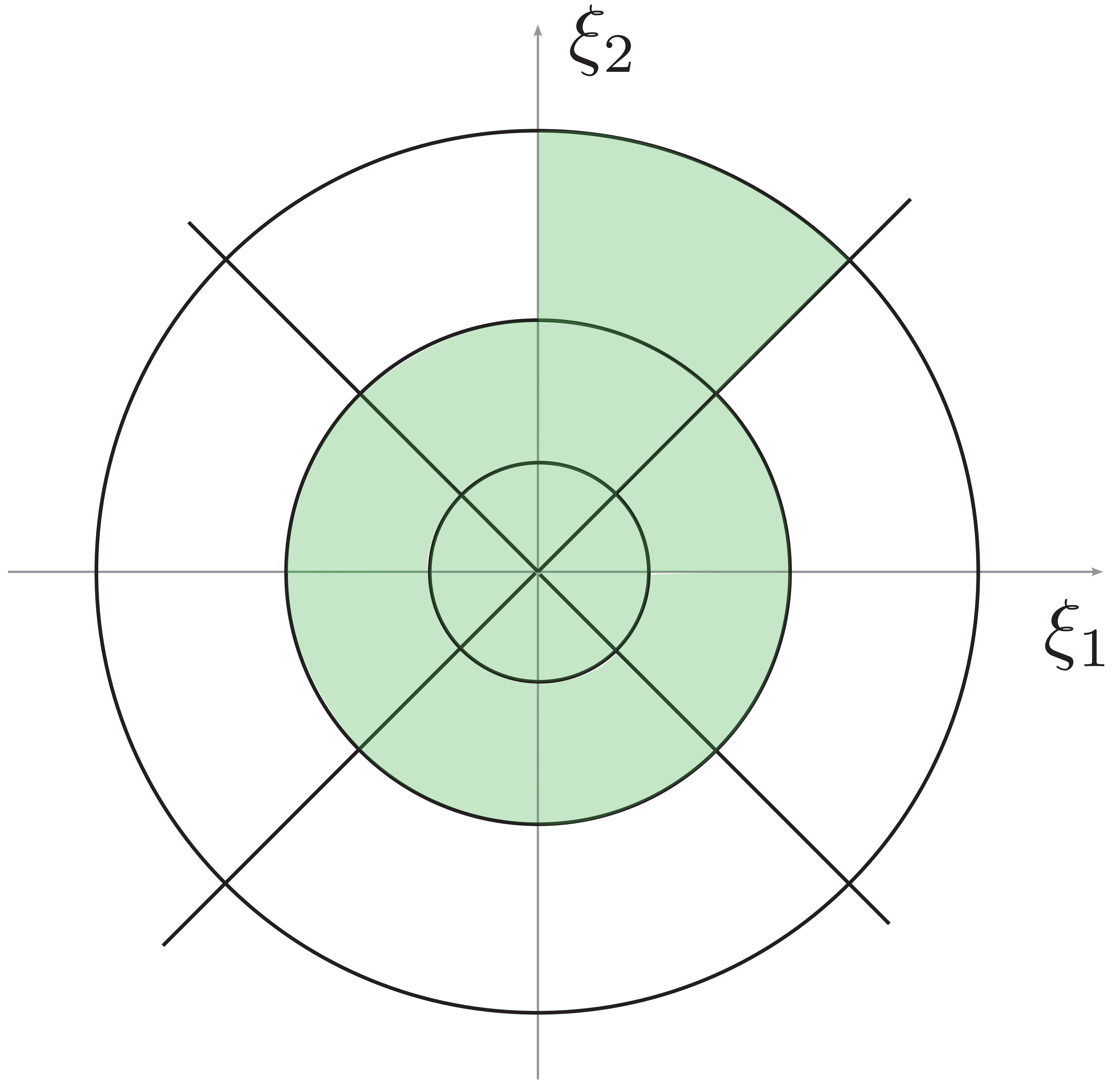

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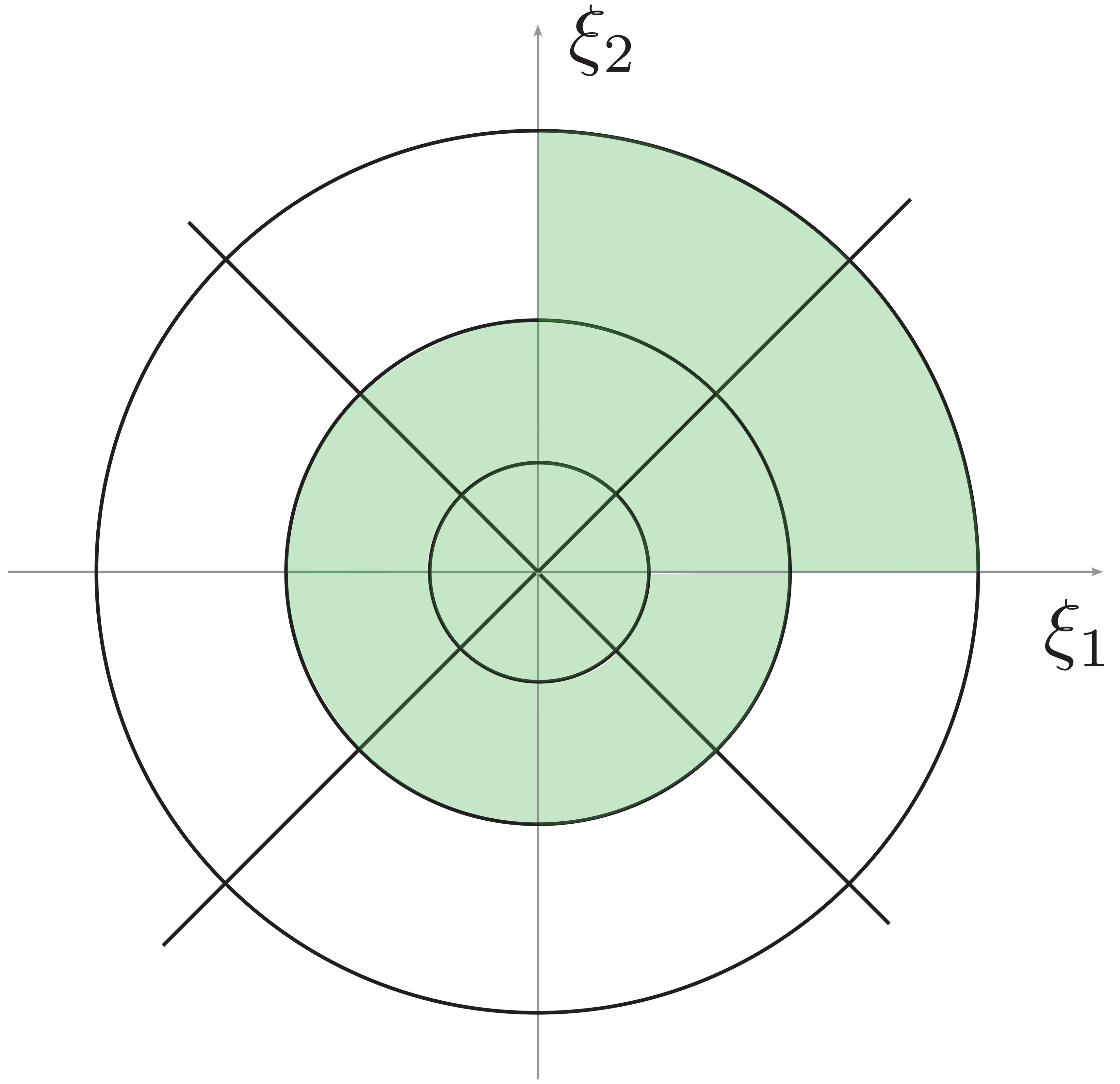

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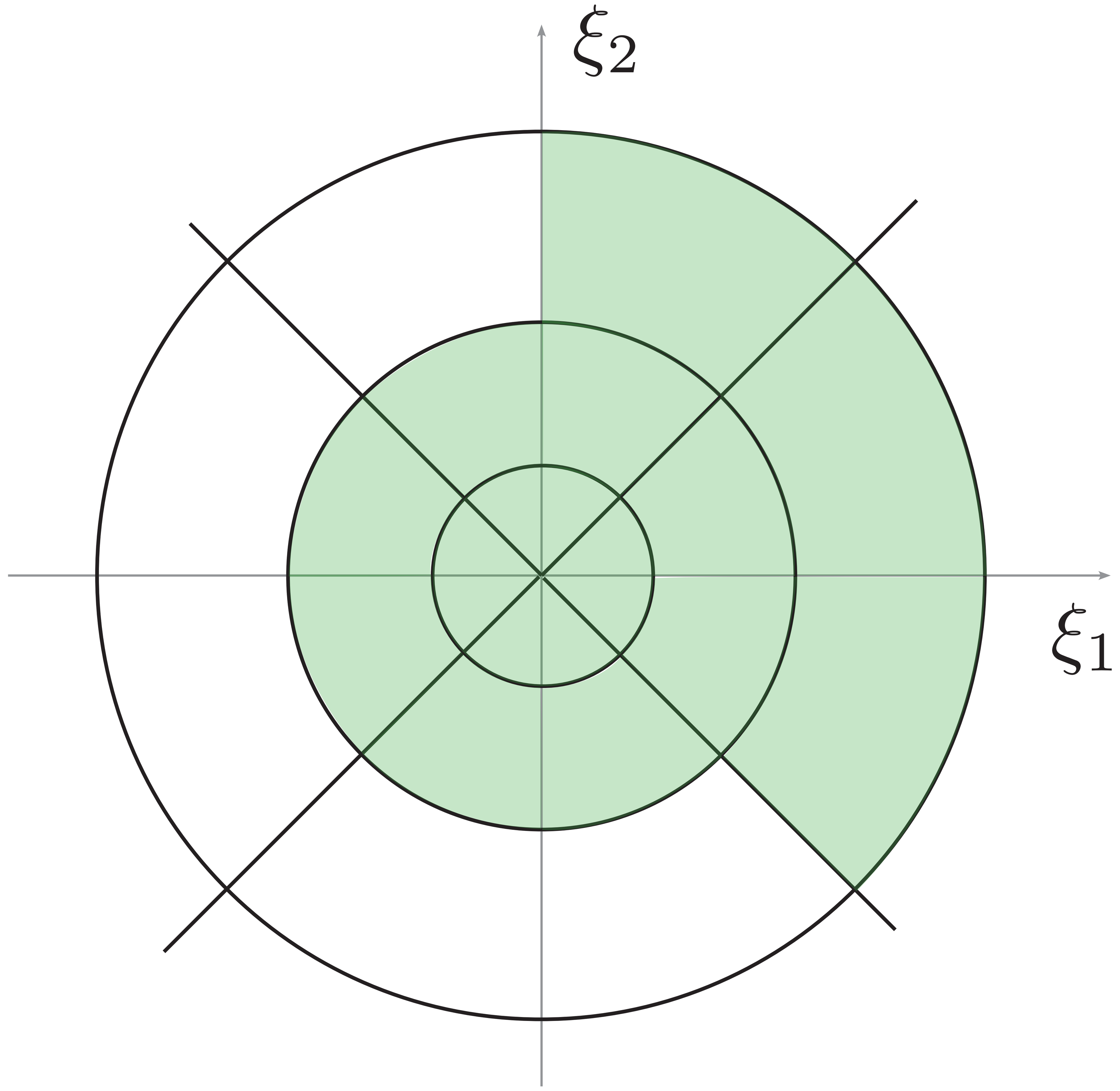

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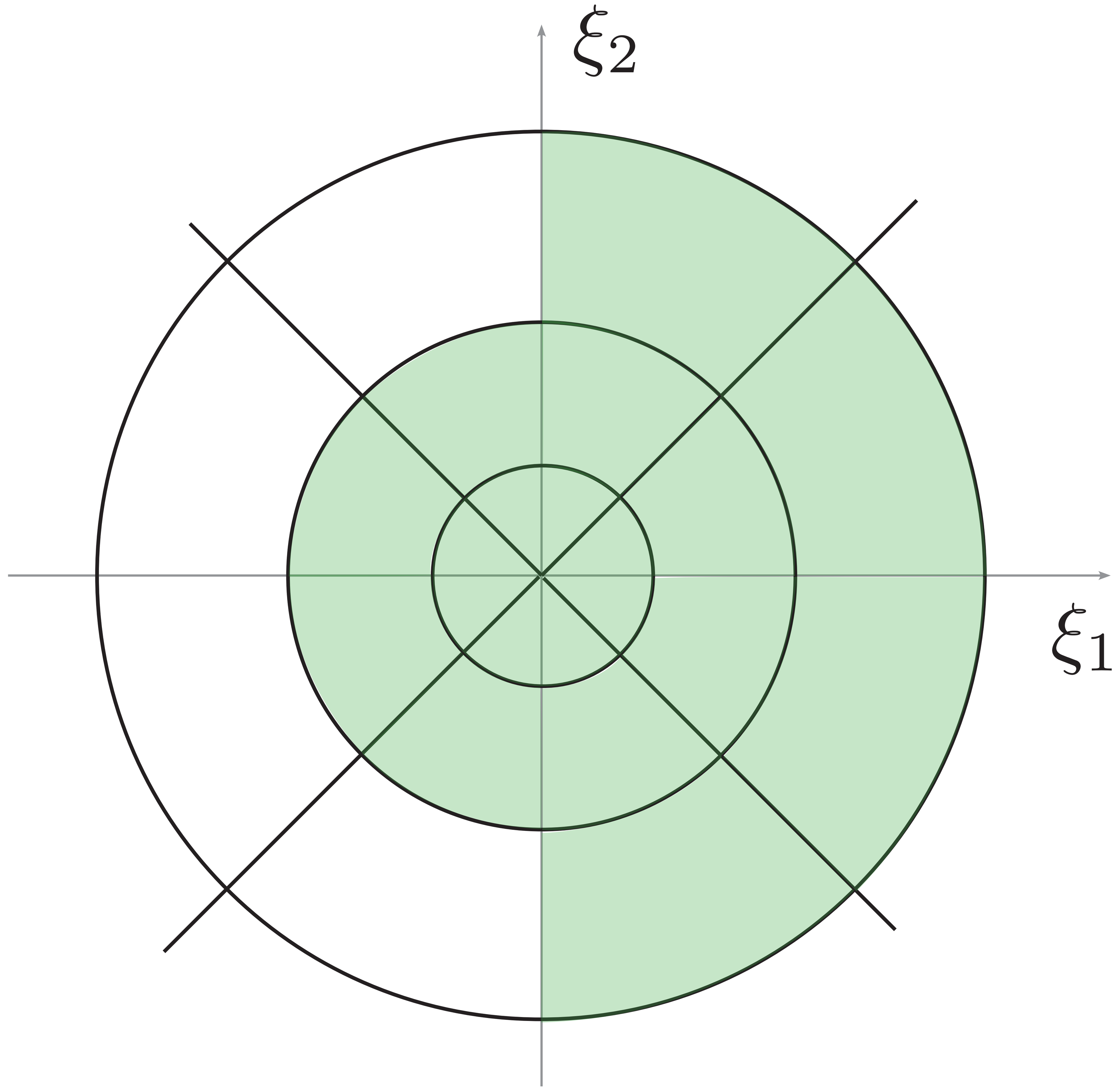

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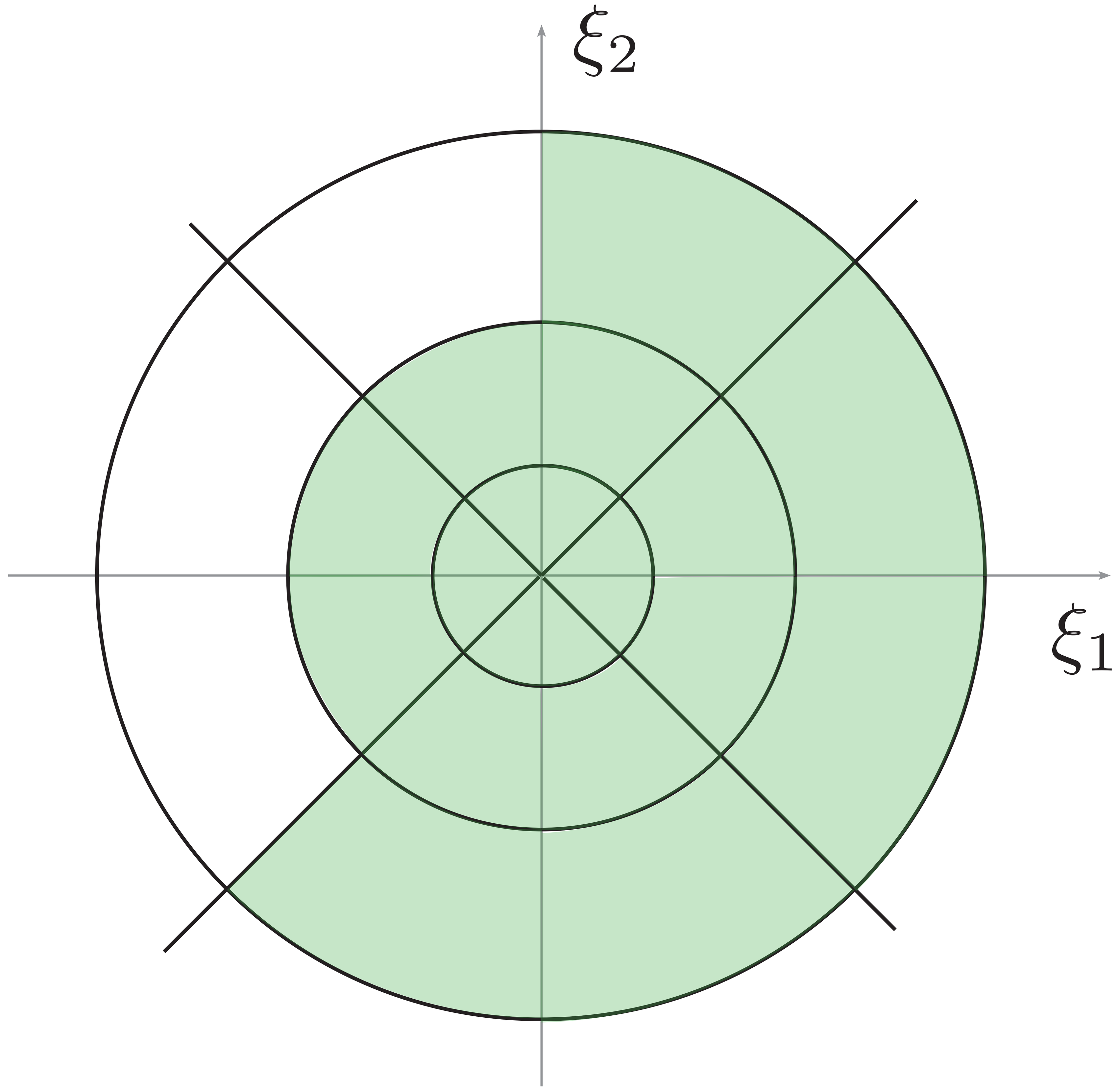

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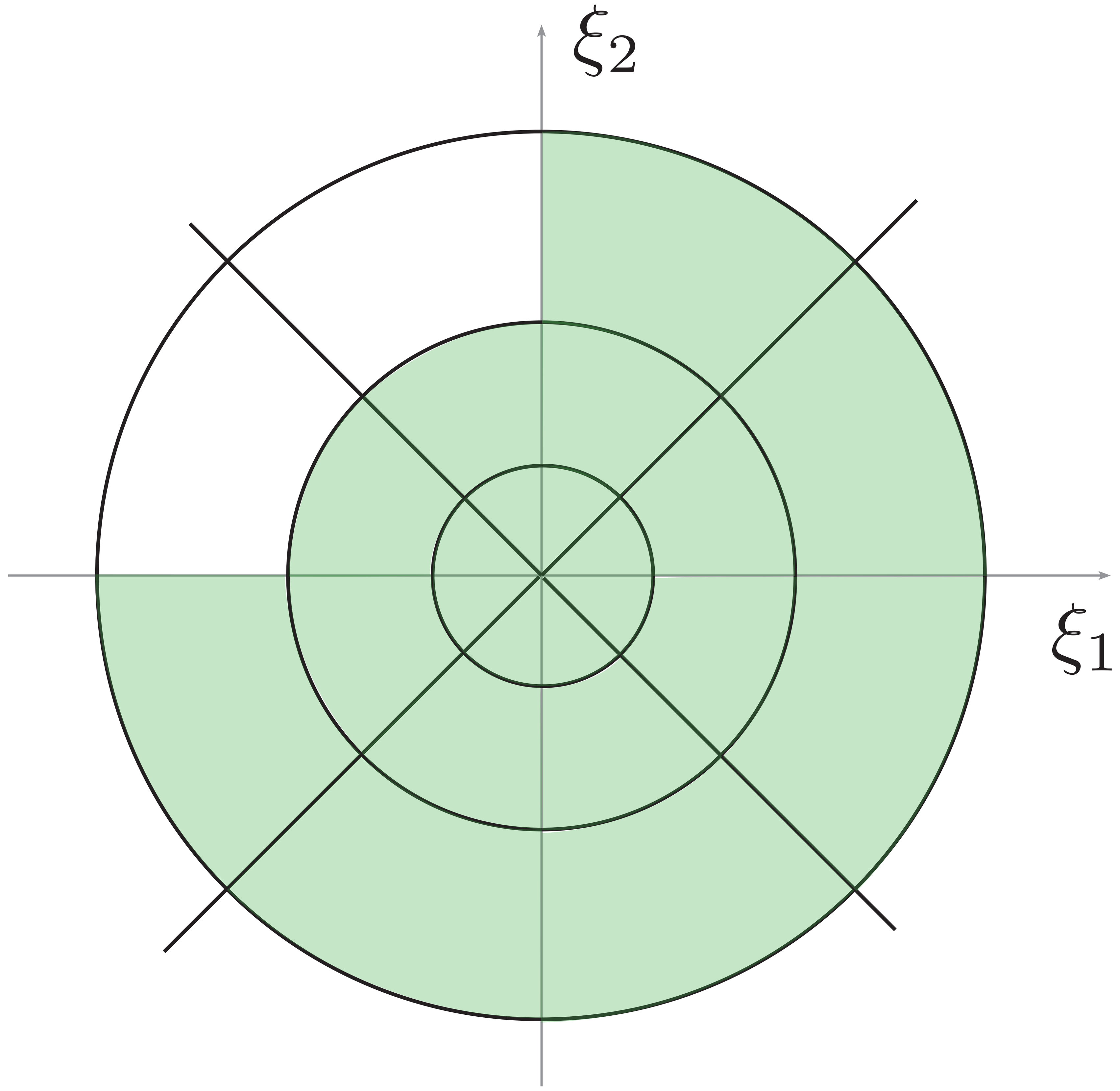

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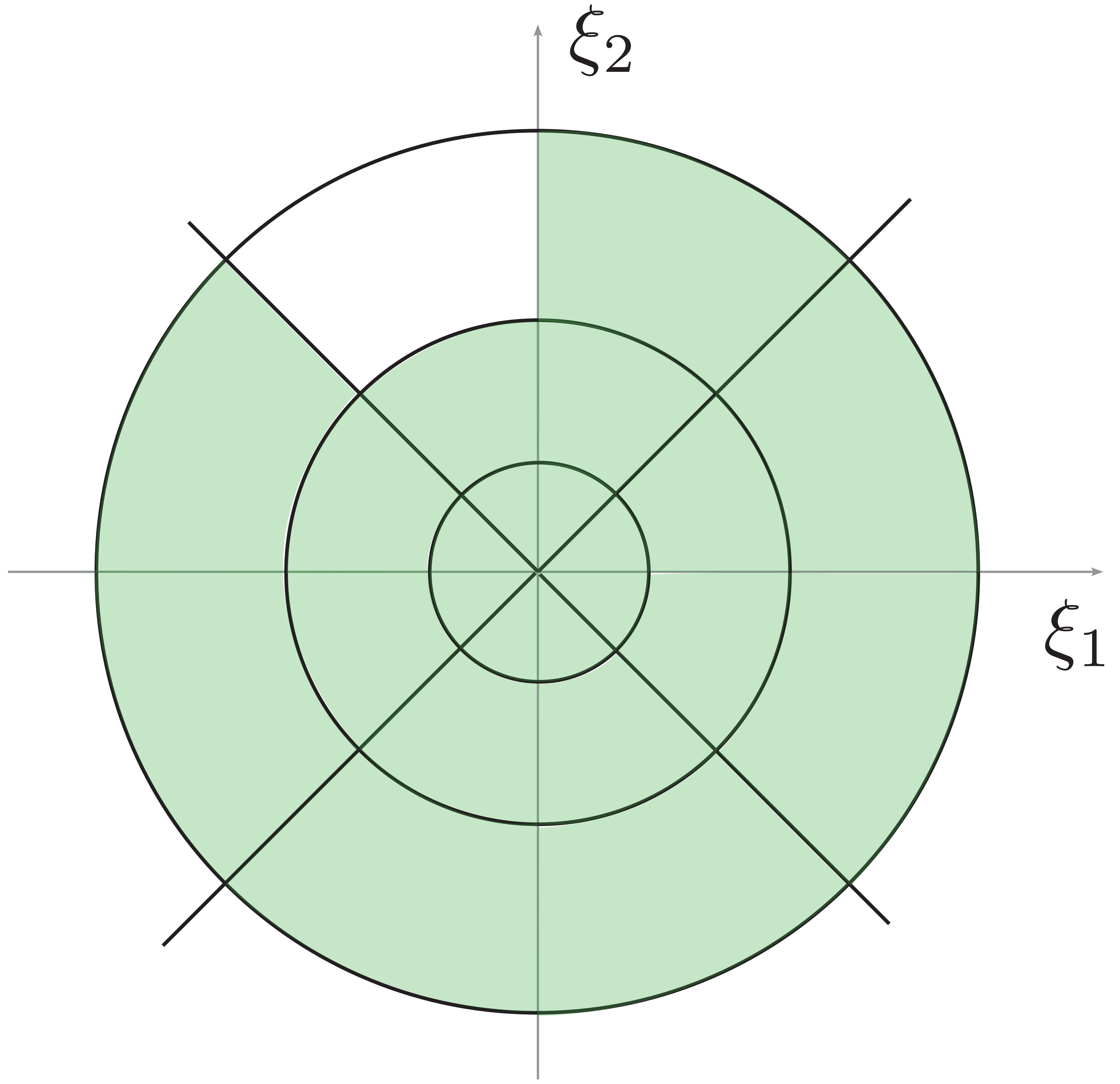


Ψec: A local spectral exterior calculus

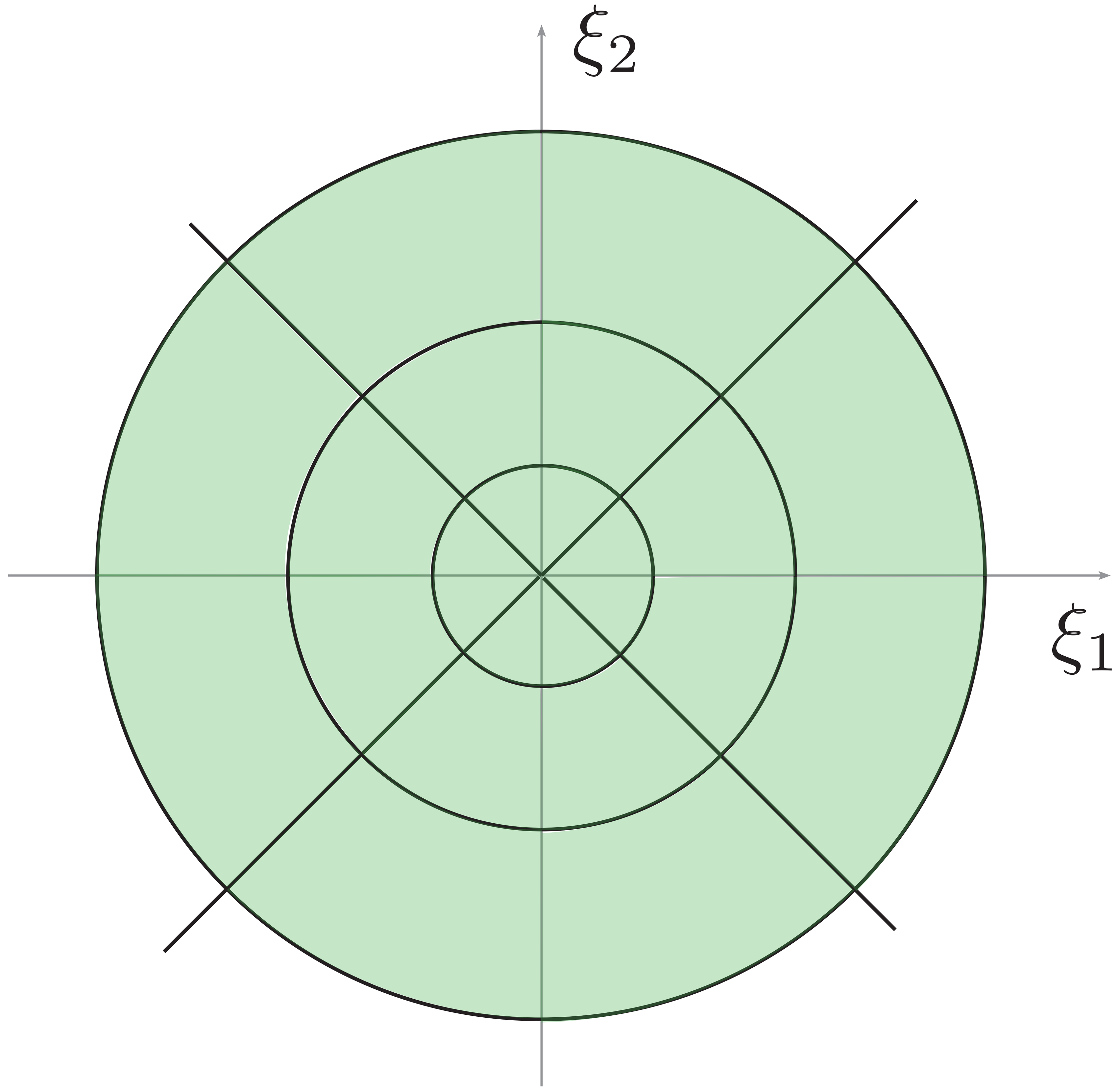
- Differential form basis functions:

$$\hat{\psi}^{2,\mathrm{d}}(\xi)\,\frac{\partial}{\partial\xi^{\theta}}\wedge\frac{\partial}{\partial\xi^{r}}$$

$$\hat{\psi}^{1,\delta}(\xi)\,\frac{\partial}{\partial\xi^{\theta}}$$

$$\hat{\psi}^{1,\mathrm{d}}(\xi)\,\frac{\partial}{\partial\xi^{r}}$$

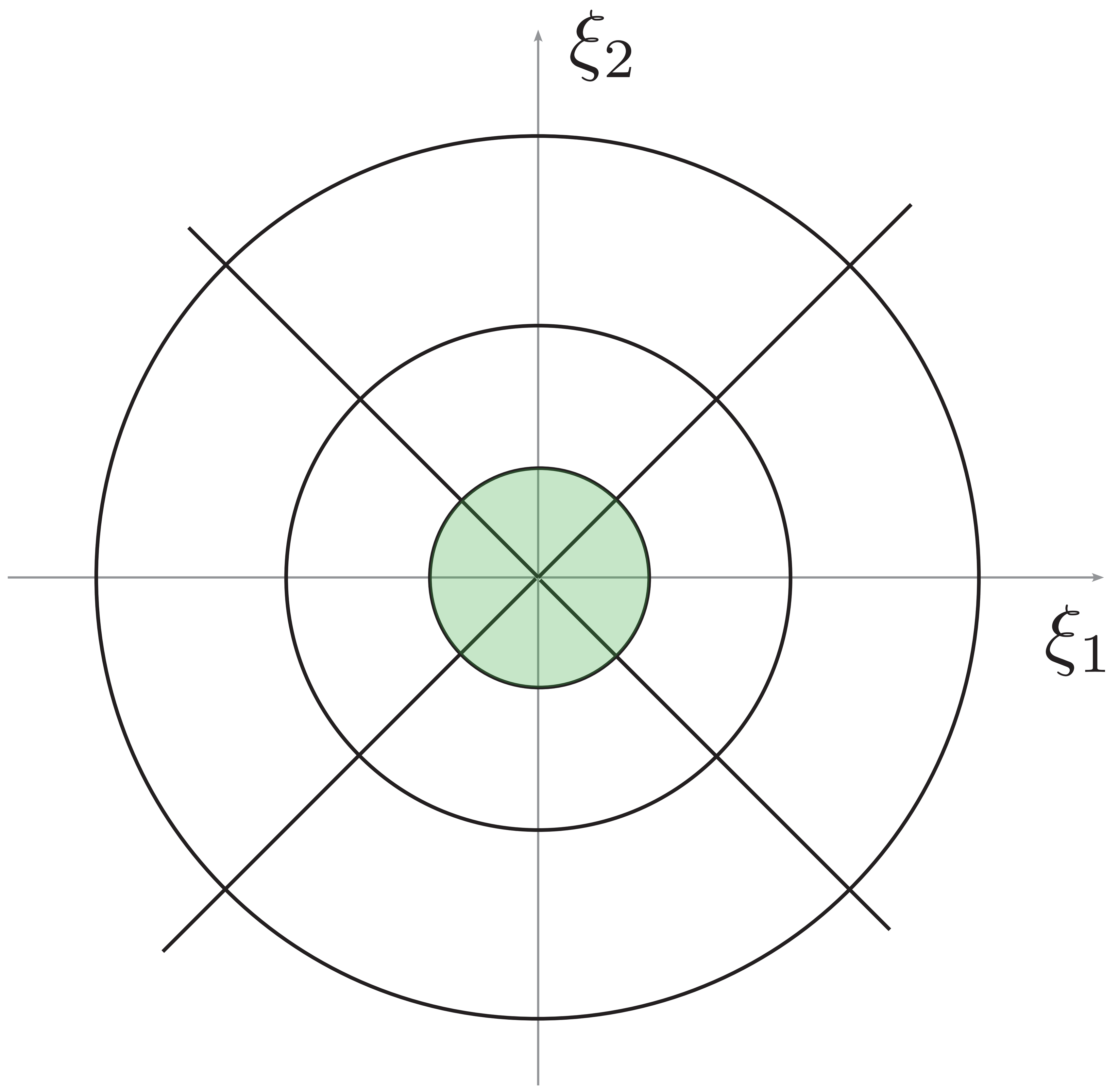
$$\hat{\psi}^{0,\delta}(\xi)\,1_{\xi}$$



Ψec: A local spectral exterior calculus

- Differential form basis functions:

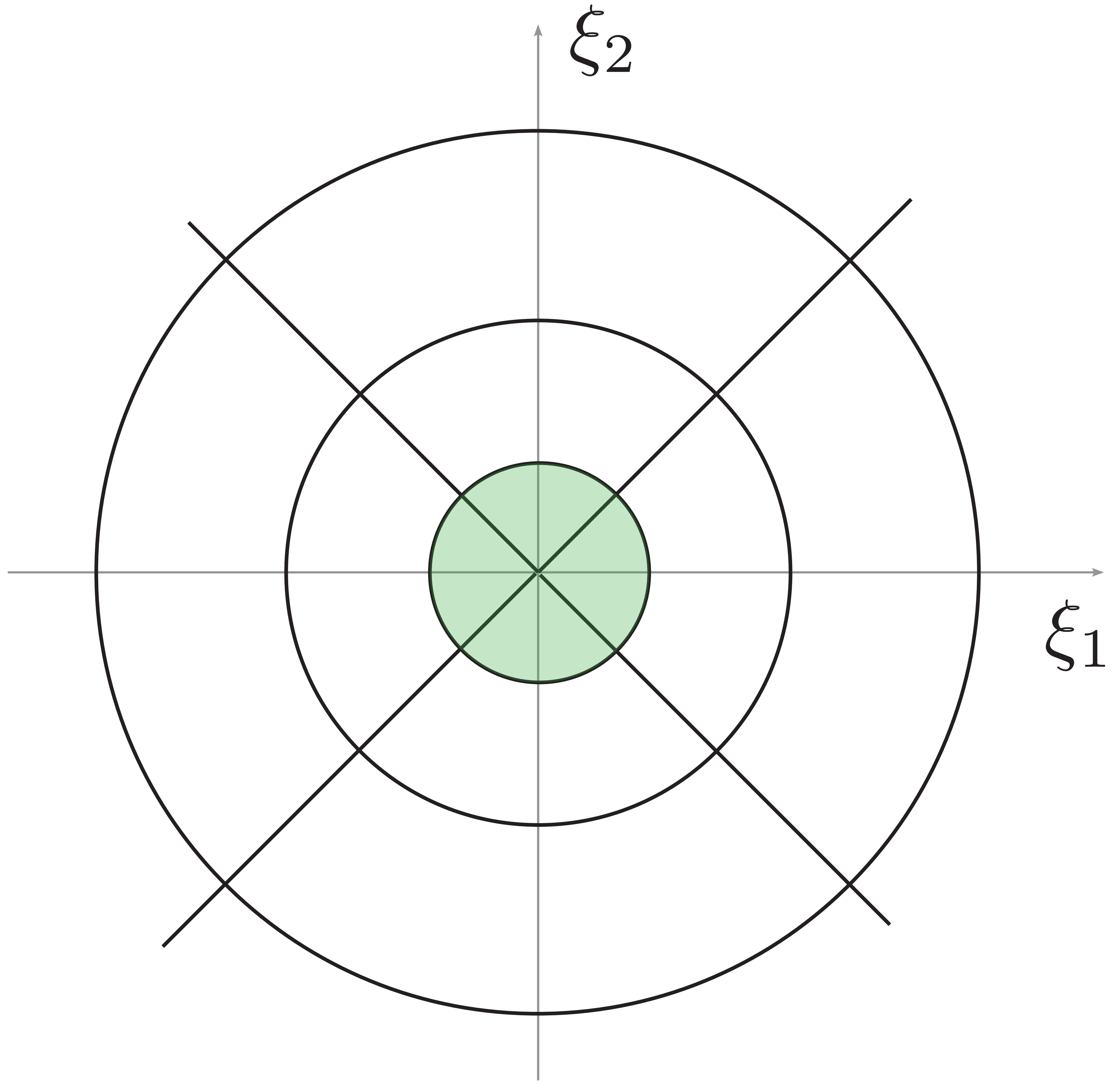
$$\hat{\psi}^{0,\delta}(\xi) = i\hat{h}(|\xi|)$$



Ψec: A local spectral exterior calculus

- Differential form basis functions:

$$\hat{\psi}_0^{0,\delta}(\xi) = i\hat{h}(|\xi|)$$

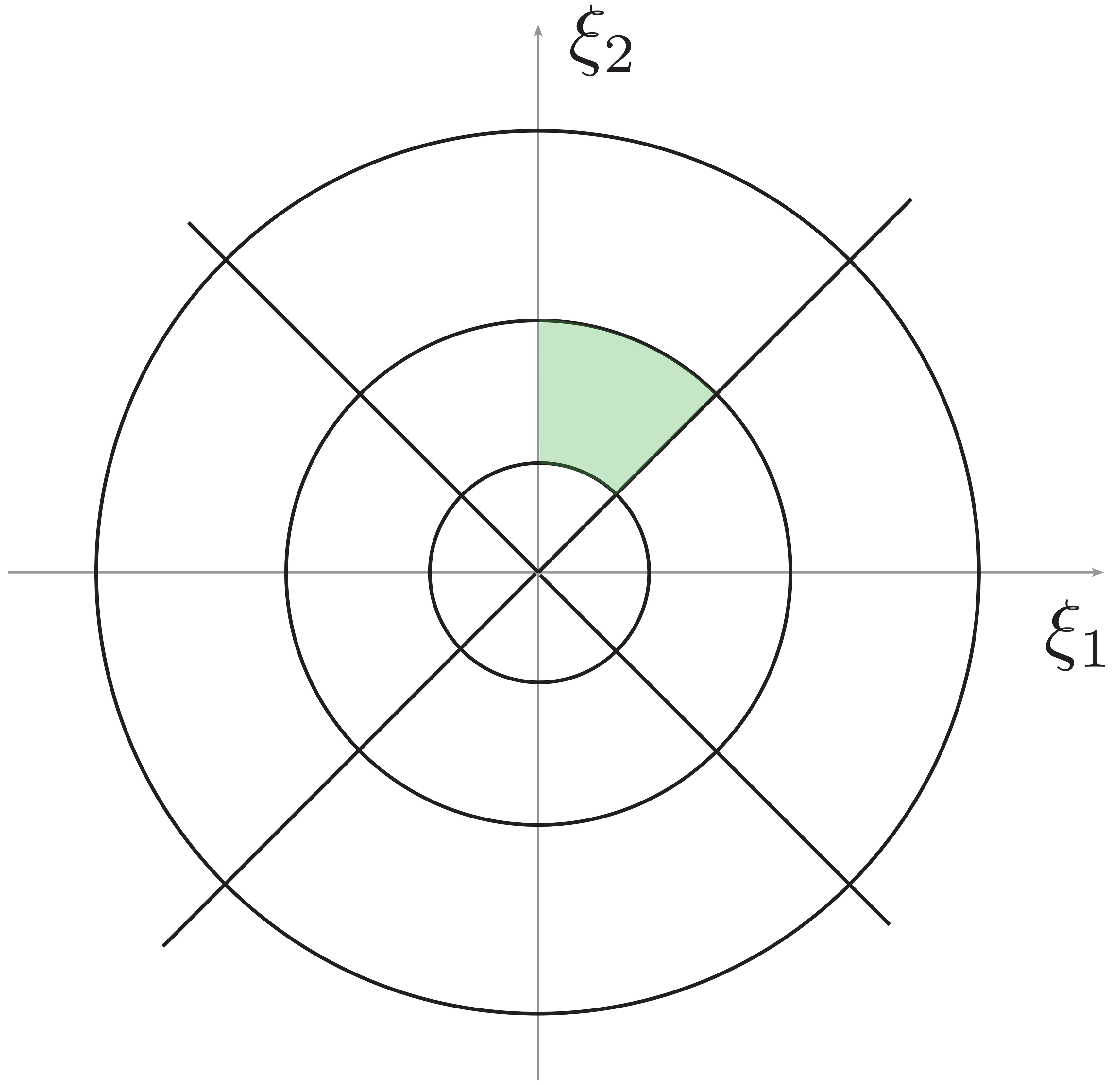


Ψec: A local spectral exterior calculus

- Differential form basis functions:

$$\hat{\psi}_0^{0,\delta}(\xi) = i\hat{h}(|\xi|)$$

$$\hat{\psi}_j^{0,\delta}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_j(\theta_\xi)$$



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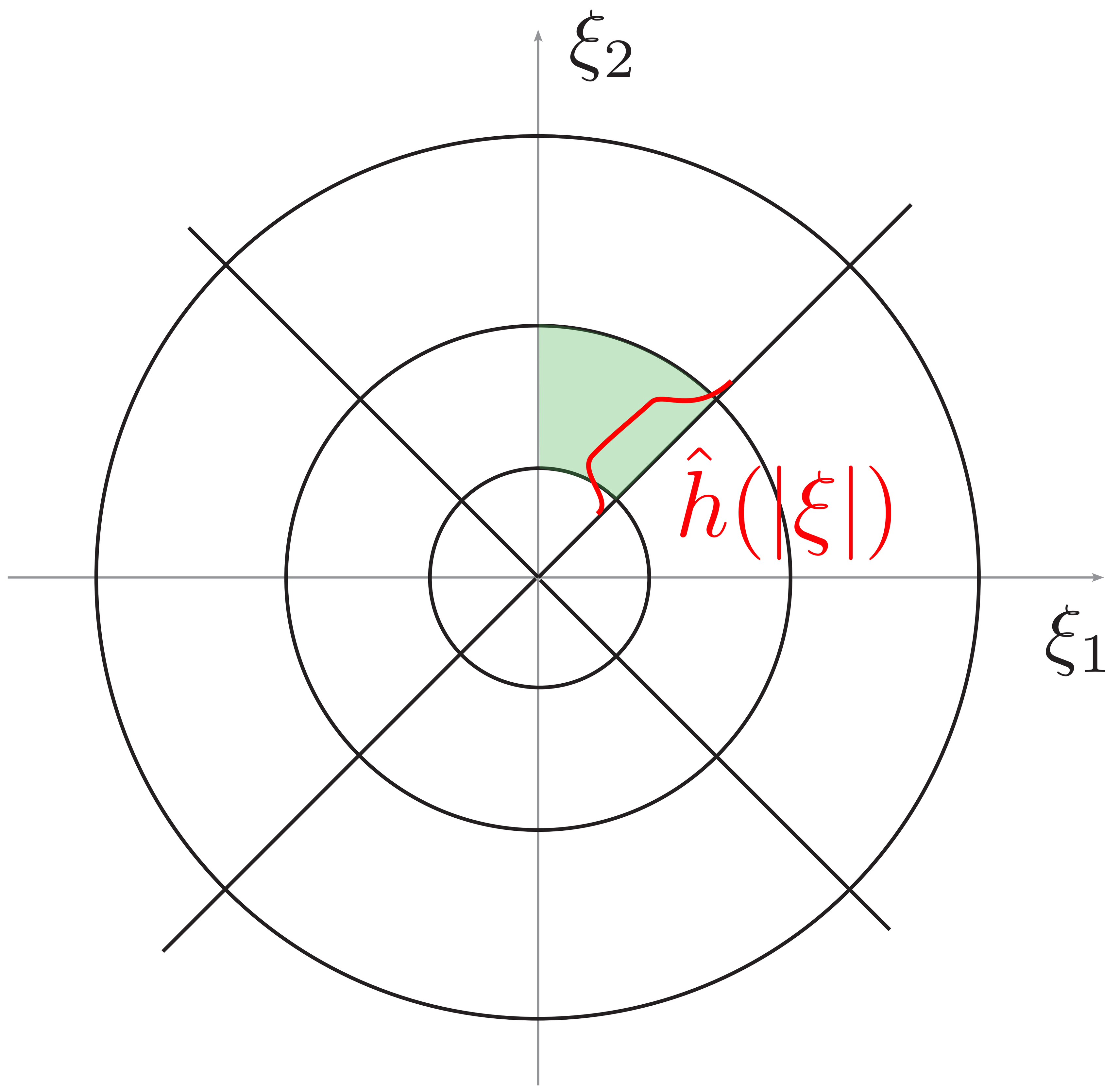
- Differential form basis functions:

$$\hat{\psi}_0^{0,\delta}(\xi) = i\hat{h}(|\xi|)$$

$$\hat{\psi}_j^{0,\delta}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_j(\theta_\xi)$$



Ψec: A local spectral exterior calculus

- Differential form basis functions:

$$\hat{\psi}_0^{0,\delta}(\xi) = i\hat{h}(|\xi|)$$

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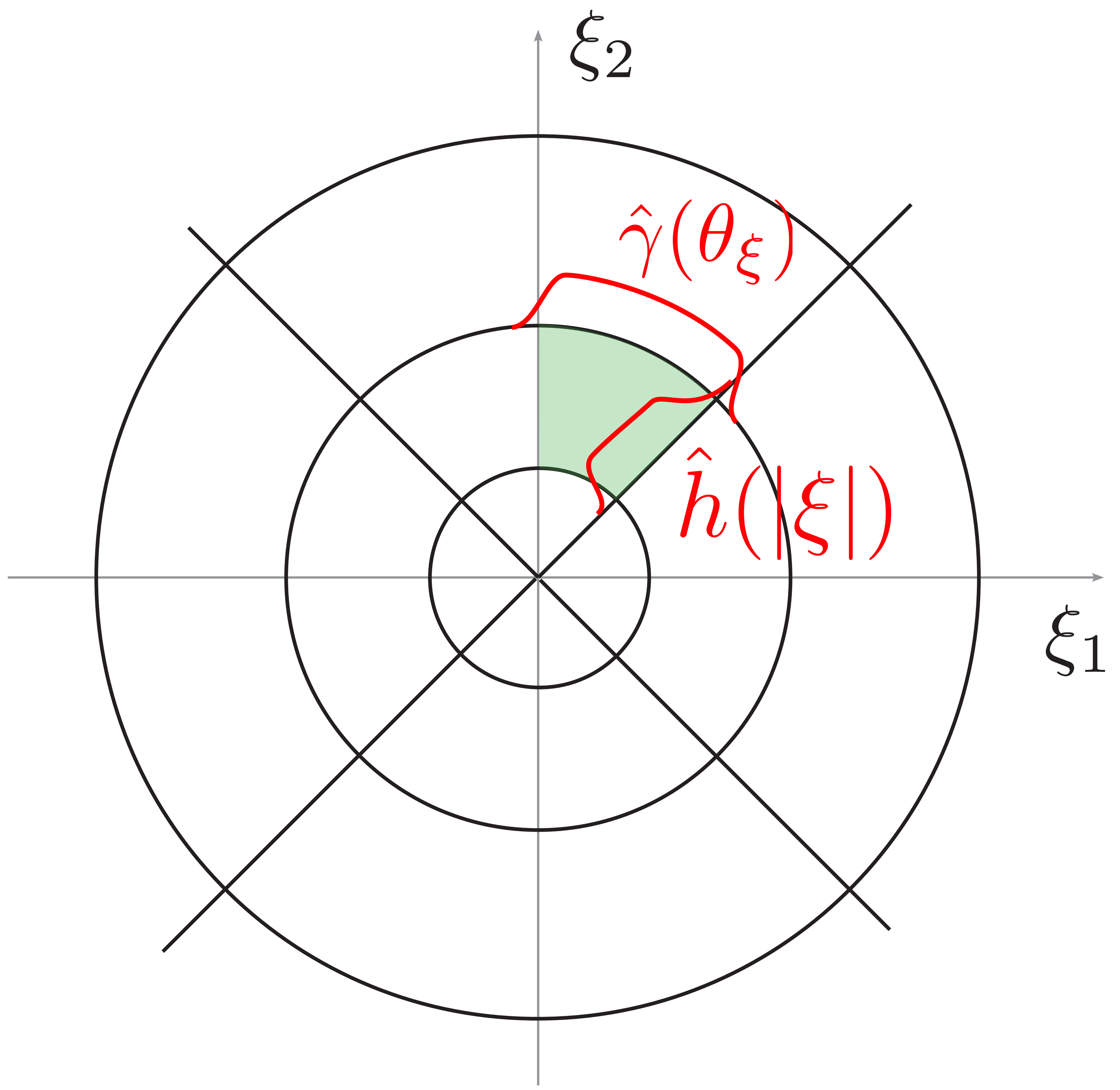
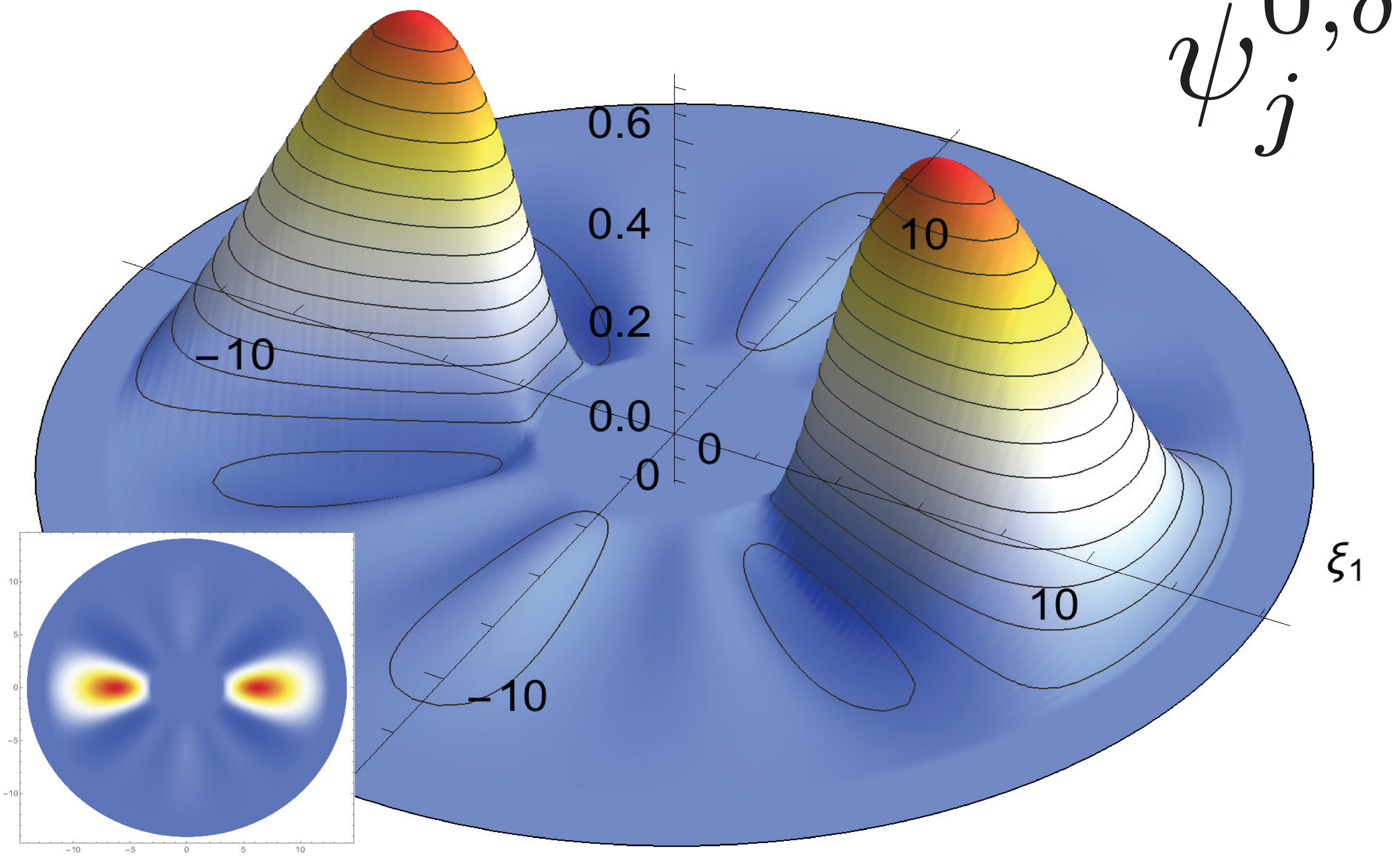


Ψec: A local spectral exterior calculus

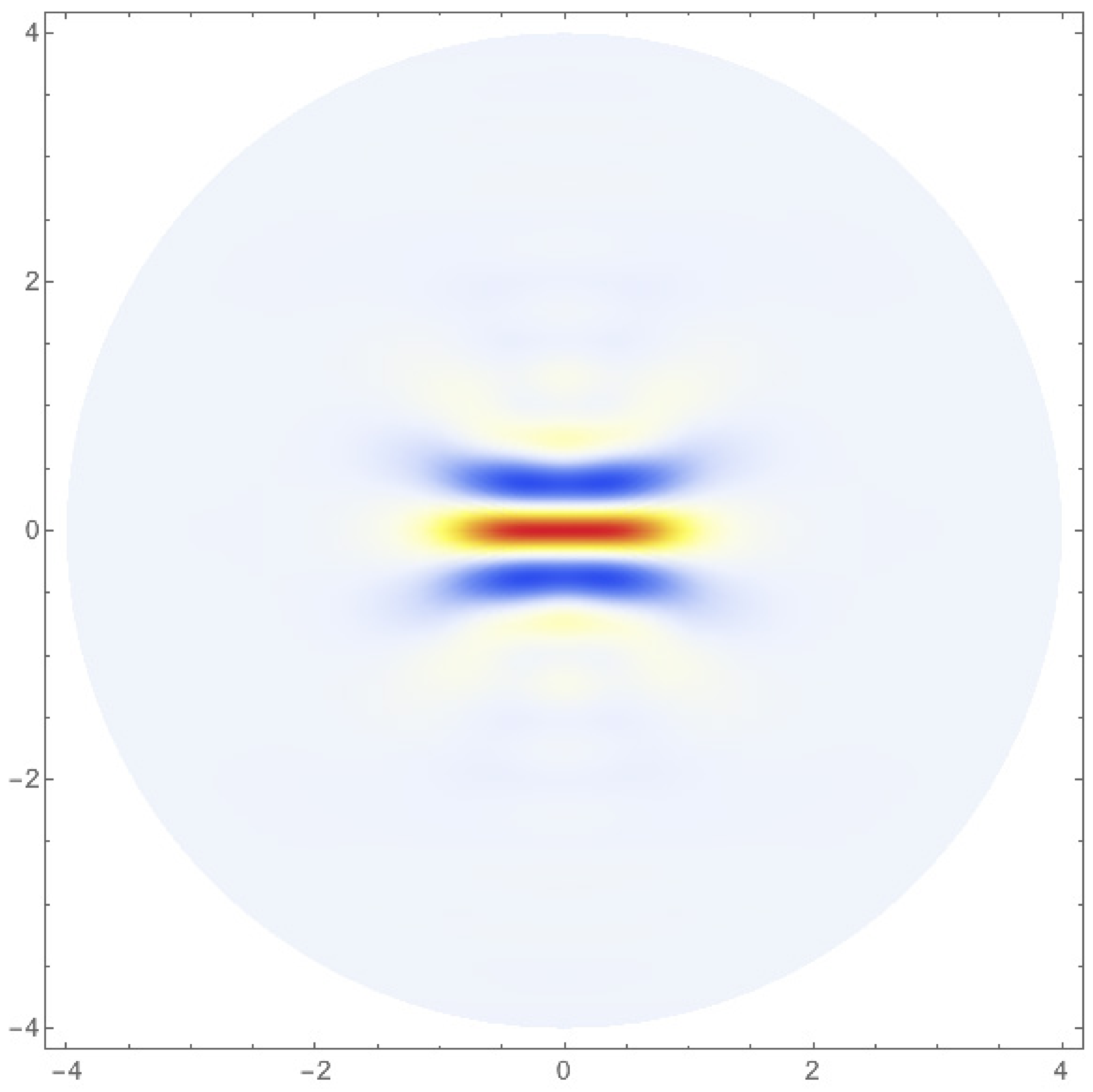
- Differential form basis functions:

$\hat{\psi}_j^{0,\delta}(\xi)$

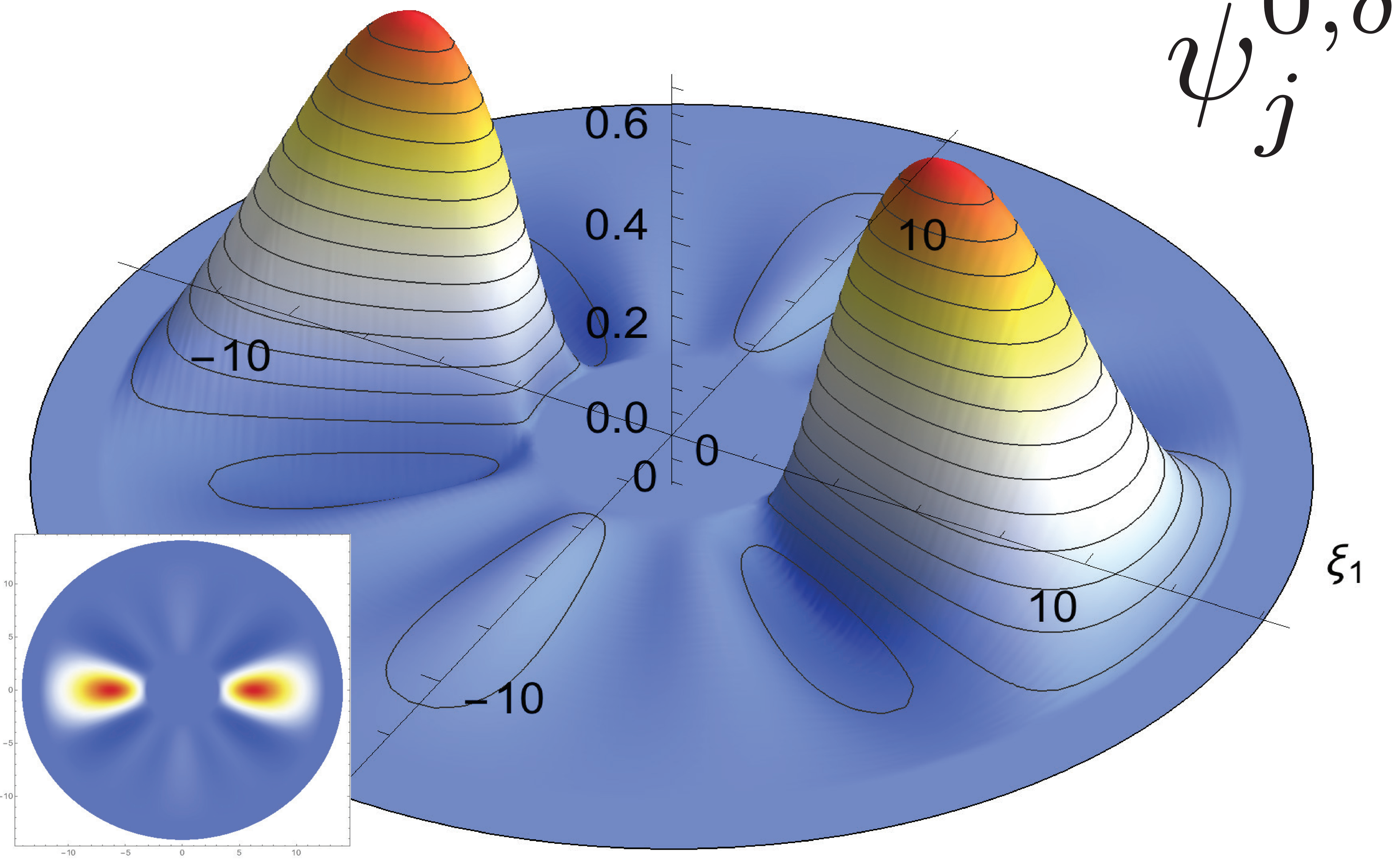


Ψec: A local spectral exterior calculus

- Differential form basis functions:



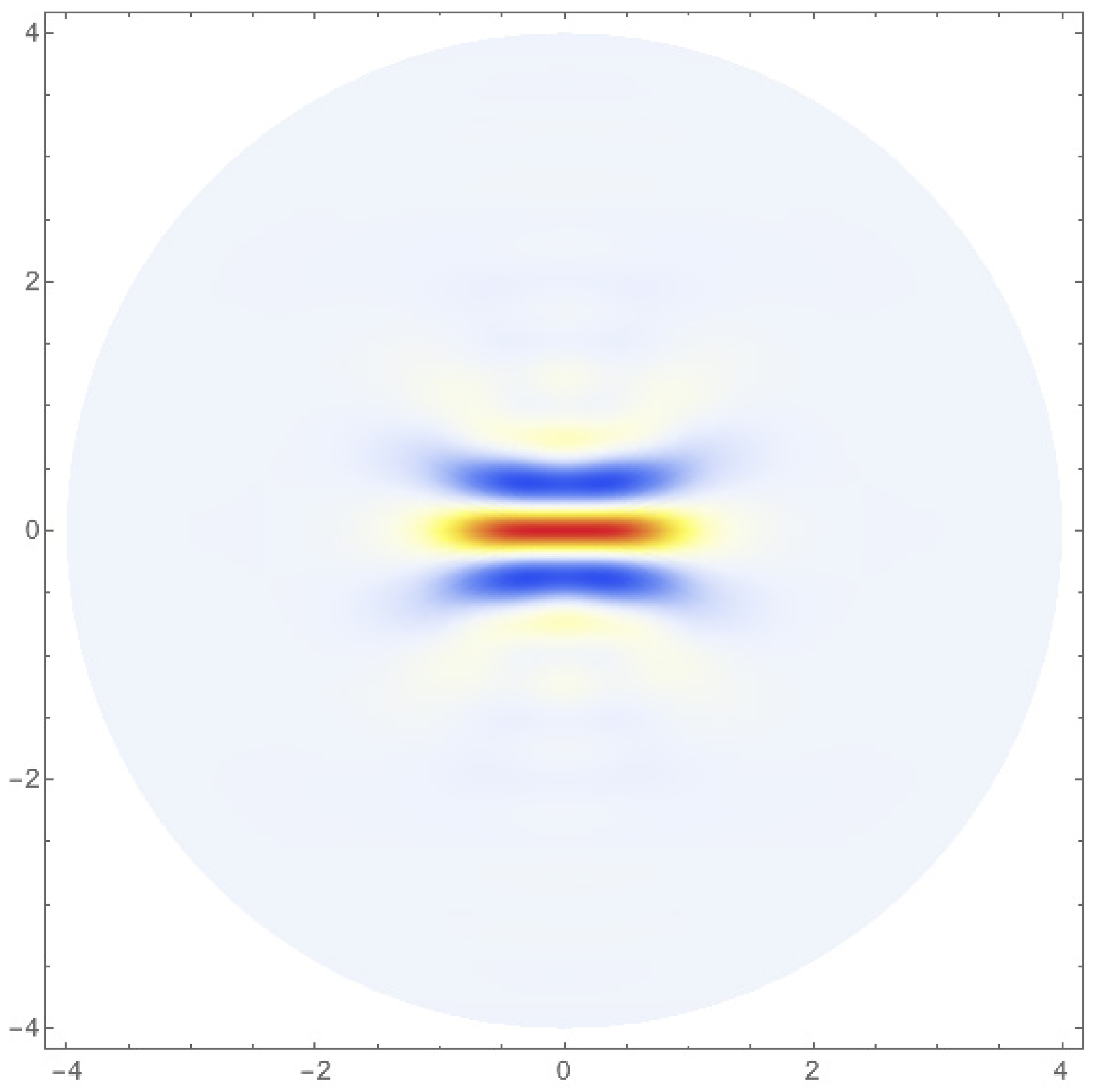
$$\psi_j^{0,\delta}(\xi)$$



$$\hat{\psi}_j^{0,\delta}(\xi)$$

Ψec: A local spectral exterior calculus

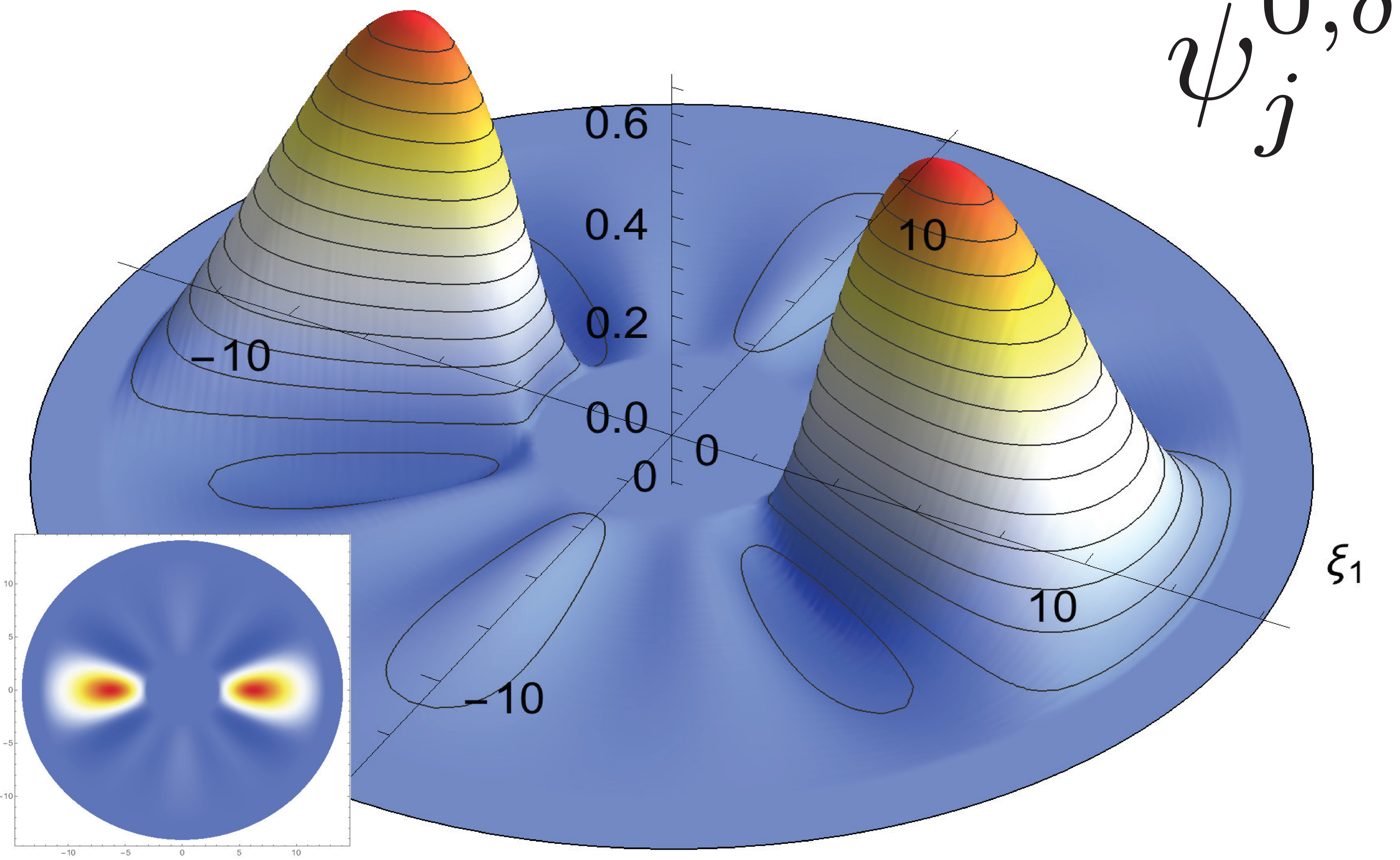
- Differential form basis functions:



$$\psi_j^{0,\delta}(\xi)$$

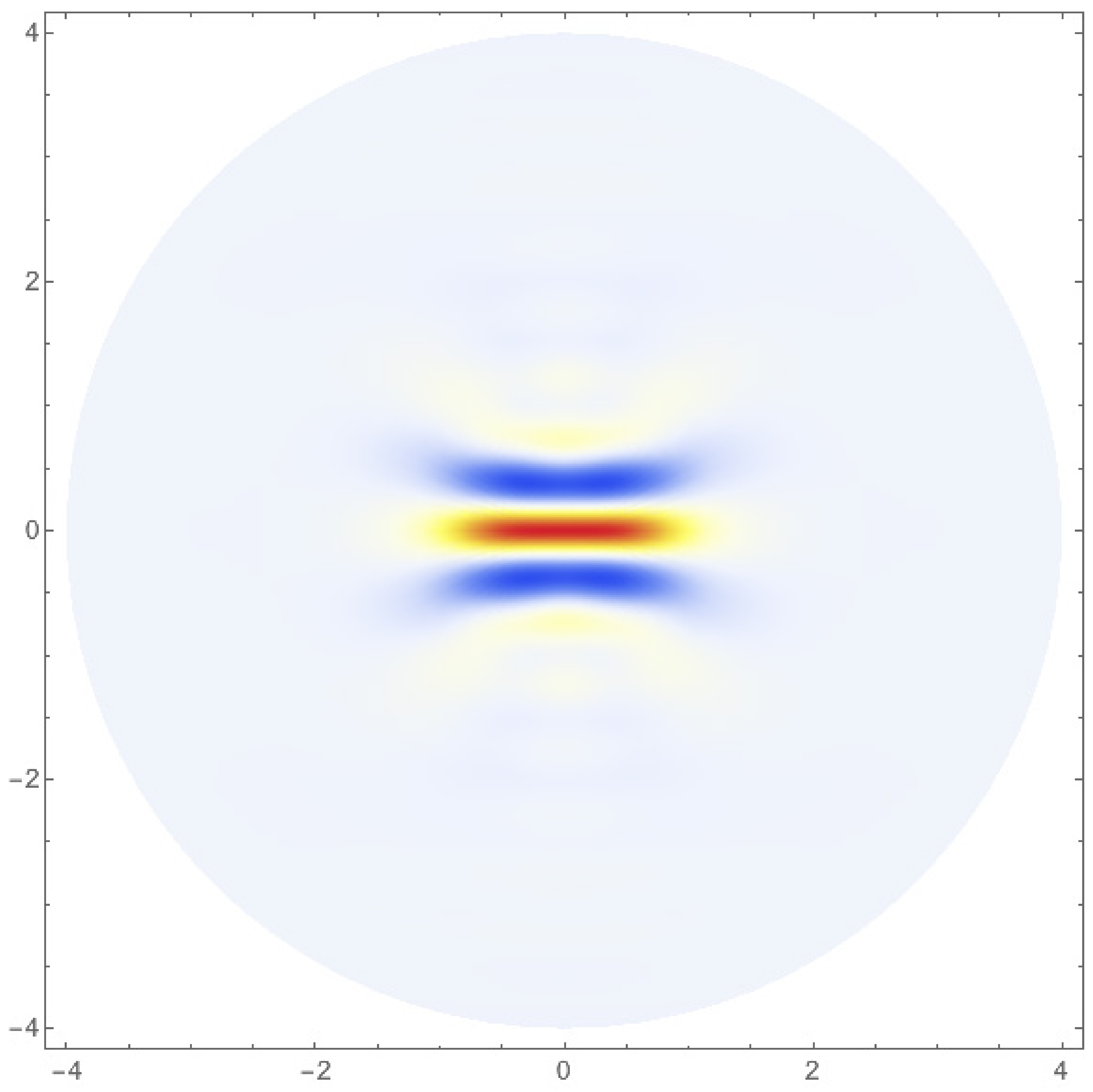
$$\xleftarrow{\mathfrak{F}^{-1}}$$

$$\hat{\psi}_j^{0,\delta}(\xi)$$



Ψec: A local spectral exterior calculus

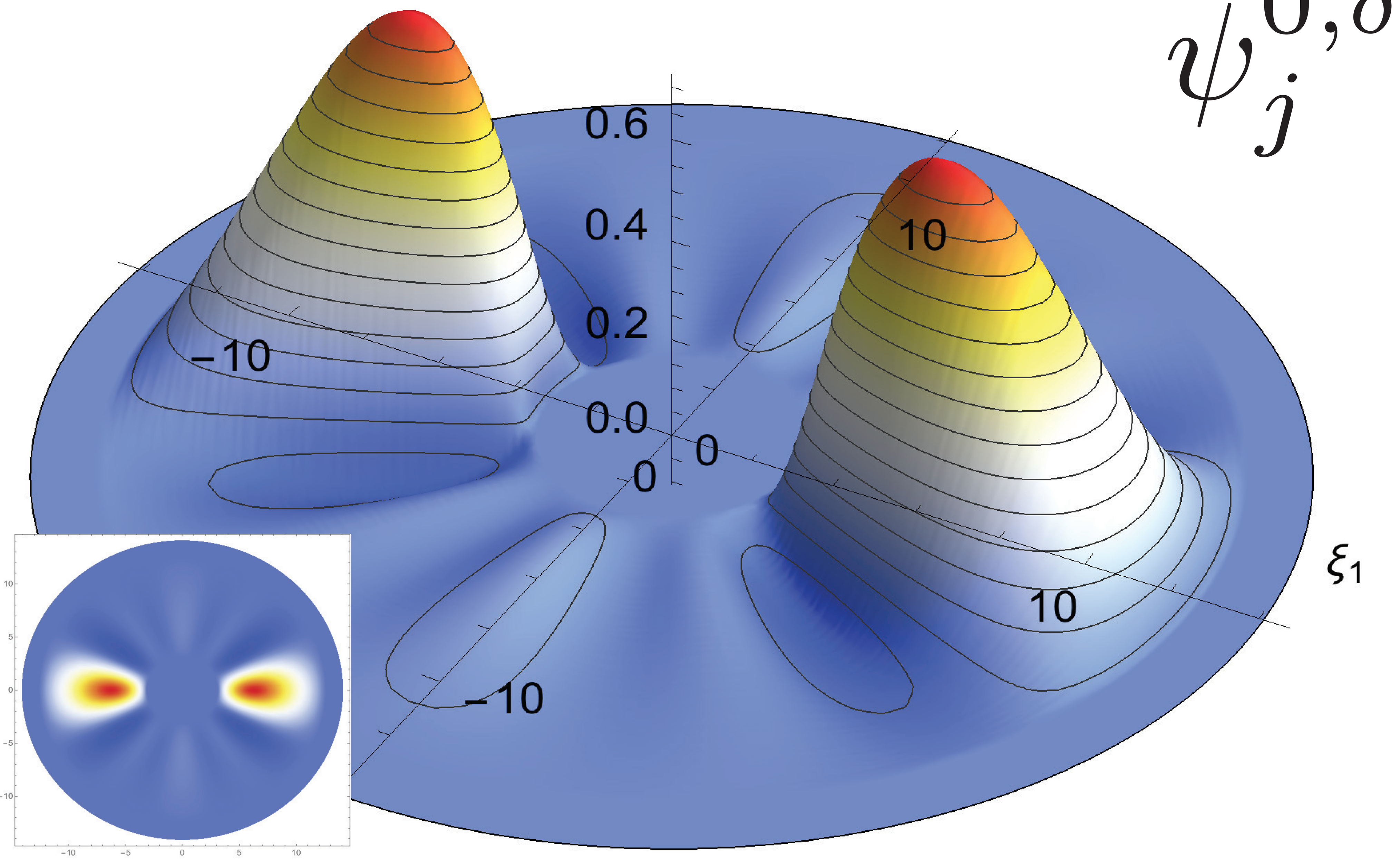
- Differential form basis functions:



$\psi_j^{0,\delta}(\xi)$

$\xleftarrow{\mathfrak{F}^{-1}}$

$\hat{\psi}_j^{0,\delta}(\xi)$



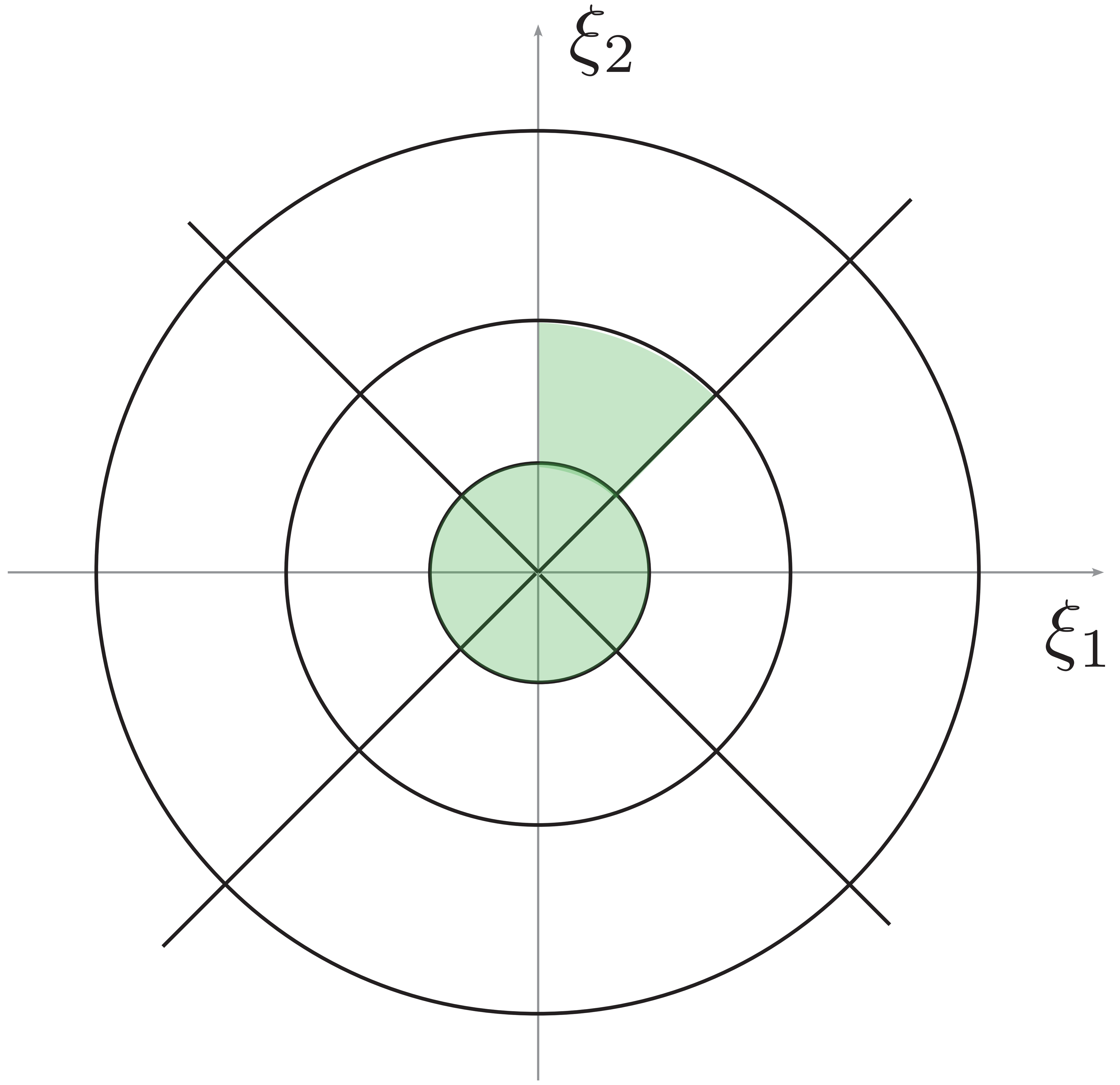
Can be computed in
closed form

Ψec: A local spectral exterior calculus

- Differential form basis functions:

$$\hat{\psi}_0^{0,\delta}(\xi) = i\hat{h}(|\xi|)$$

$$\hat{\psi}_{j,k,t}^{0,\delta}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_{j,t}(\theta_\xi)\,e^{-ik(x)}$$



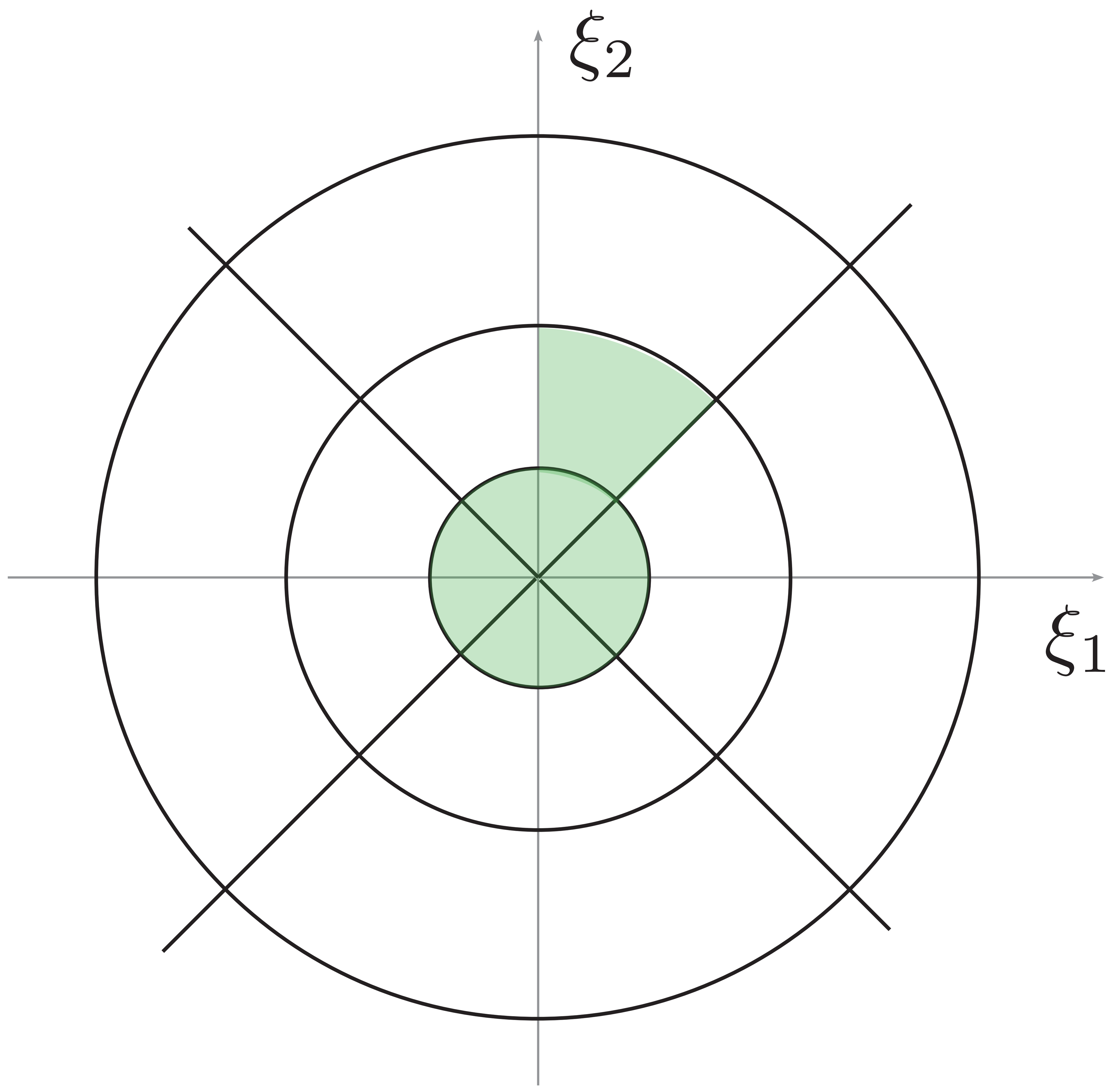
Ψec: A local spectral exterior calculus

- Differential form basis functions:

$$\hat{\psi}_0^{0,\delta}(\xi) = i\hat{h}(|\xi|)$$

$$\hat{\psi}_{j,\boxed{k},t}^{0,\delta}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_{j,t}(\theta_\xi)\,e^{-i\boxed{k}(x)}$$

translation



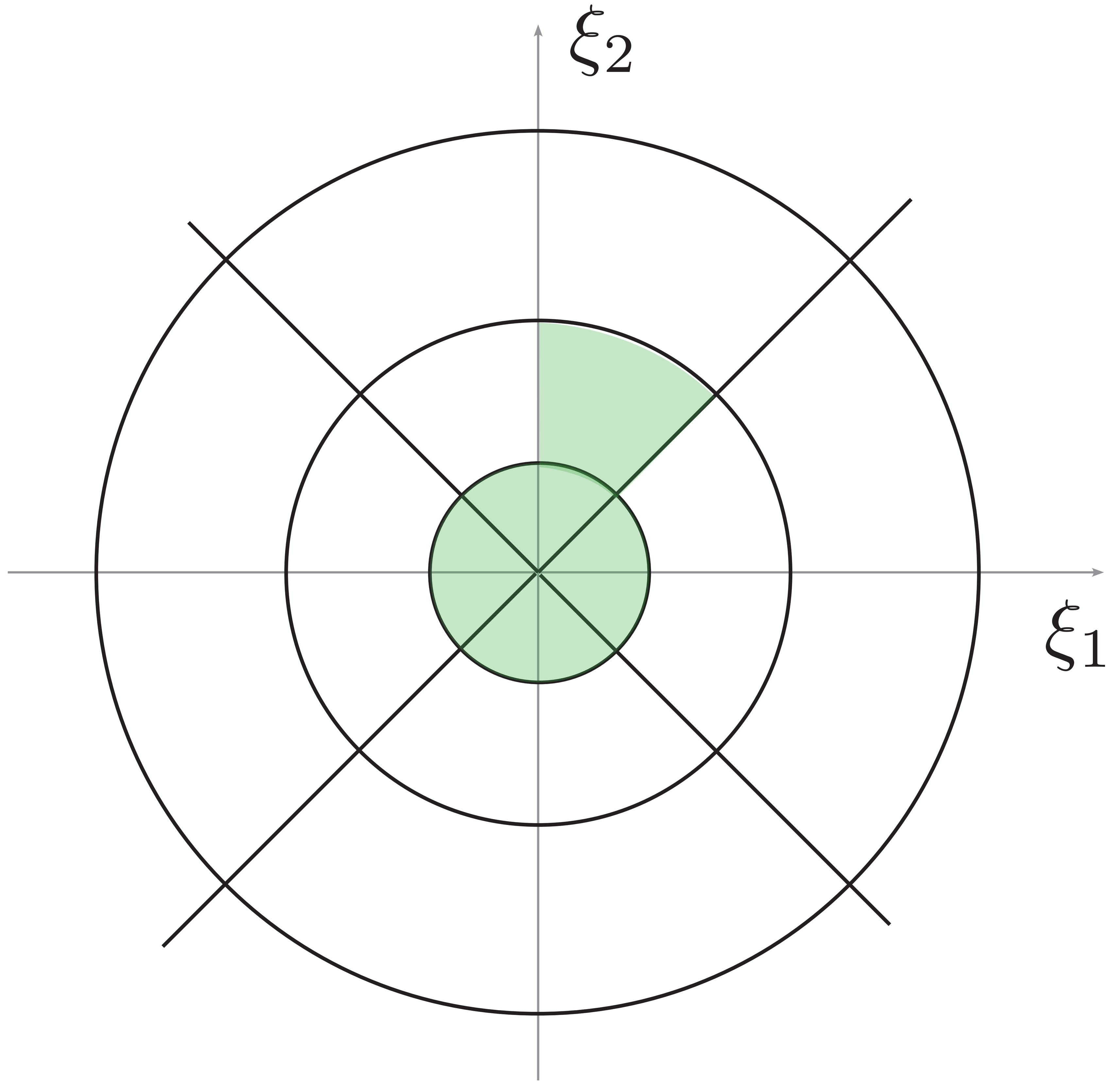
Ψec: A local spectral exterior calculus

- Differential form basis functions:

$$\hat{\psi}_0^{0,\delta}(\xi) = i\hat{h}(|\xi|)$$

$$\hat{\psi}_{j,k,\boxed{t}}^{0,\delta}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_{j,\boxed{t}}(\theta_\xi)\,e^{-ik(x)}$$

orientation



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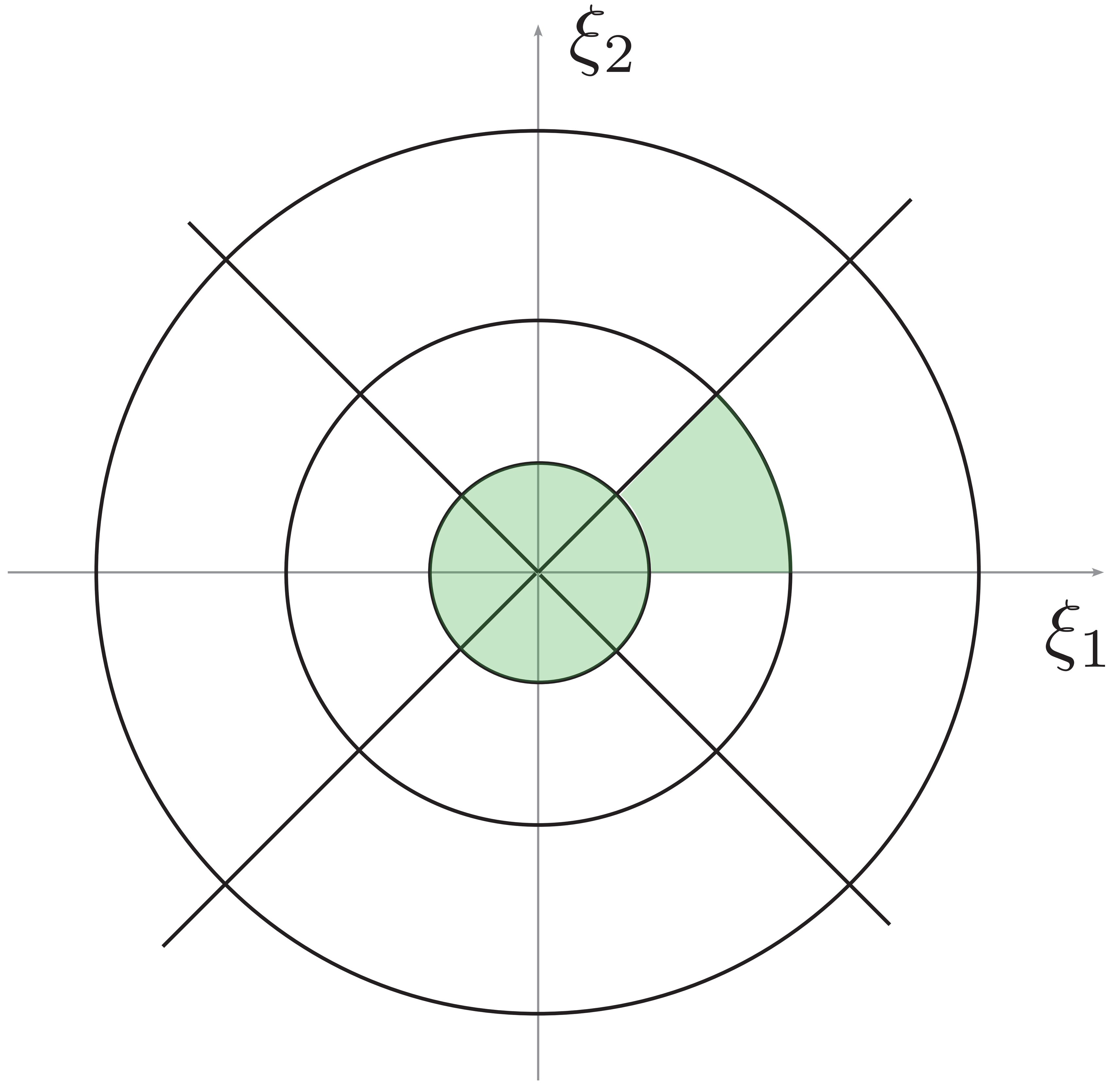
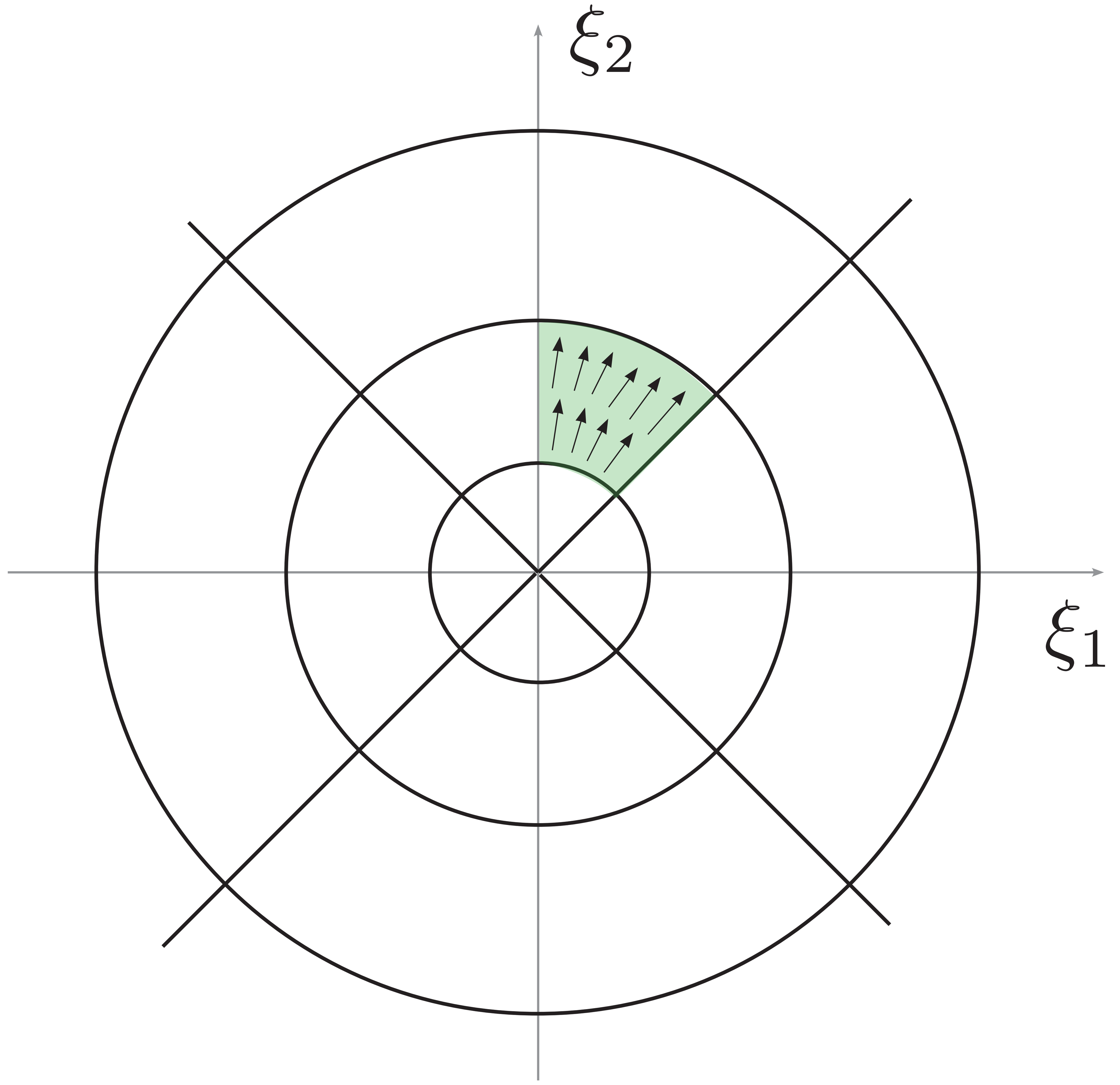
orientation



Ψec: A local spectral exterior calculus

- Differential form basis functions:

$$\hat{\psi}_j^{1,\mathrm{d}}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_j(\theta_\xi)\,\frac{\partial}{\partial r_\xi}$$

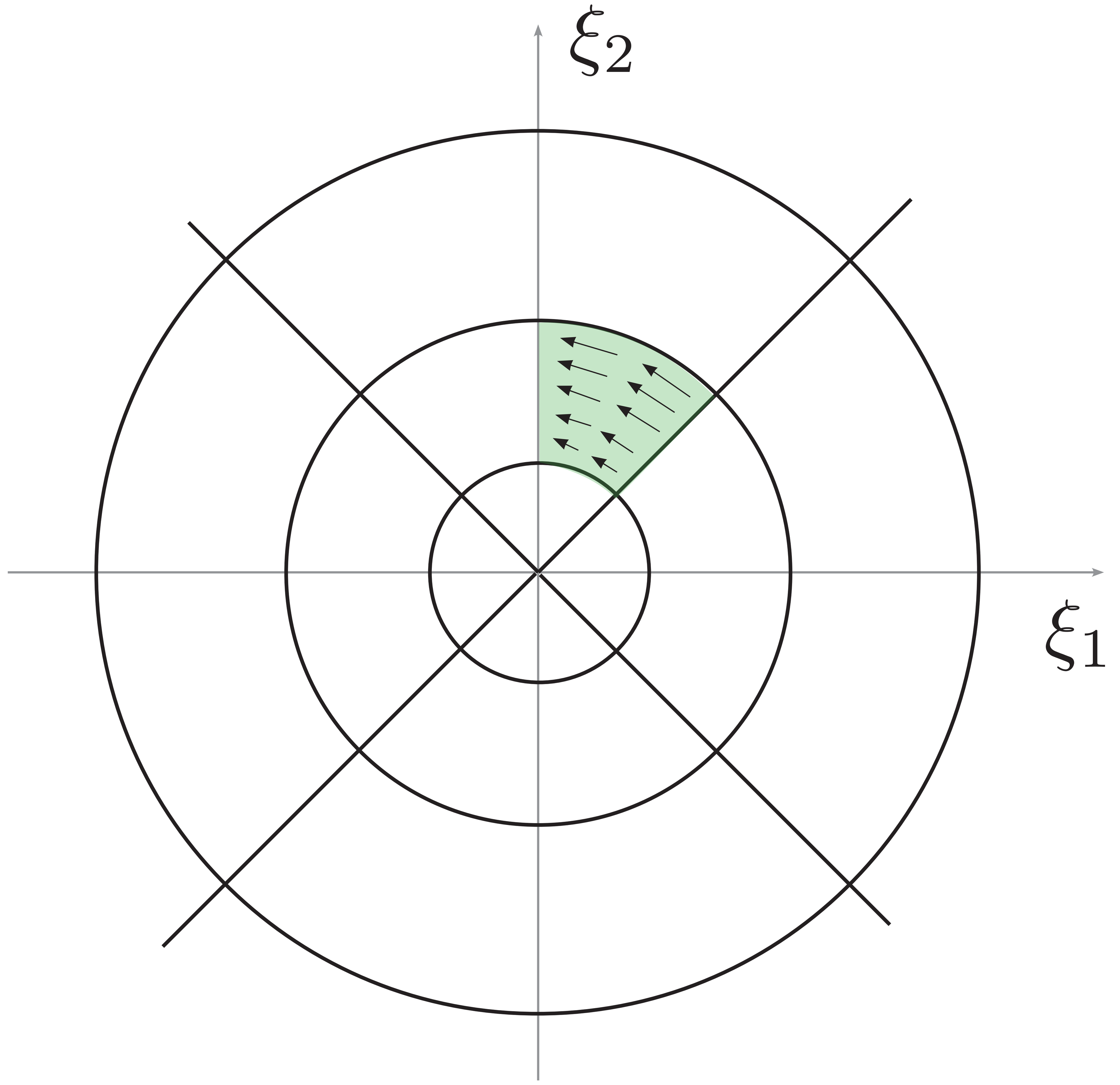


Ψec: A local spectral exterior calculus

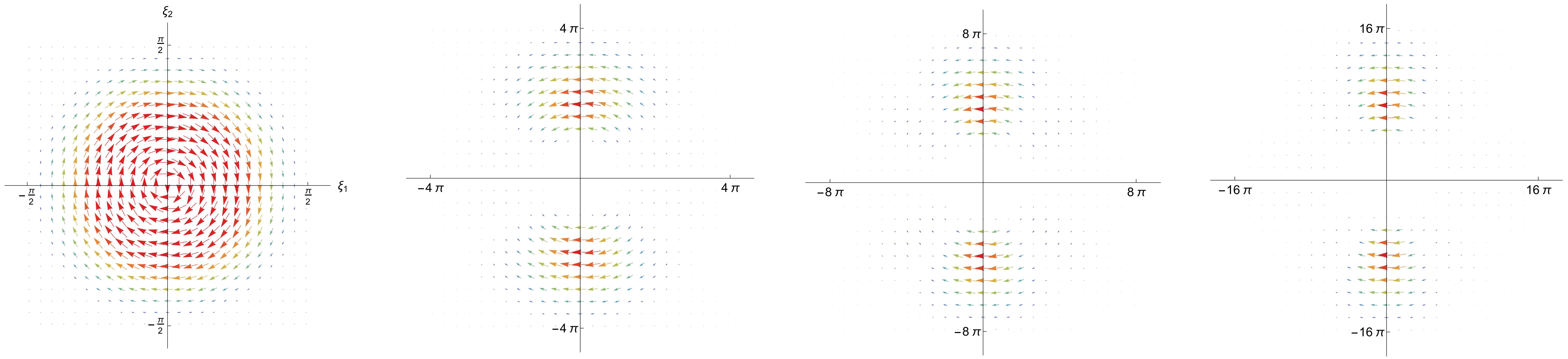
- Differential form basis functions:

$$\hat{\psi}_j^{1,\mathrm{d}}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_j(\theta_\xi)\,\frac{\partial}{\partial r_\xi}$$

$$\hat{\psi}_j^{1,\delta}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_j(\theta_\xi)\,\frac{\partial}{\partial \theta_\xi}$$

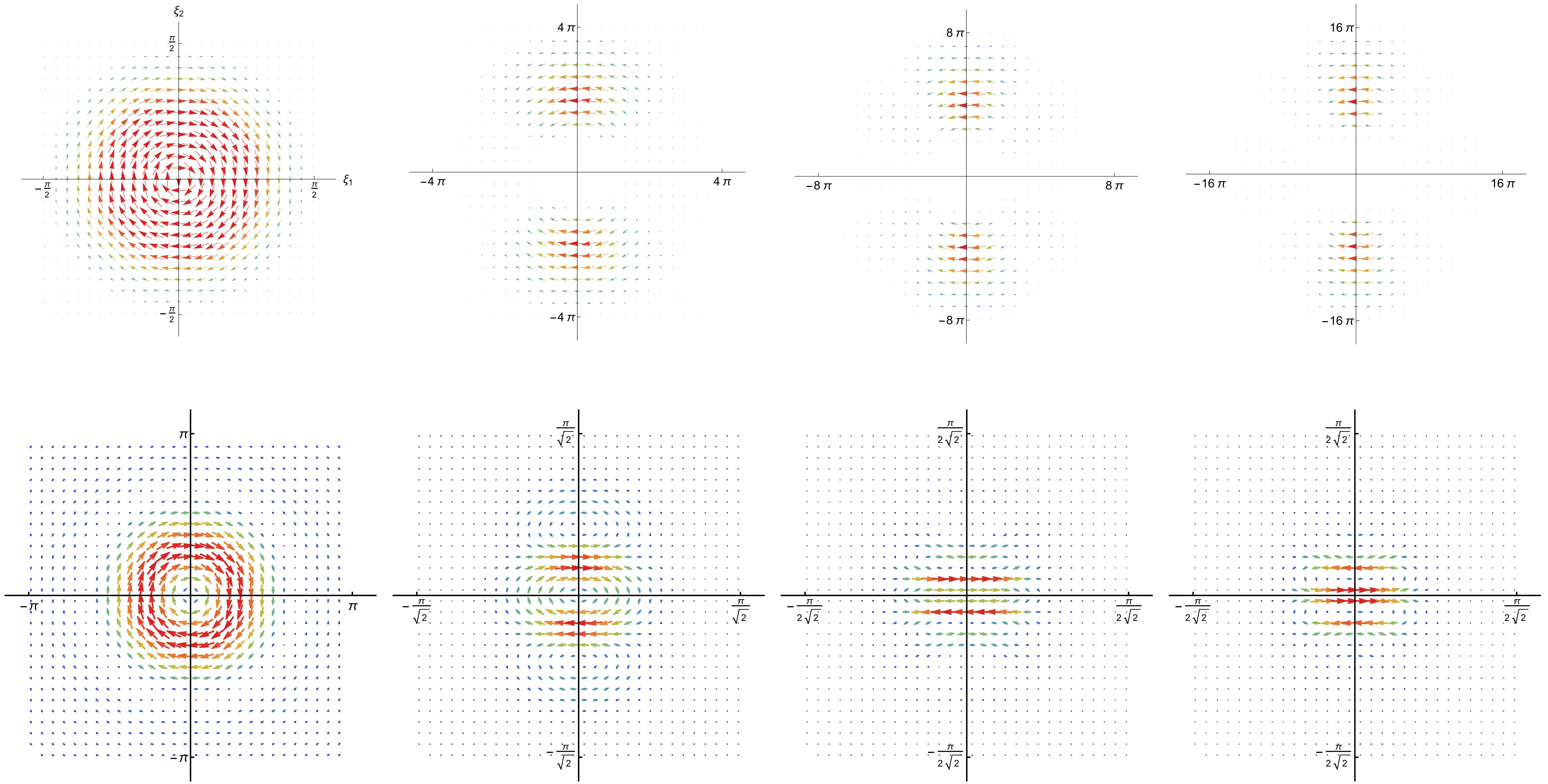


Ψec: A local spectral exterior calculus



$$\hat{\psi}_j^{1,\delta}$$

Ψec: A local spectral exterior calculus



$\hat{\psi}_j^{1,\delta}$

$\psi_j^{1,\delta}$

Ψec: A local spectral exterior calculus

reference

Ψec: A local spectral exterior calculus

$j = -1$

Ψec: A local spectral exterior calculus

$j = 0$

Ψec: A local spectral exterior calculus

$j = 1$

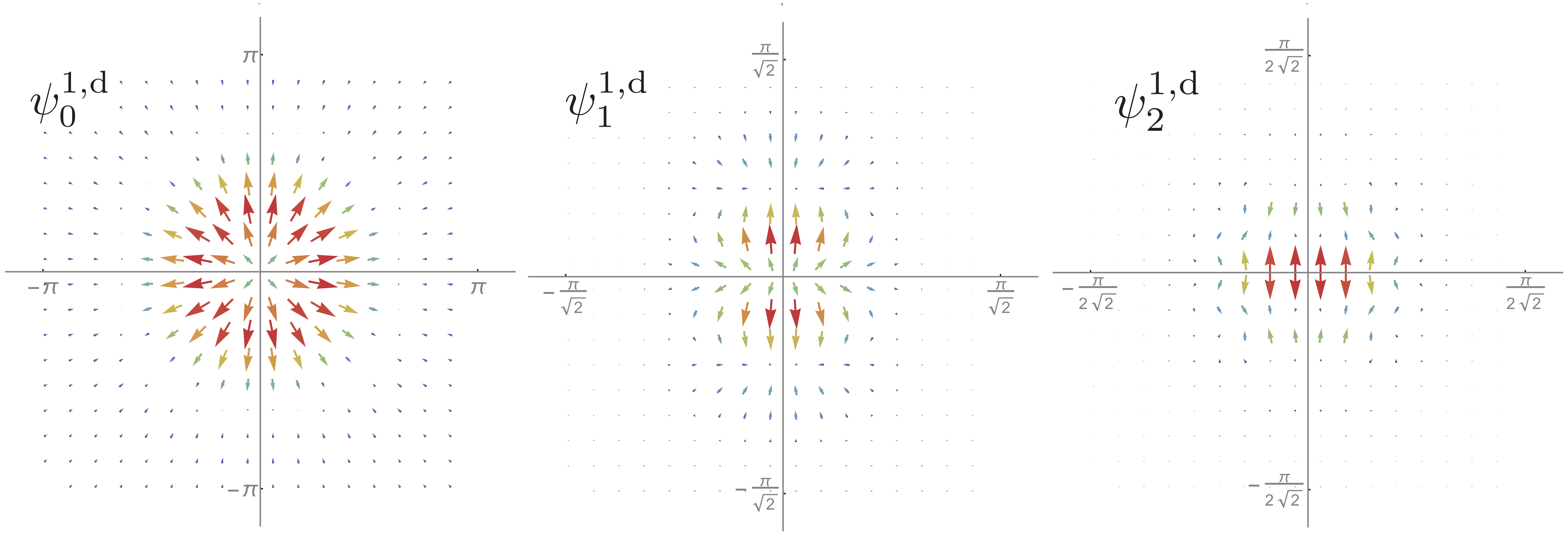
Ψec: A local spectral exterior calculus

$j = 2$

Ψec: A local spectral exterior calculus

reference

Ψec: A local spectral exterior calculus



Ψec: A local spectral exterior calculus

- Differential form basis functions:

$$\hat{\psi}_j^{1,\mathrm{d}}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_j(\theta_\xi)\,\frac{\partial}{\partial r_\xi}$$

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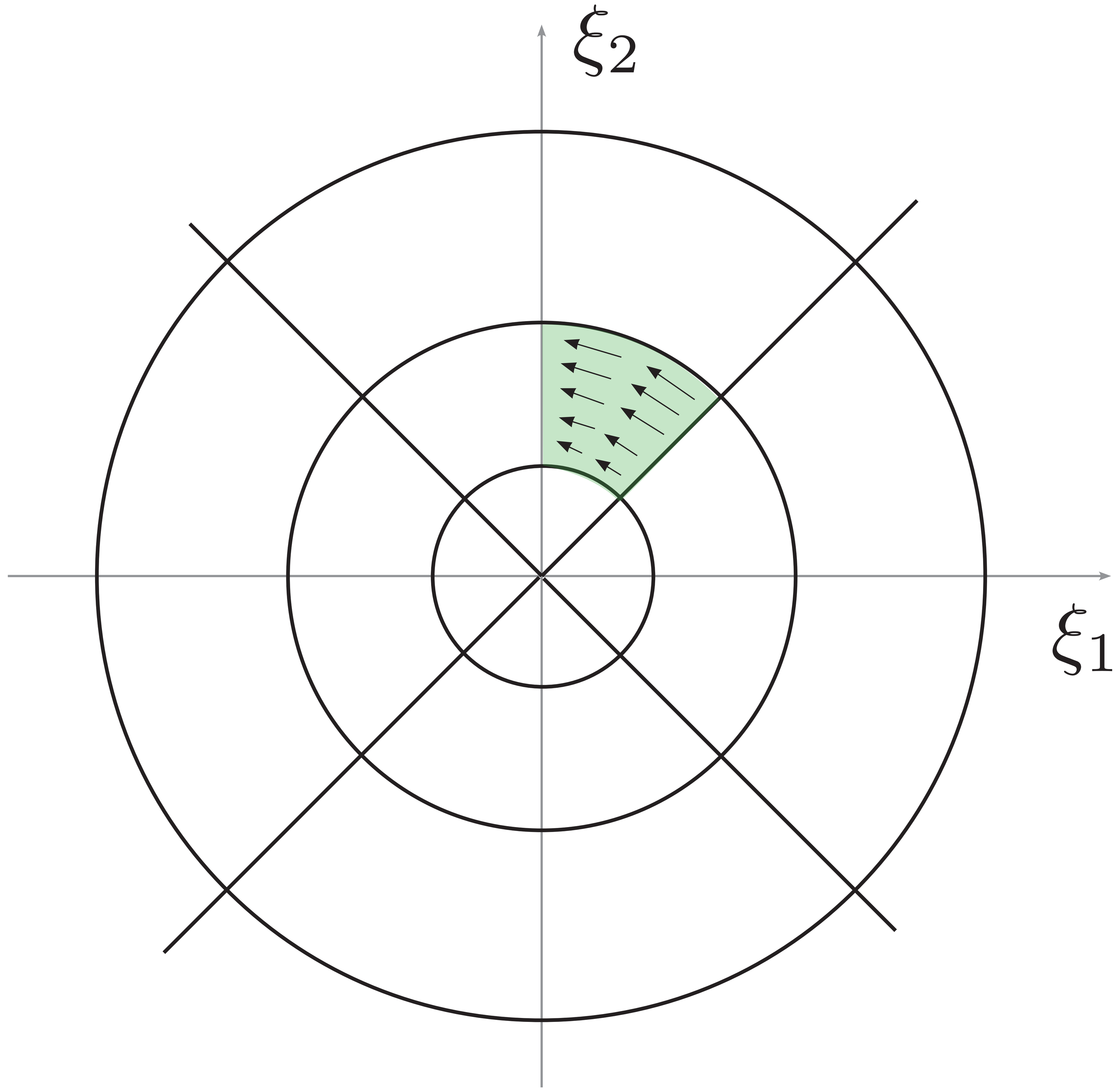

Ψec: A local spectral exterior calculus

- Differential form $\boxed{\text{basis}}$ functions:

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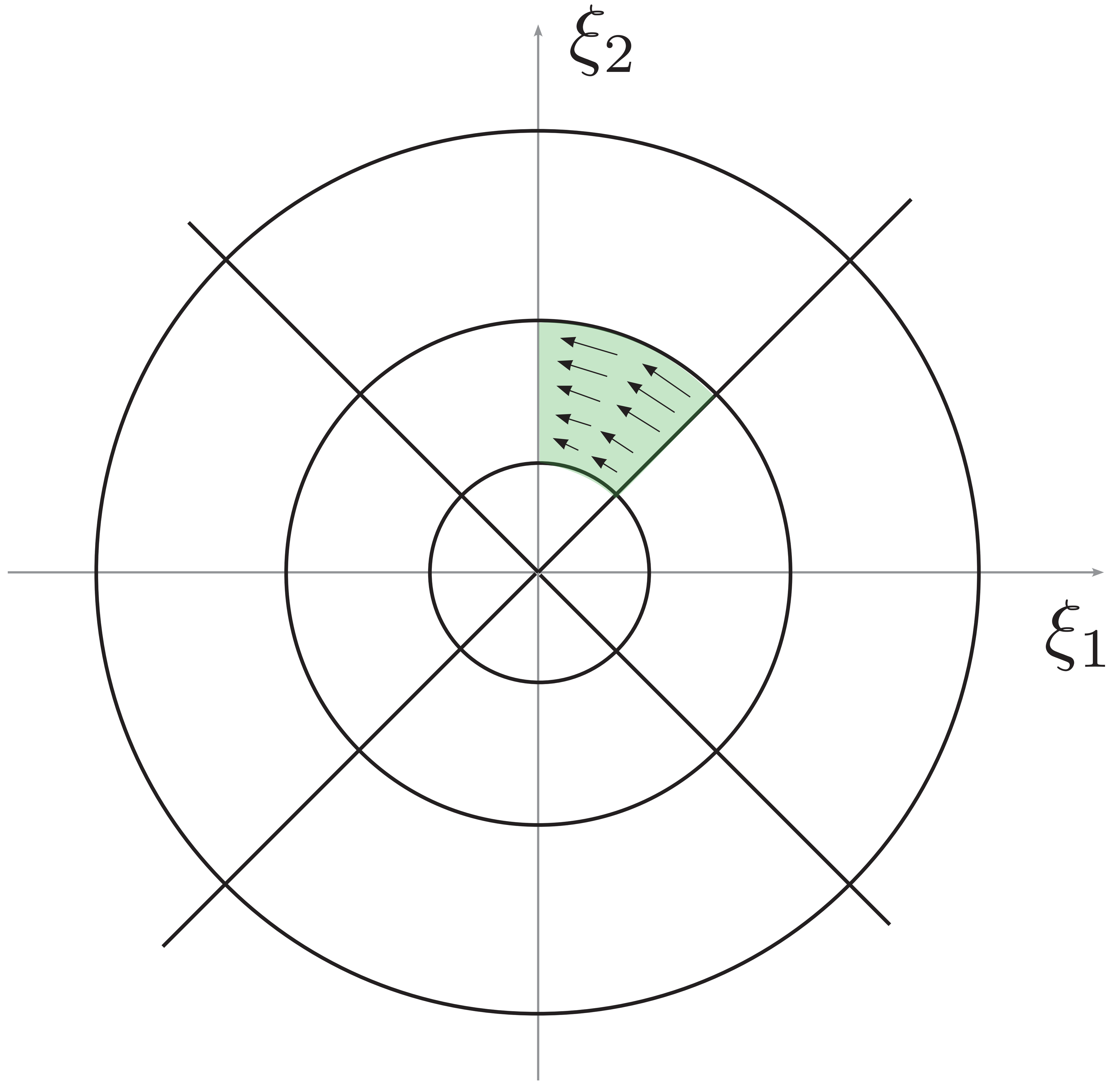
$$\hat{\psi}_j^{1,\delta}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_j(\theta_\xi)\,\frac{\partial}{\partial \theta_\xi}$$



Ψec: A local spectral exterior calculus

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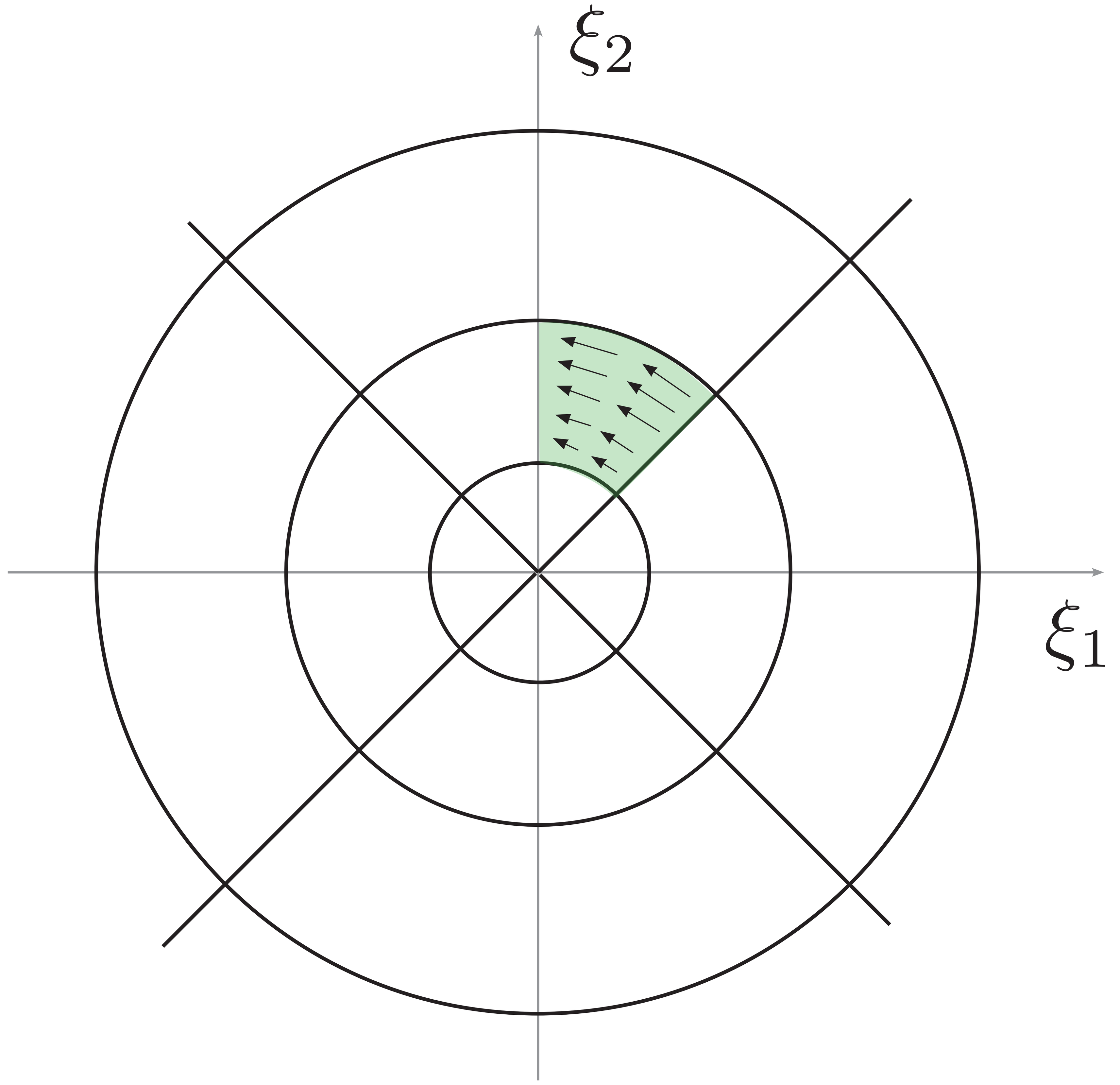


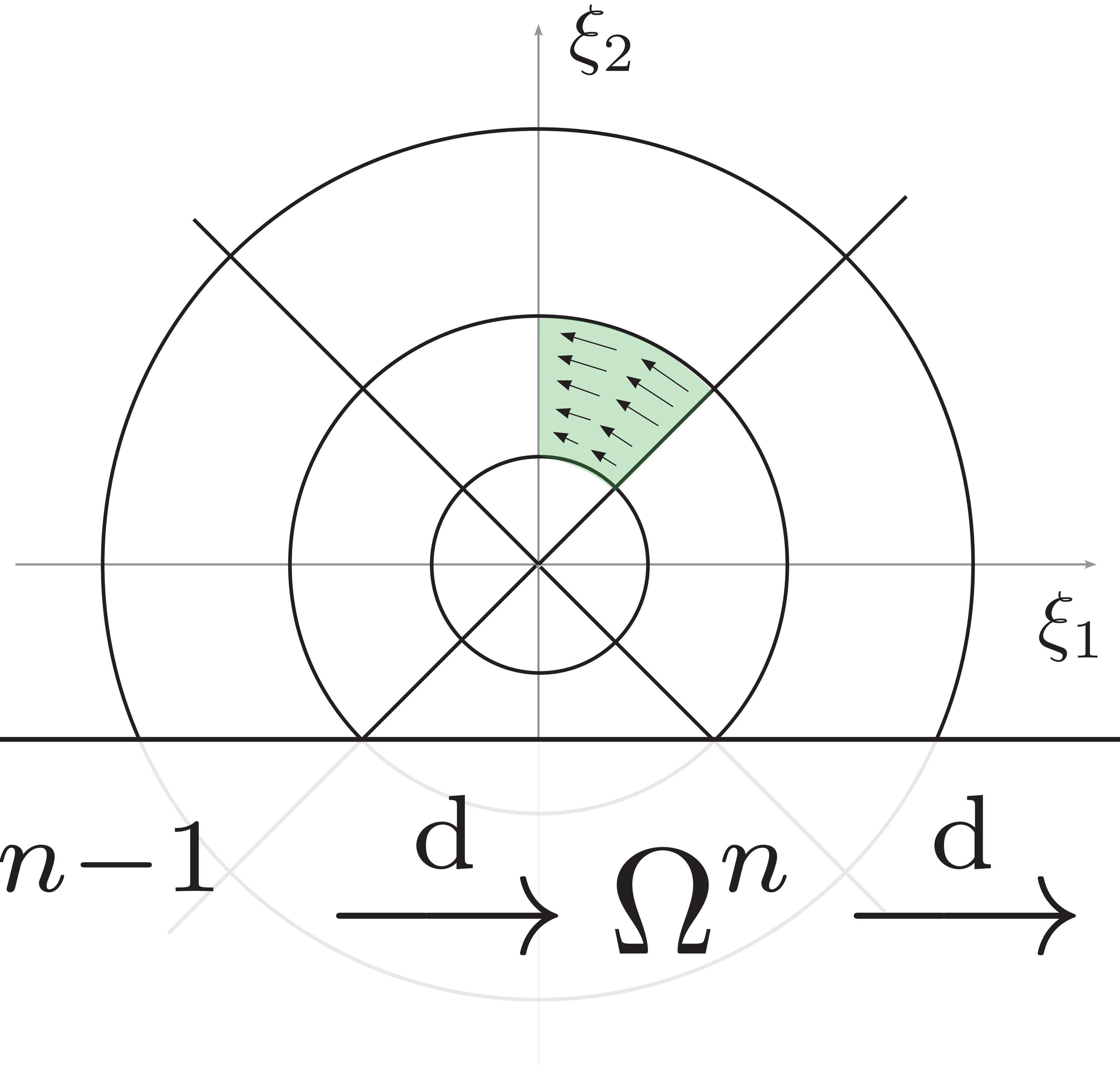
Basis for which space?

Ψec: A local spectral exterior calculus

- Differential form $\boxed{\text{basis}}$ functions:

$$\hat{\psi}_j^{1,\mathrm{d}}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_j(\theta_\xi)\,\frac{\partial}{\partial r_\xi}$$

$$\hat{\psi}_j^{1,\delta}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_j(\theta_\xi)\,\frac{\partial}{\partial \theta_\xi}$$



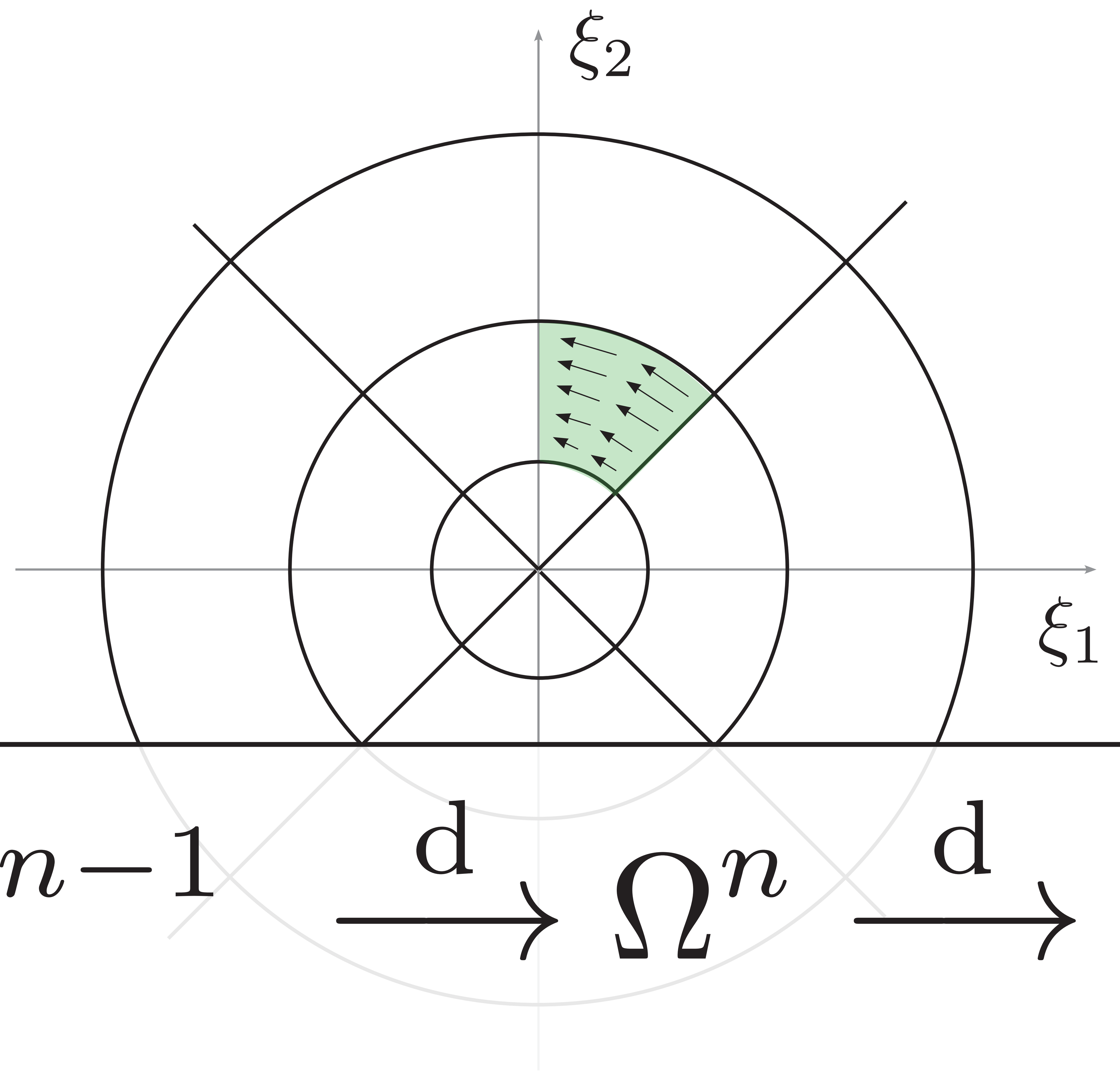
$$0 \xrightarrow{\mathrm{d}} \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \rightarrow \cdots \rightarrow \Omega^{n-1} \xrightarrow{\mathrm{d}} \Omega^n \xrightarrow{\mathrm{d}} 0$$

Ψec: A local spectral exterior calculus

- Differential form basis functions:

$$\hat{\psi}_j^{1,\mathrm{d}}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_j(\theta_\xi)\,\frac{\partial}{\partial r_\xi}$$

$$\hat{\psi}_j^{1,\delta}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_j(\theta_\xi)\,\frac{\partial}{\partial \theta_\xi}$$



$$0 \xrightarrow{\mathrm{d}} \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \rightarrow \cdots \rightarrow \Omega^{n-1} \xrightarrow{\mathrm{d}} \Omega^n \xrightarrow{\mathrm{d}} 0$$

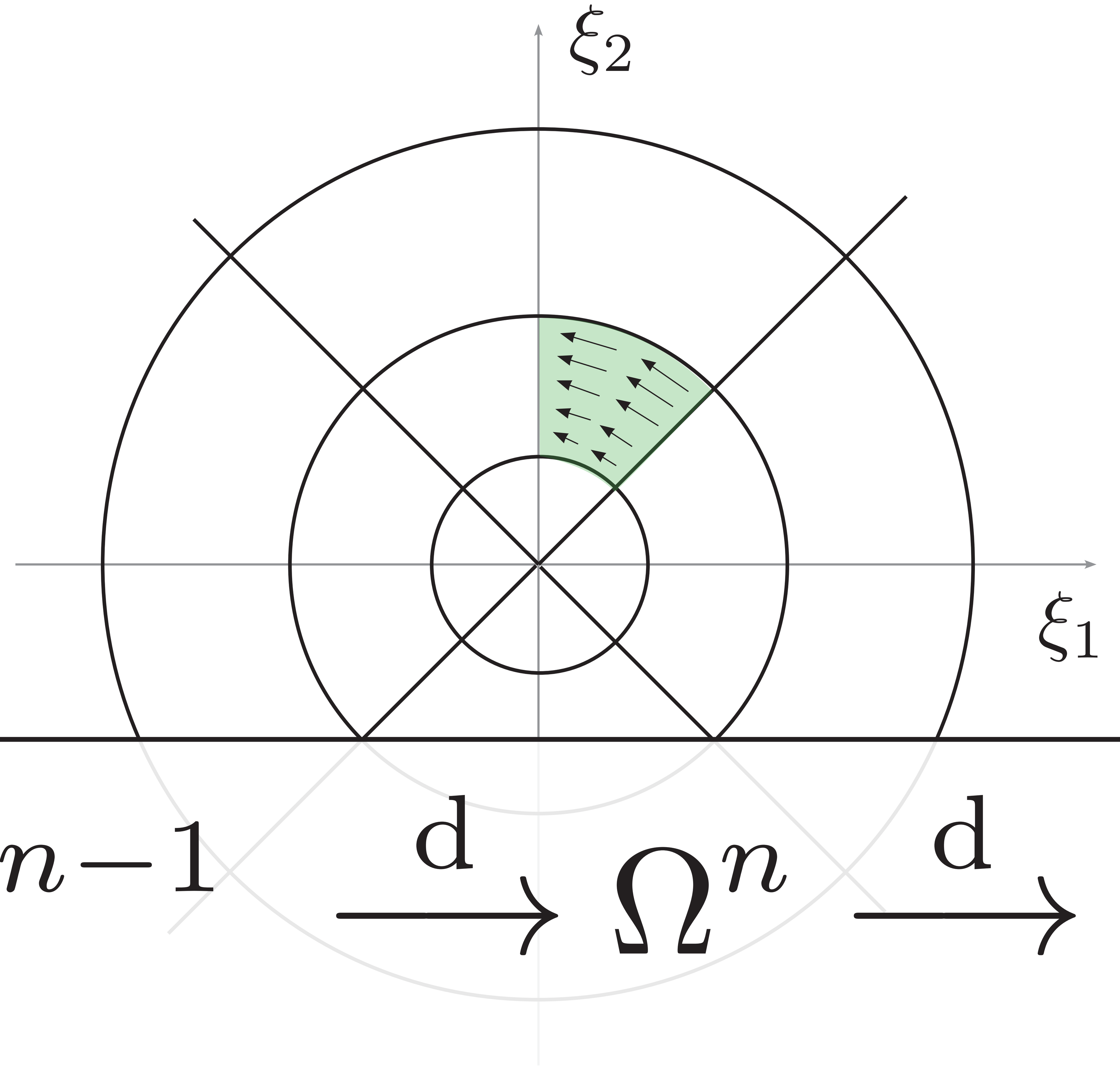
L_2

Ψec: A local spectral exterior calculus

- Differential form $\boxed{\text{basis}}$ functions:

$$\hat{\psi}_j^{1,\mathrm{d}}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_j(\theta_\xi)\,\frac{\partial}{\partial r_\xi}$$

$$\hat{\psi}_j^{1,\delta}(\xi) = i\hat{h}(|2^{-j}\xi|)\,\gamma_j(\theta_\xi)\,\frac{\partial}{\partial \theta_\xi}$$



$$\boxed{0 \xrightarrow{\mathrm{d}} \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \rightarrow \cdots \rightarrow \Omega^{n-1} \xrightarrow{\mathrm{d}} \Omega^n \xrightarrow{\mathrm{d}} 0}$$

L_2 $\quad$ H^{-1}

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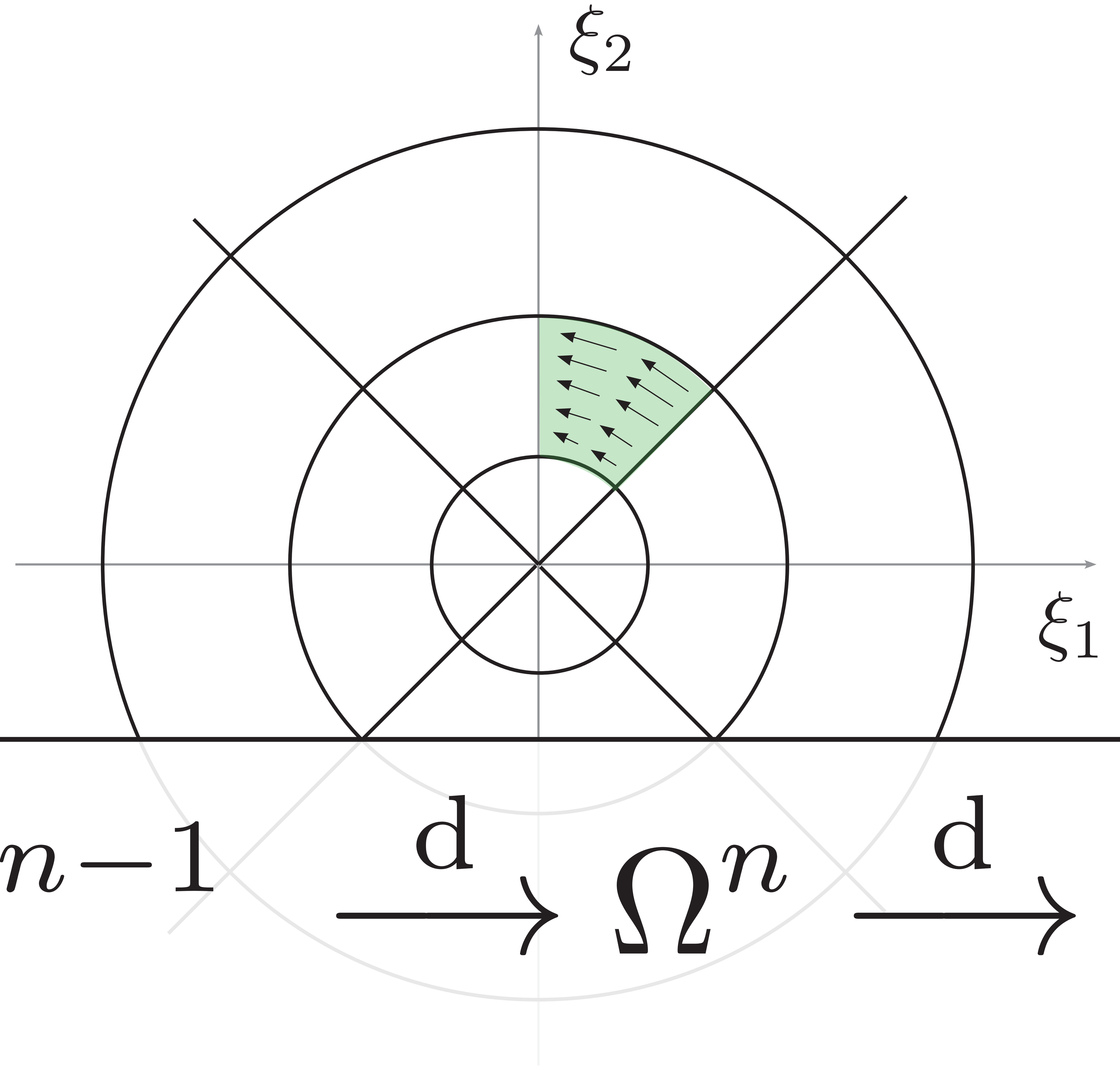

$$0 \xrightarrow{\mathrm{d}} \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \rightarrow \cdots \rightarrow \Omega^{n-1} \xrightarrow{\mathrm{d}} \Omega^n \xrightarrow{\mathrm{d}} 0$$

$$L_2 \qquad H^{-1} \qquad H^{-2}$$

Ψec: A local spectral exterior calculus

$$
\begin{tikzcd}
0 \arrow[r, "\mathrm{d}"] & \Omega^0 \arrow[r, shift left, harpoon, "\mathrm{d}"] \arrow[rrrr, bend right=40, dashed, leftrightarrow, "\star"'] & \Omega^1 \arrow[l, shift left, harpoon, "\delta"] \arrow[r, shift left, harpoon] \arrow[rr, bend right=30, dashed, leftrightarrow, "\star"'] & \cdots \arrow[l, shift left, harpoon] \arrow[r, shift left, harpoon] & \Omega^{n-1} \arrow[l, shift left, harpoon] \arrow[r, shift left, harpoon, "\mathrm{d}"] & \Omega^n \arrow[l, shift left, harpoon, "\delta"] \arrow[r, "\mathrm{d}"] & 0
\end{tikzcd}
$$

What is a compatible chain of function spaces?

Applications

Local Fourier slice photography

Local Fourier slice photography

C. Lessig. Local fourier slice photography. ACM Trans. Graph., 39(3), Apr. 2020.

Local Fourier slice photography

C. Lessig. Local fourier slice photography. ACM Trans. Graph., 39(3), Apr. 2020.

Local Fourier slice photography

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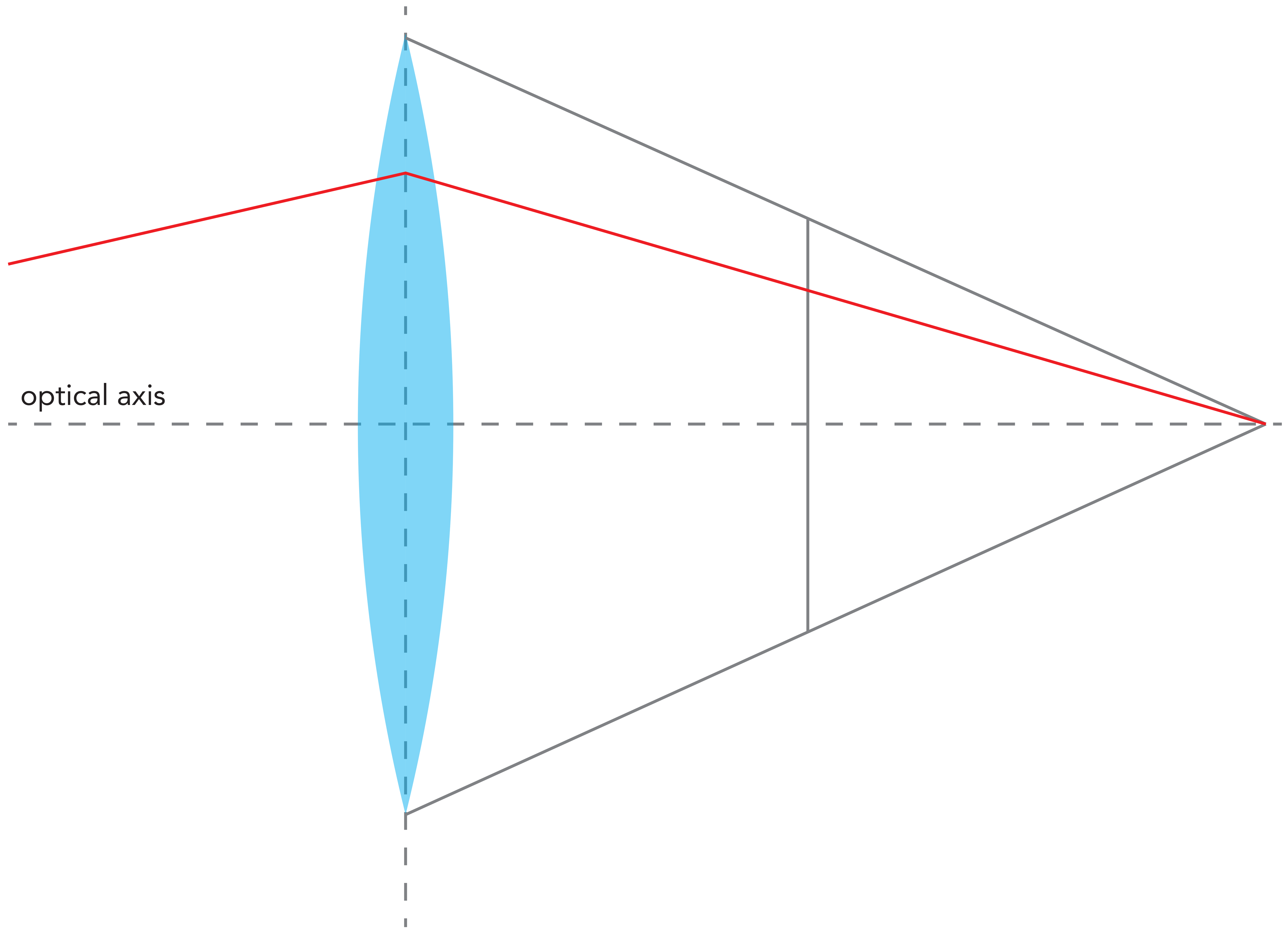
Local Fourier slice photography

C. Lessig. Local fourier slice photography. ACM Trans. Graph., 39(3), Apr. 2020.

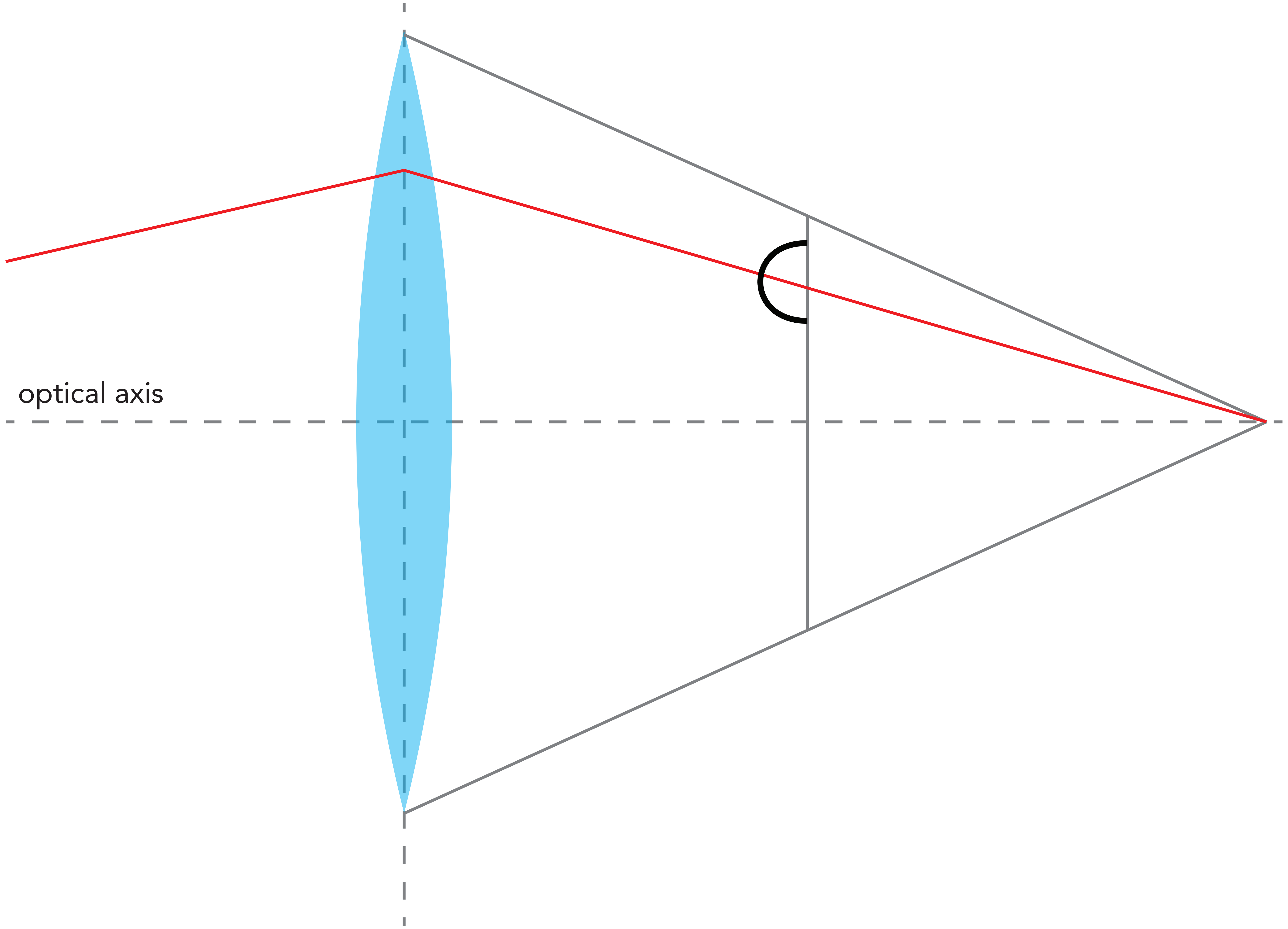
Local Fourier slice photography

- Light field camera:



Local Fourier slice photography

- Light field camera:



Local Fourier slice photography

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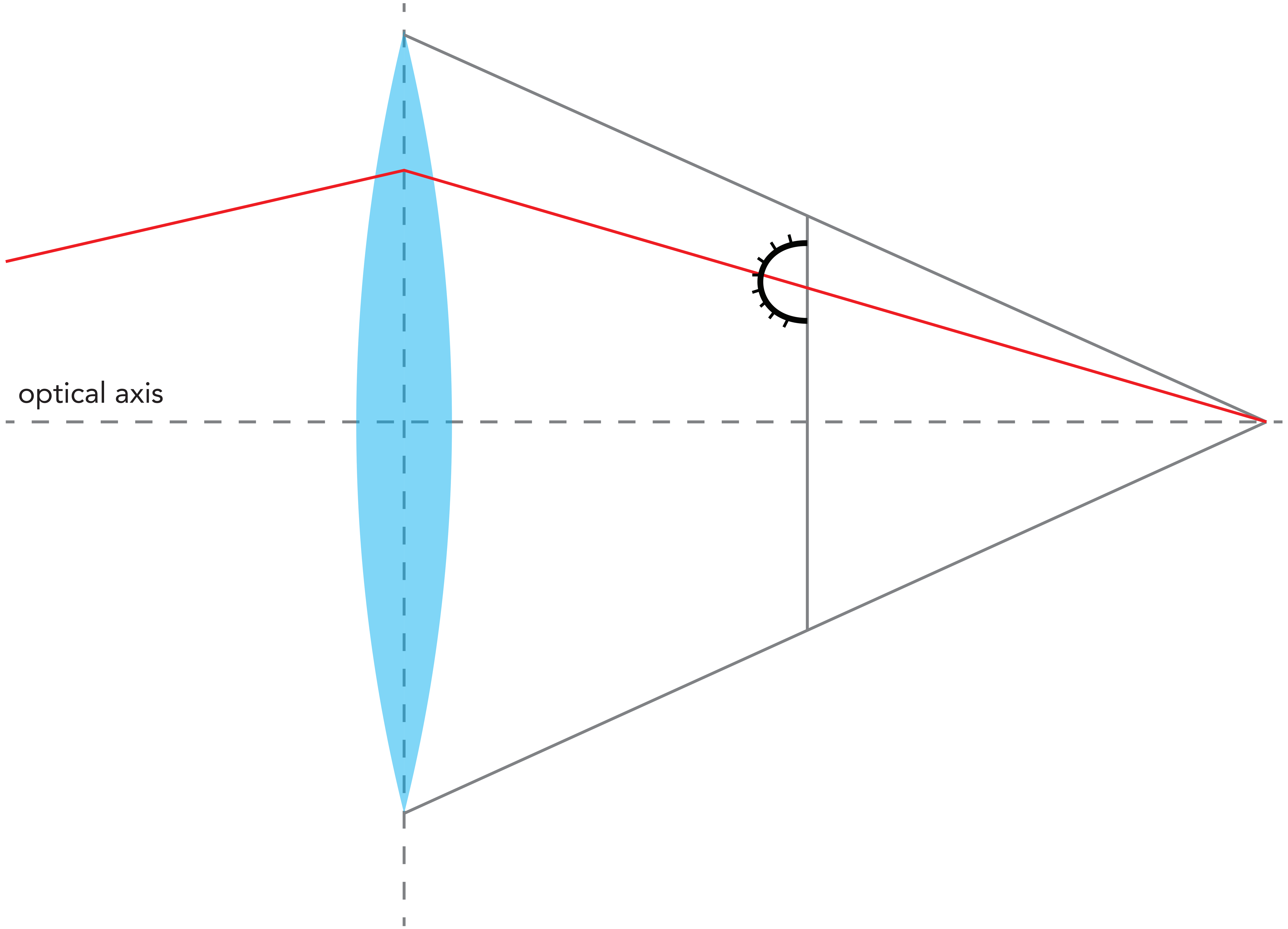

Local Fourier slice photography

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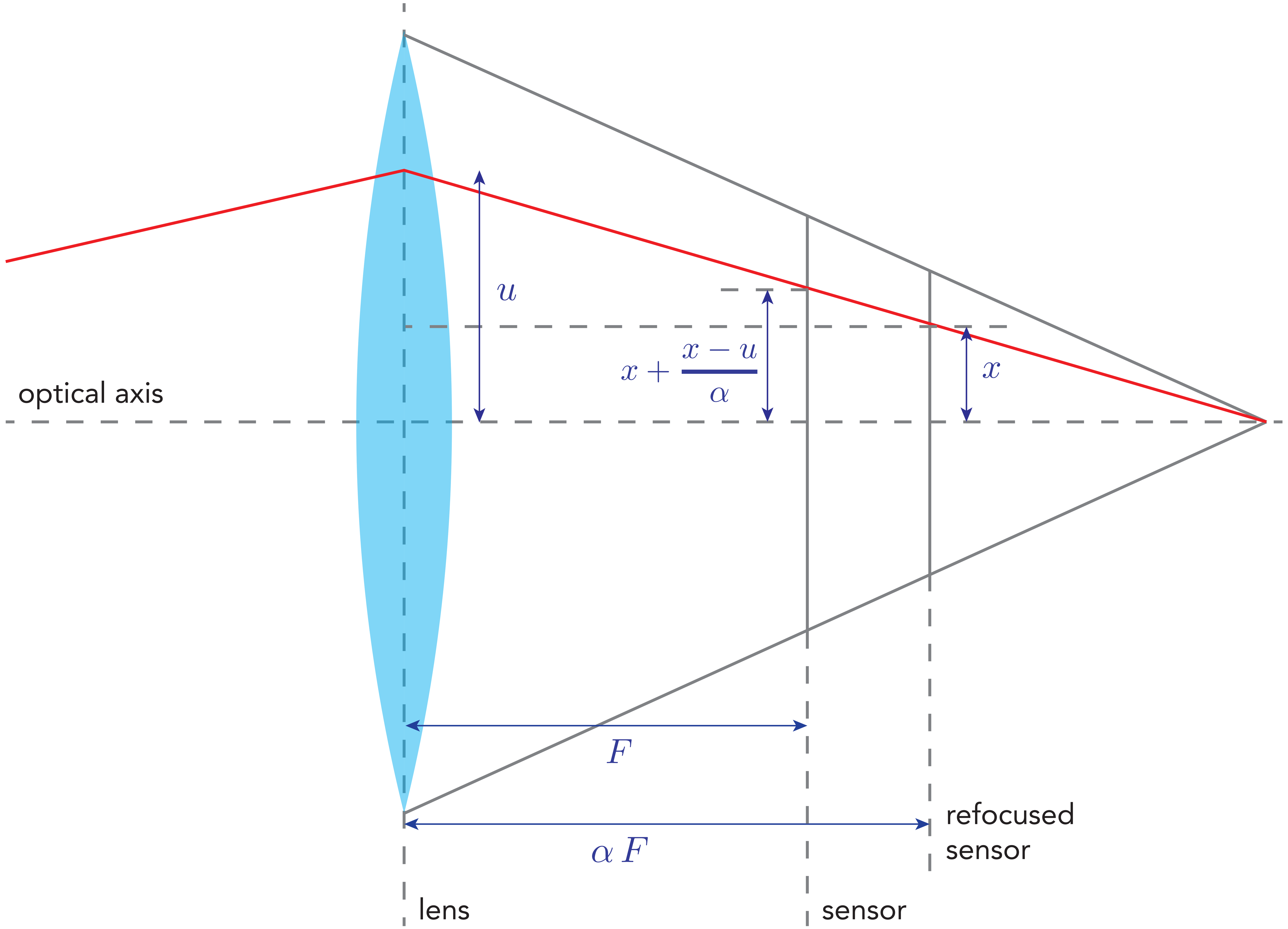
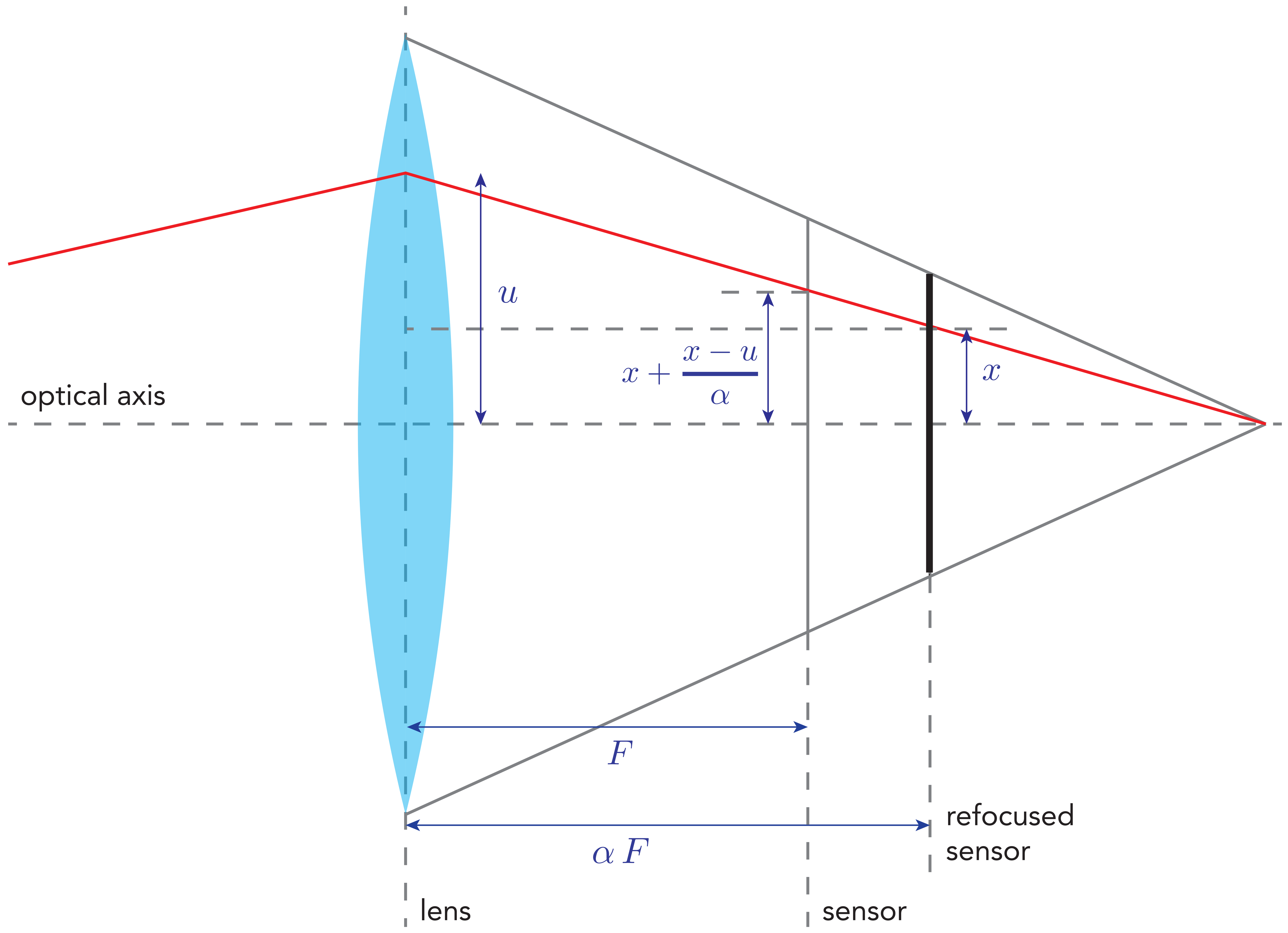


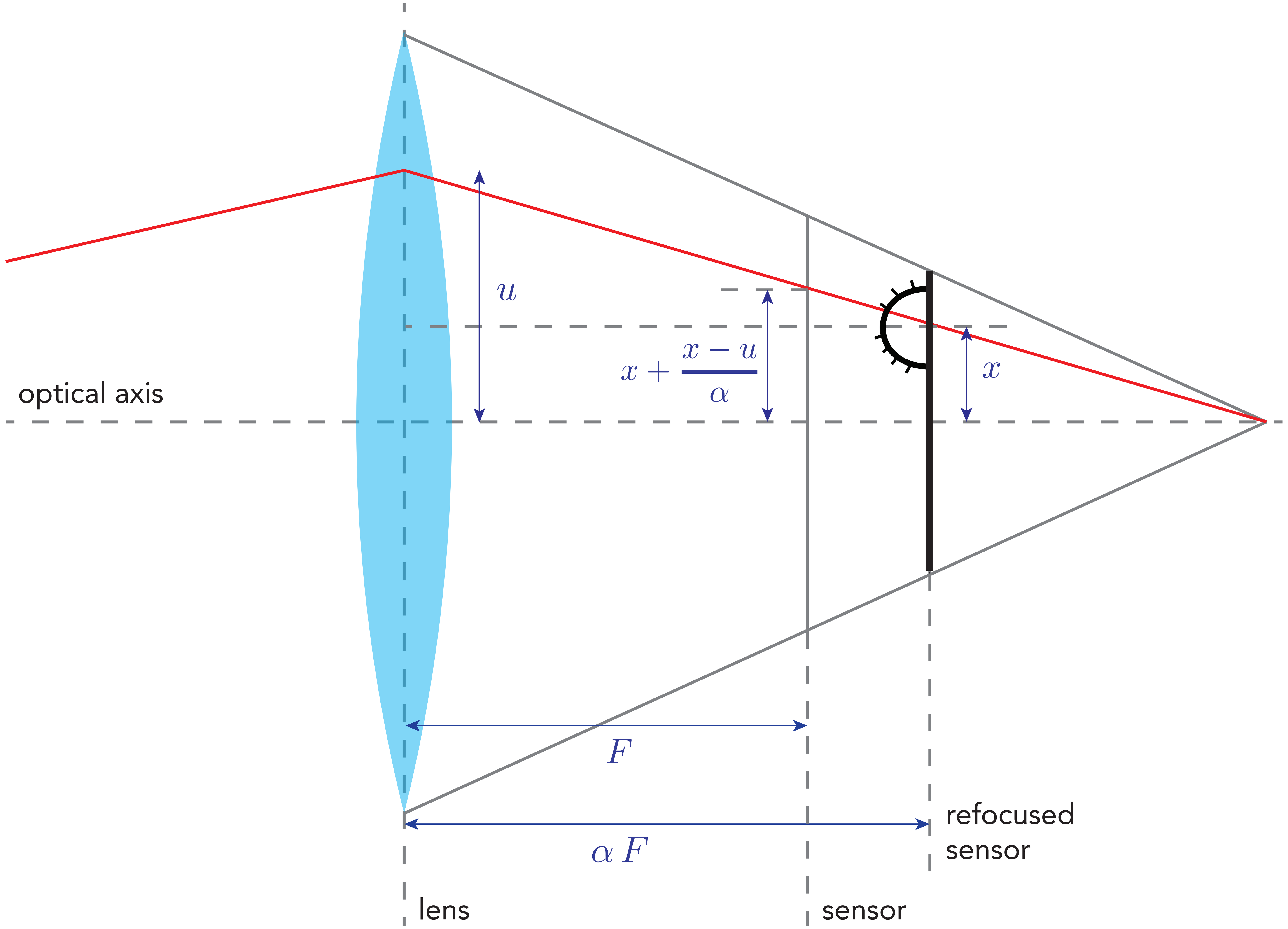
Local Fourier slice photography

- Light field camera:



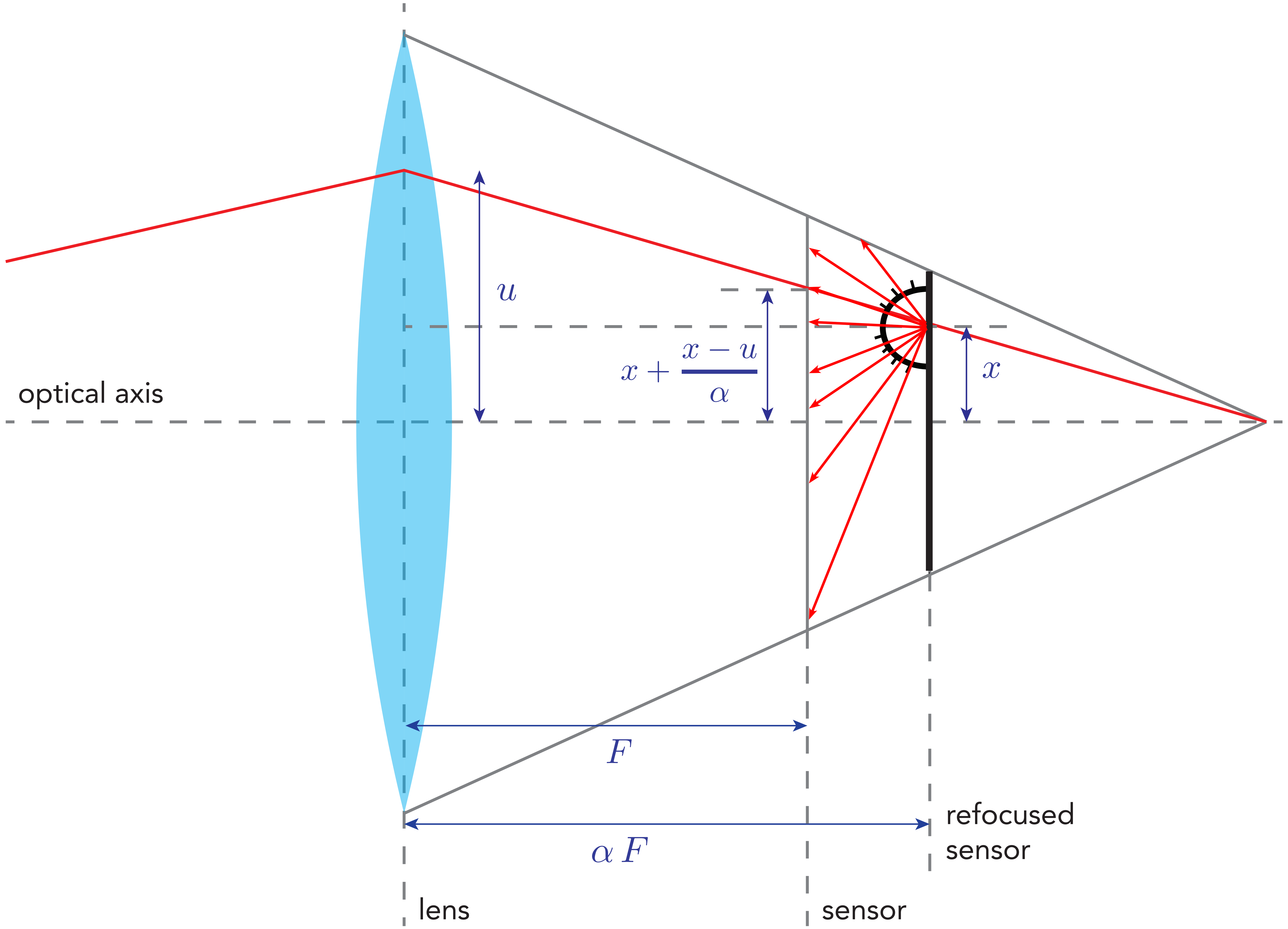
Local Fourier slice photography

- Light field camera:



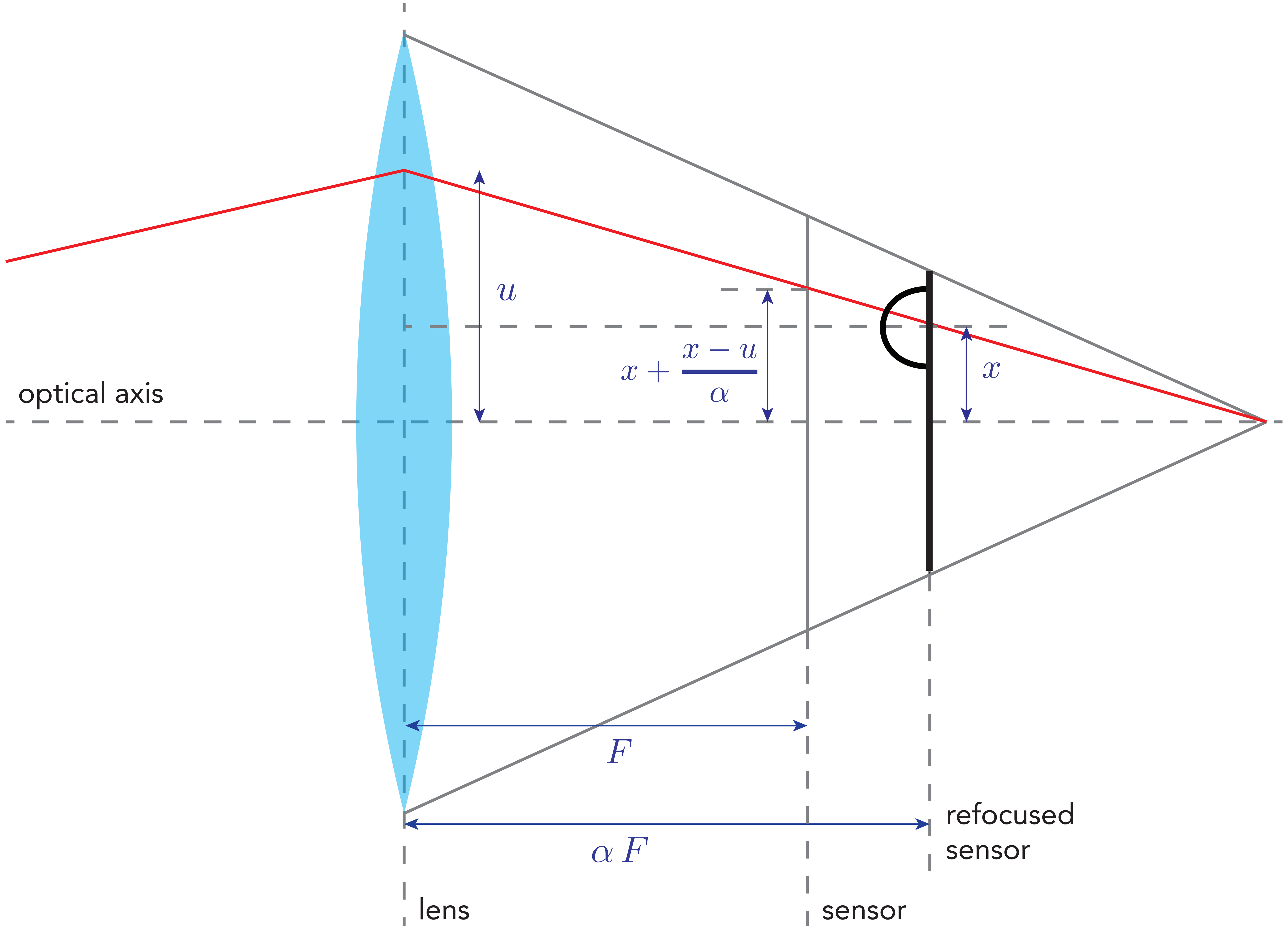
Local Fourier slice photography

- Light field camera:



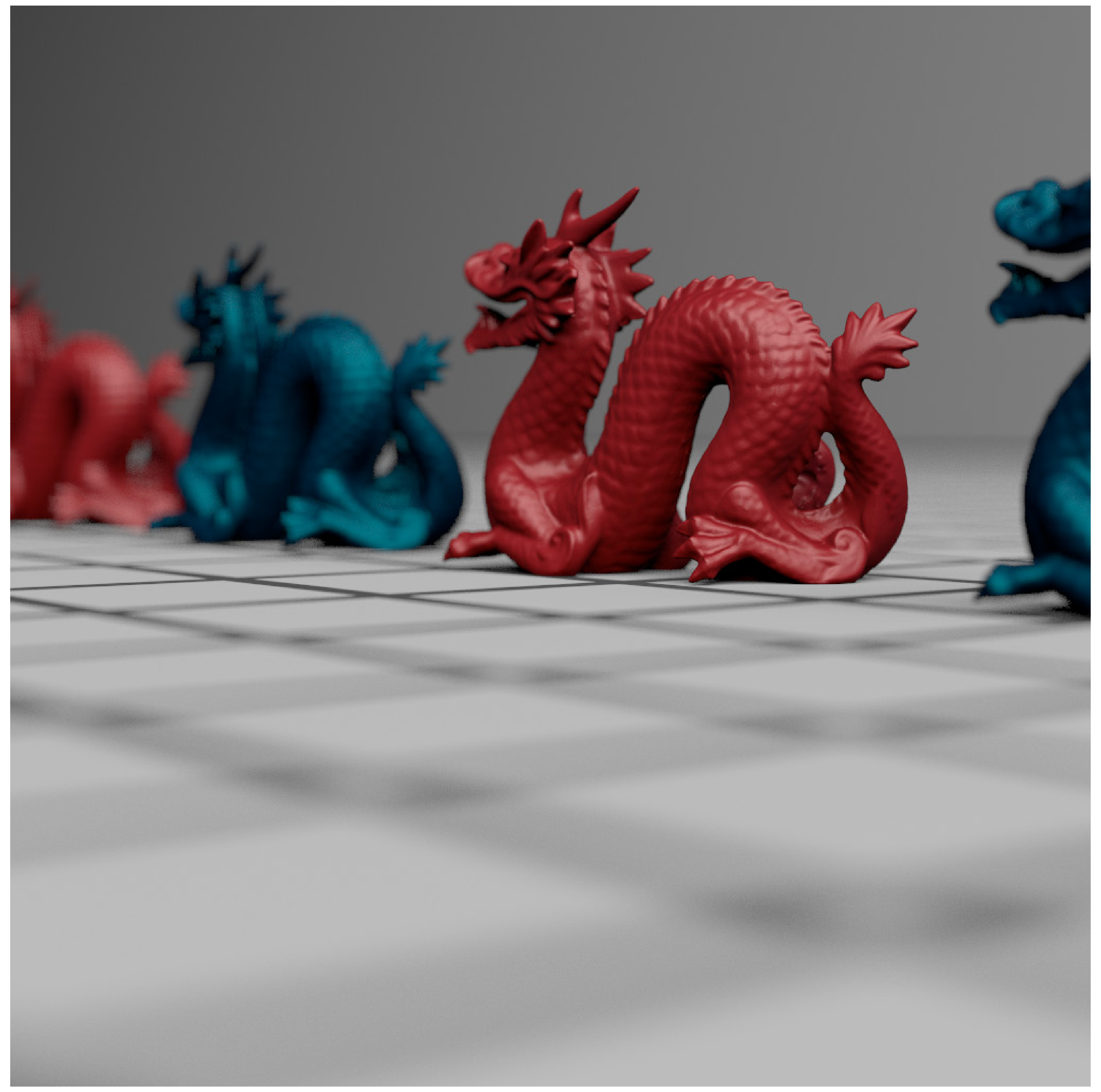
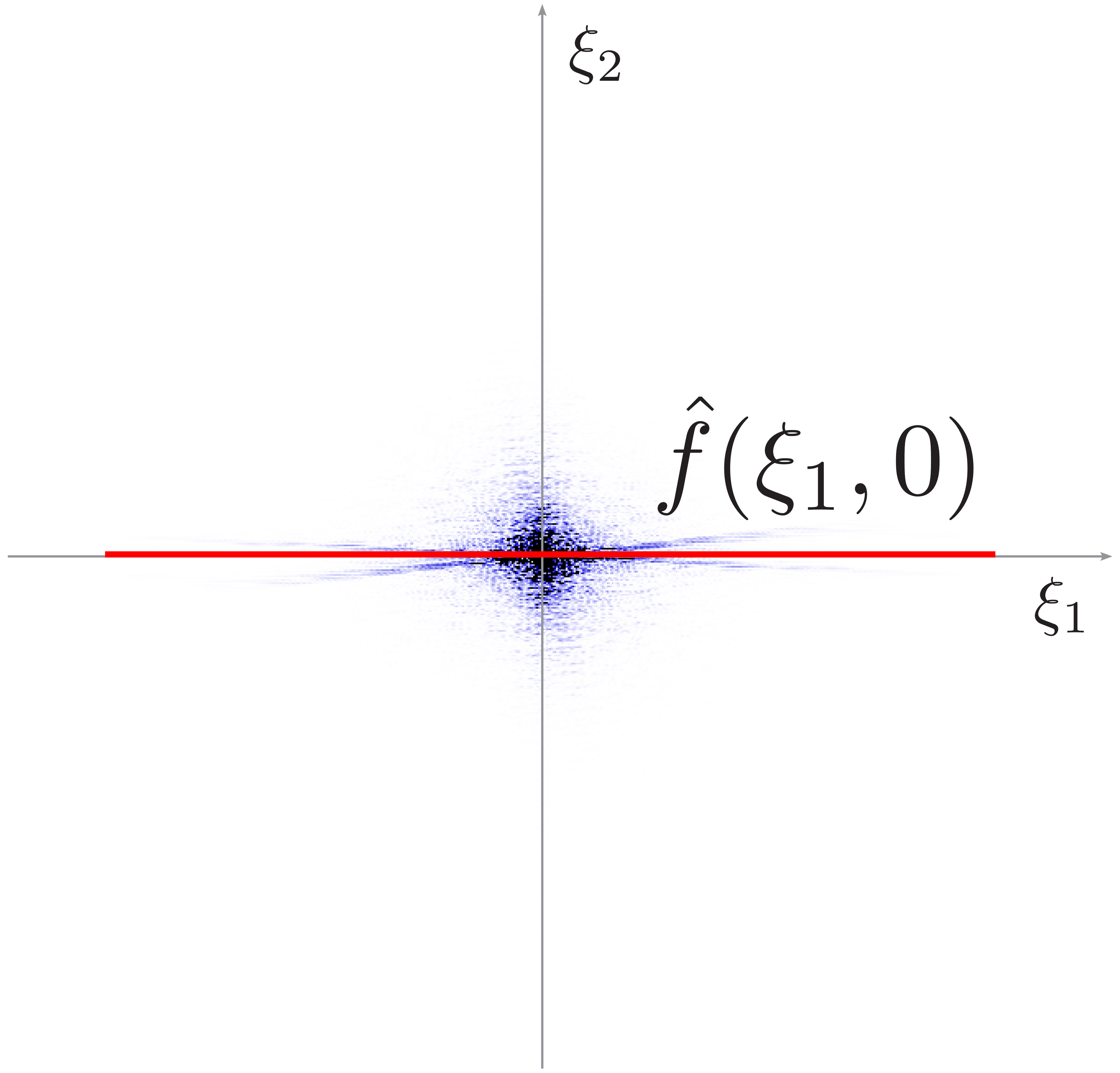
Local Fourier slice photography

- Light field camera:



Local Fourier slice photography

- Refocusing is shearing + integration:

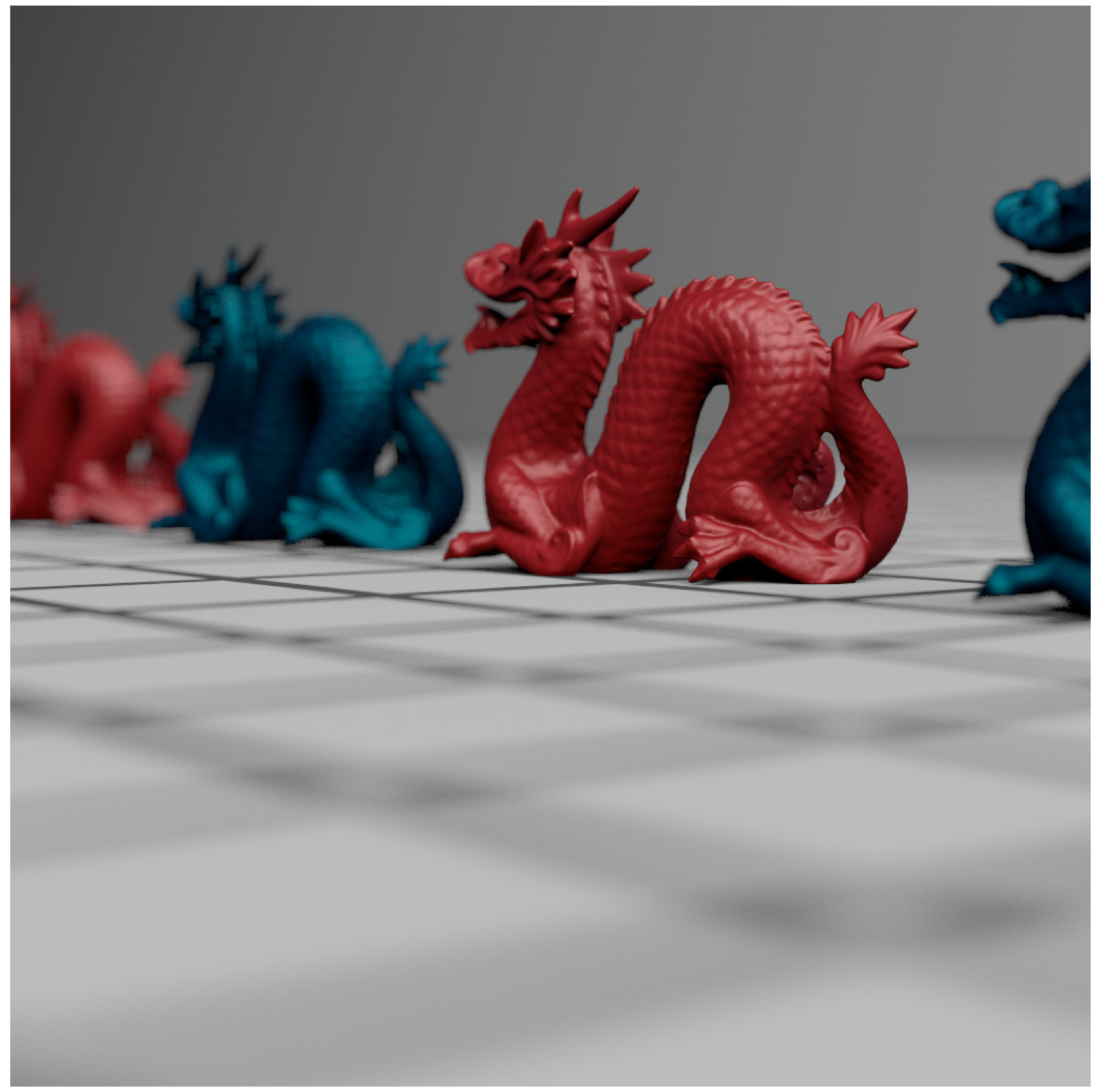
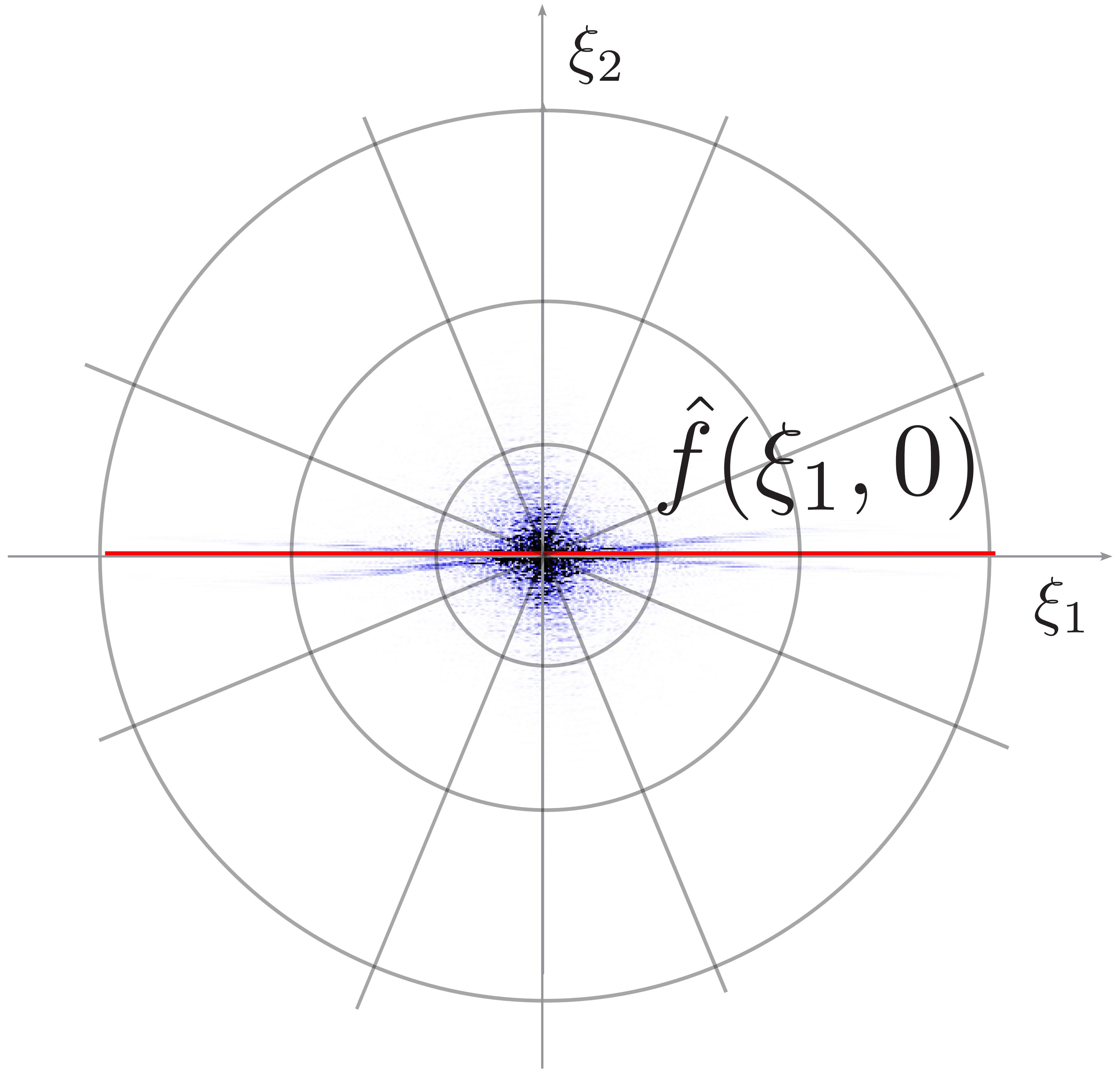
$\int$  $\mathrm{d}x_2 \quad \Longleftrightarrow$ 

1-form

1-form

Local Fourier slice photography

- Refocusing is shearing + integration:

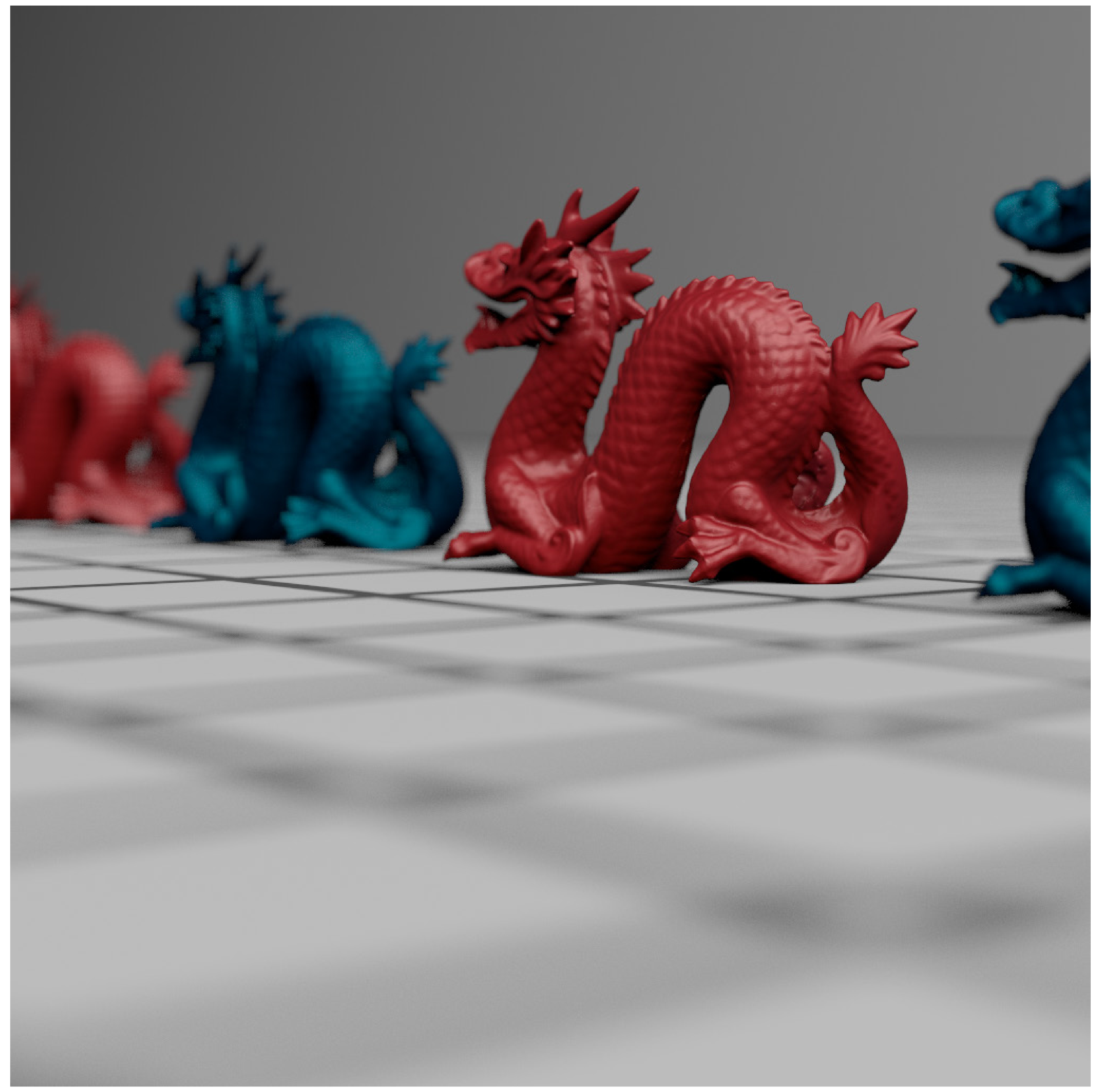
$\int$  $\mathrm{d}x_2 \iff$ 

1-form 1-form

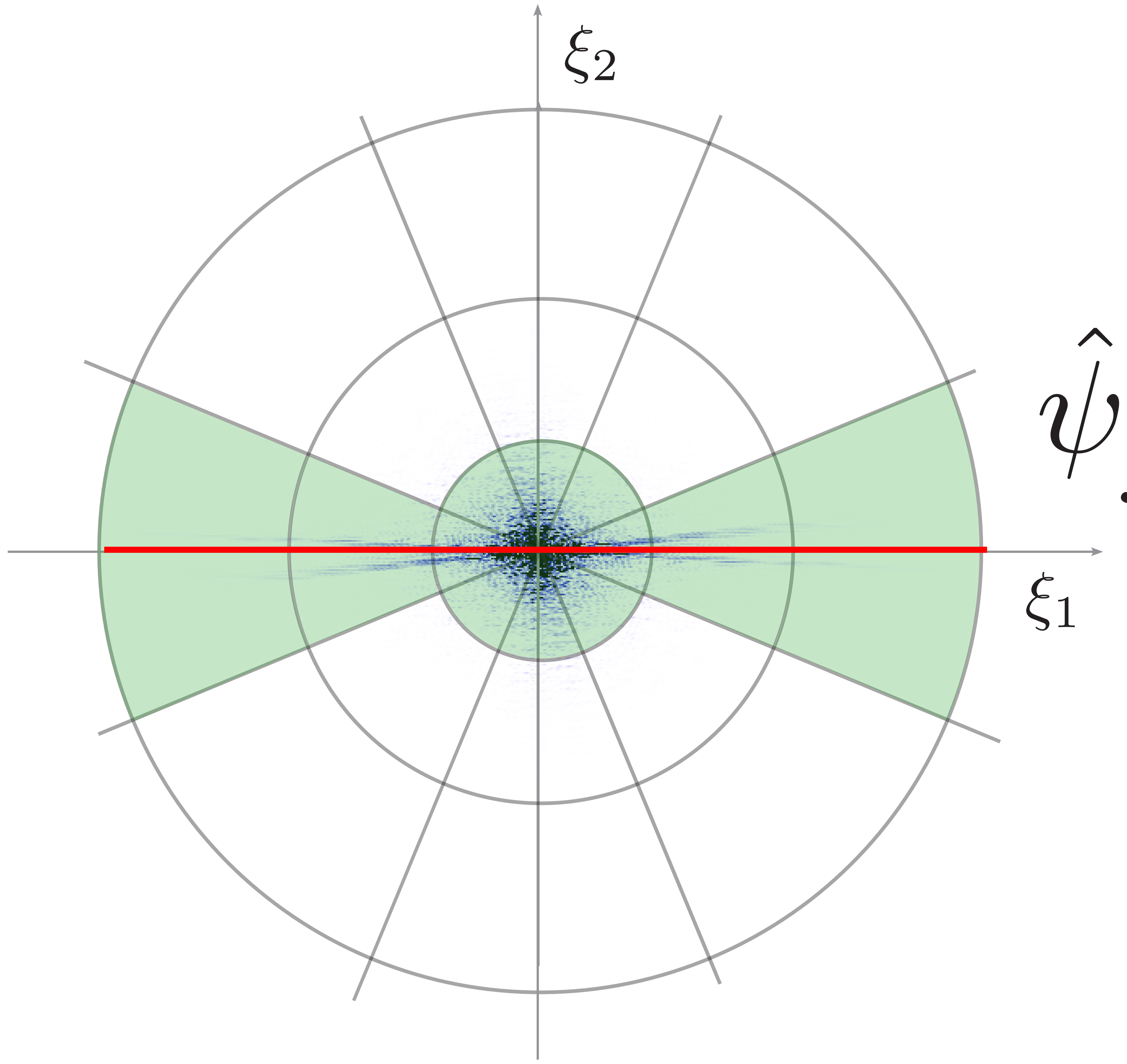
Local Fourier slice photography

- Refocusing is shearing + integration:

$\int$



$\mathrm{d}x_2 \iff$



$\hat{\psi}_j^r(\xi_1, 0)$

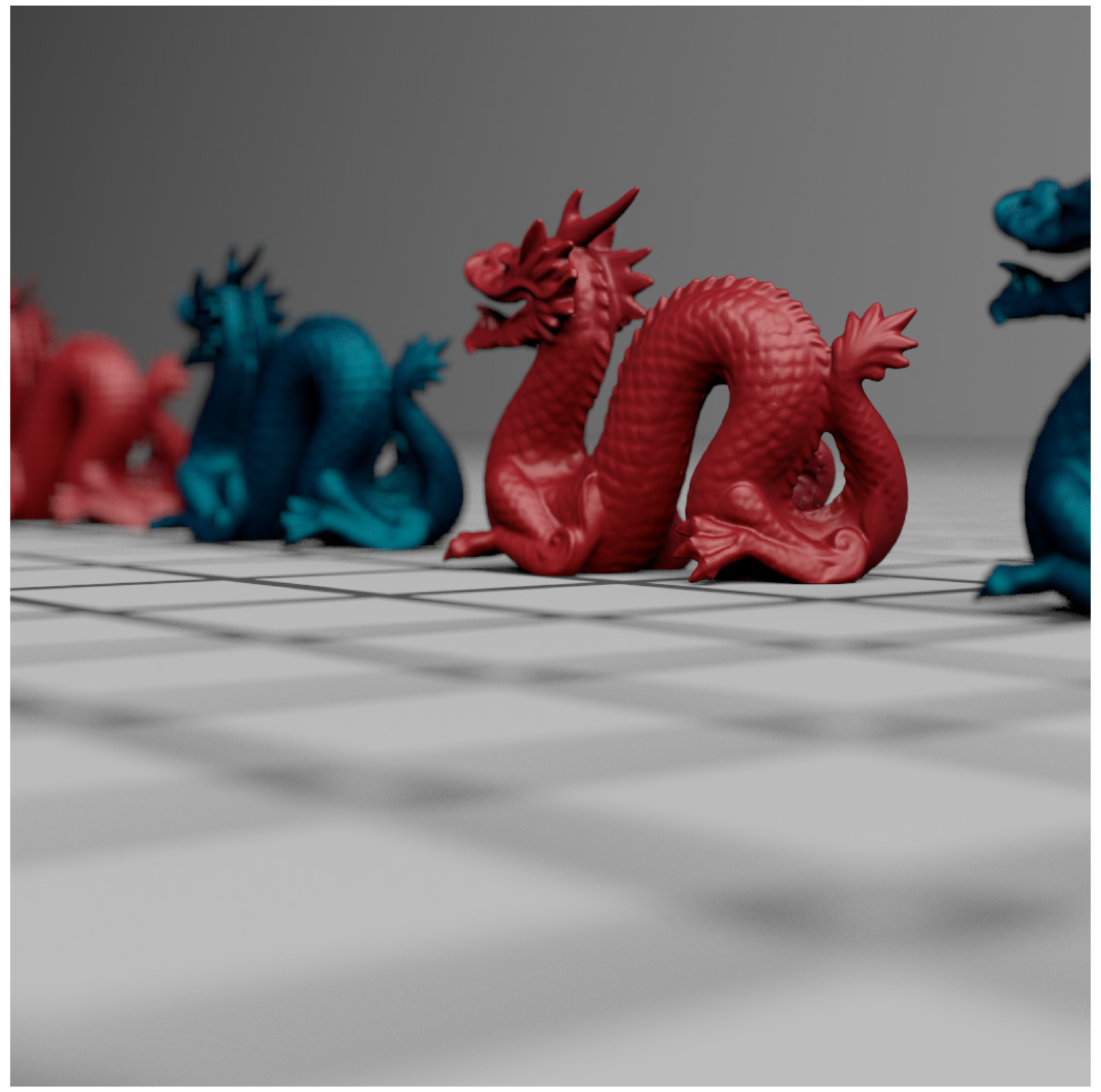
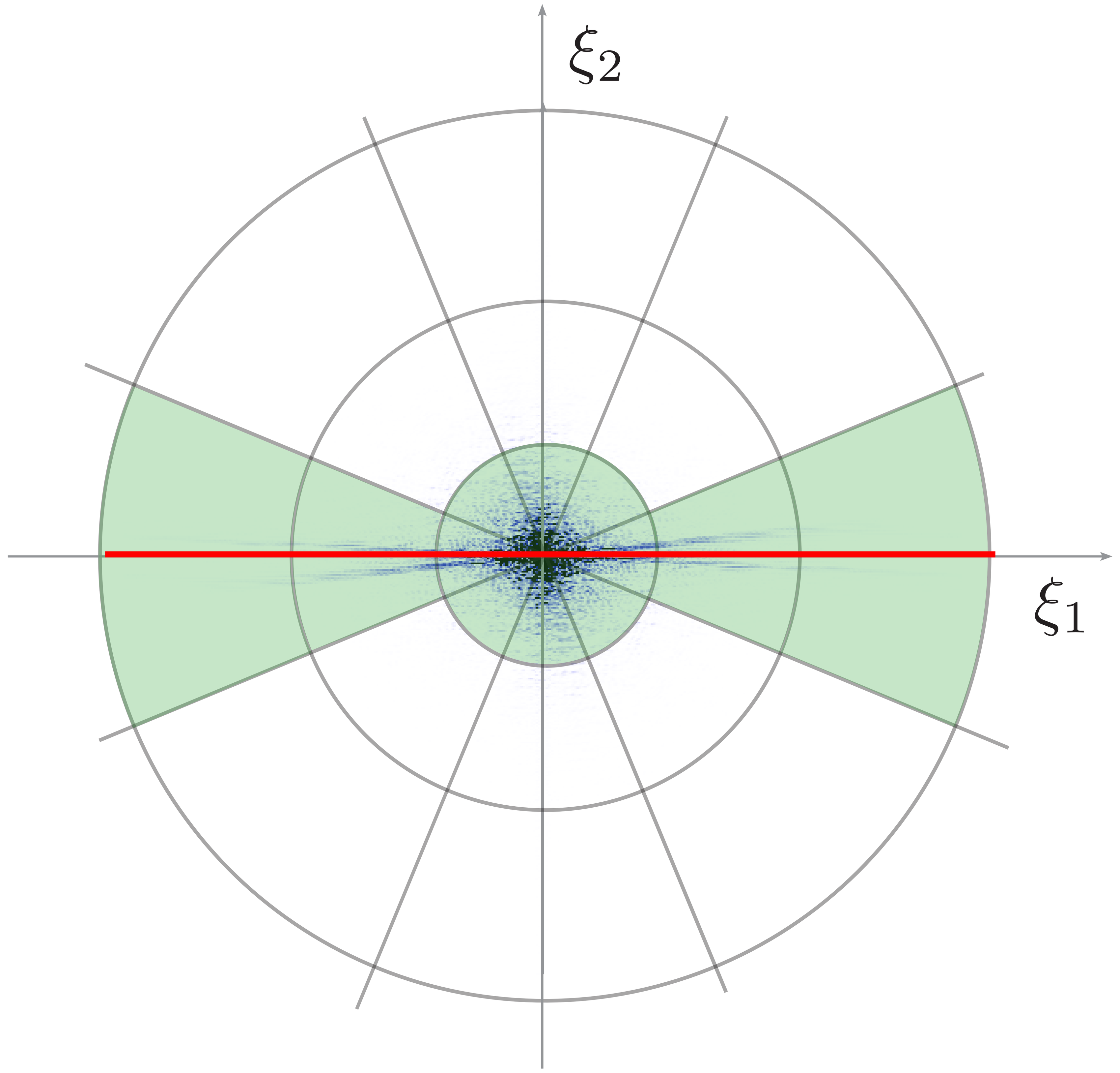
1-form

1-form

Local Fourier slice photography

- Refocusing is shearing ...

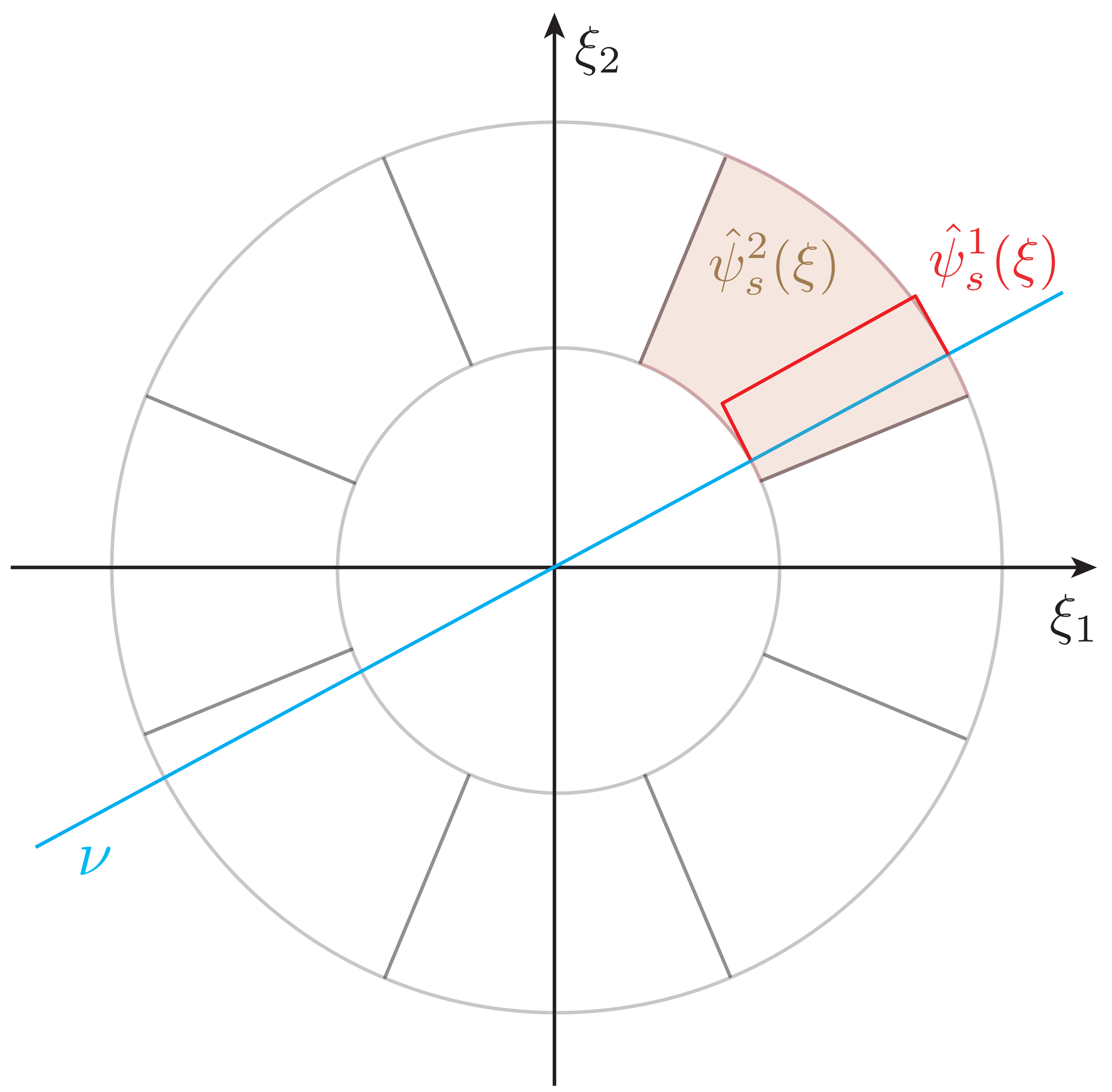
$$\hat{\psi}_j^r(\xi_1, 0) = \hat{\psi}_j^{r-1}(\xi_1)$$

$\int$  $\mathrm{d}x_2 \quad \Longleftrightarrow$ 

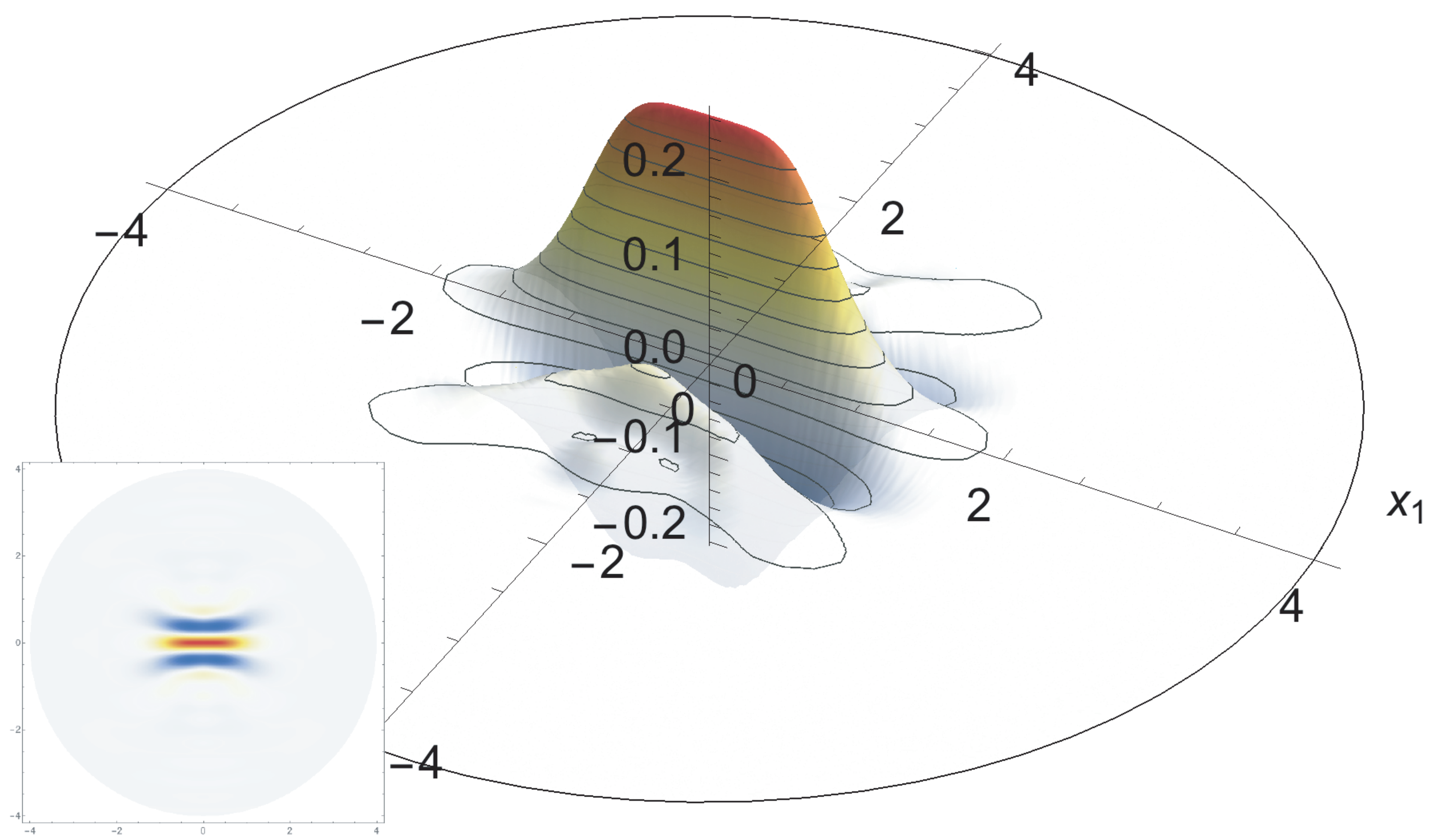
1-form

1-form

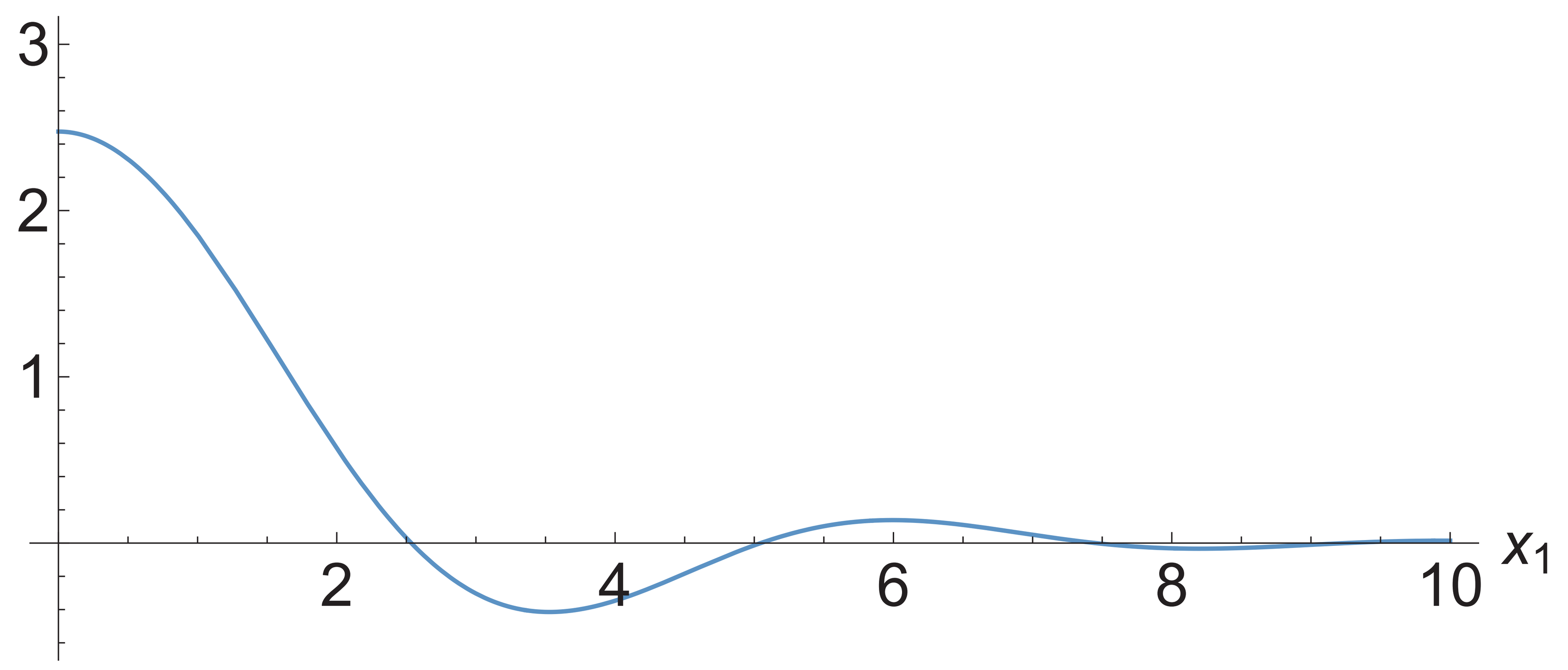
Local Fourier slice photography



$$\psi_j^{(2)}(x_1, x_2)$$



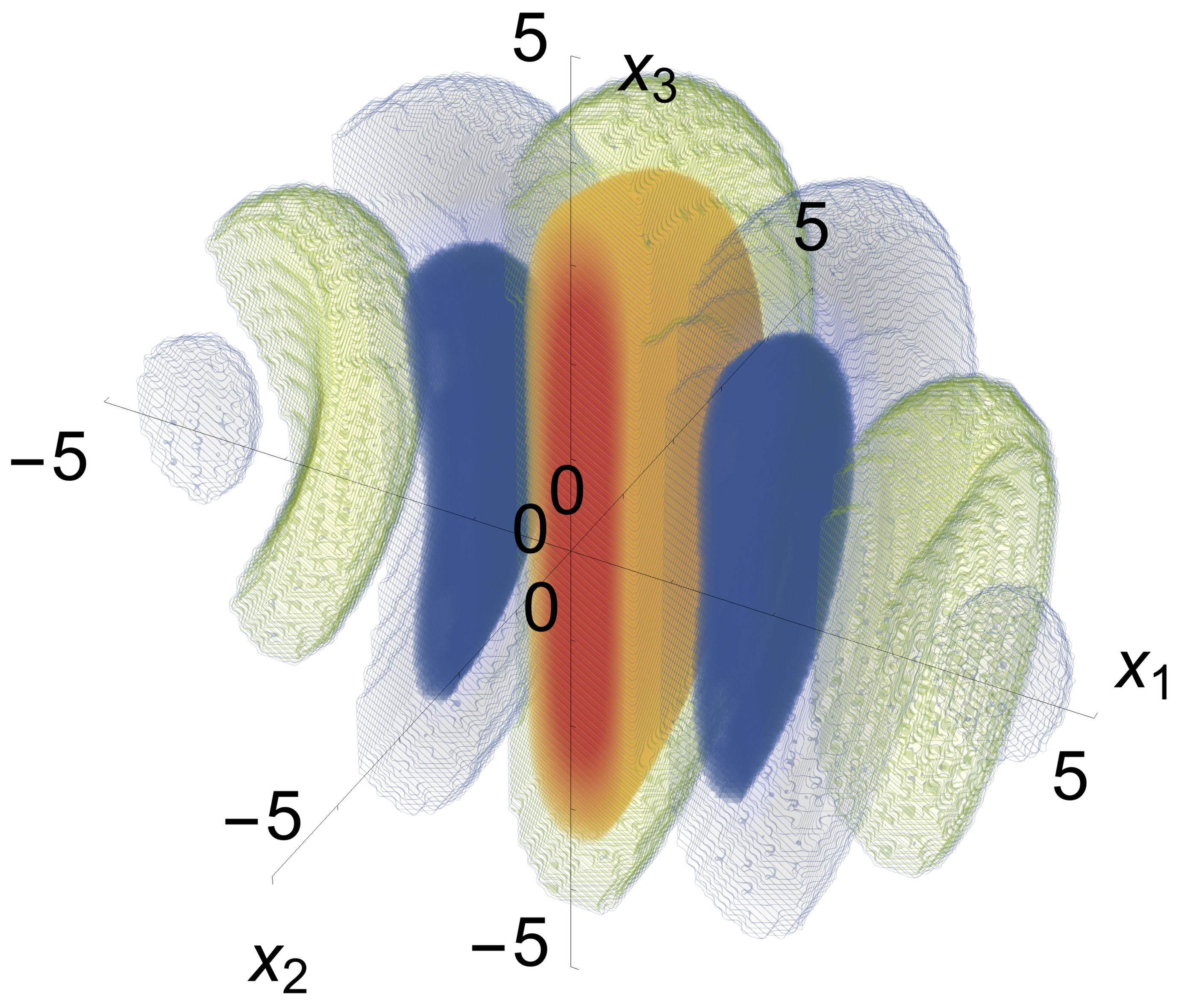
$$\psi_j^{(1)}(x_1)$$



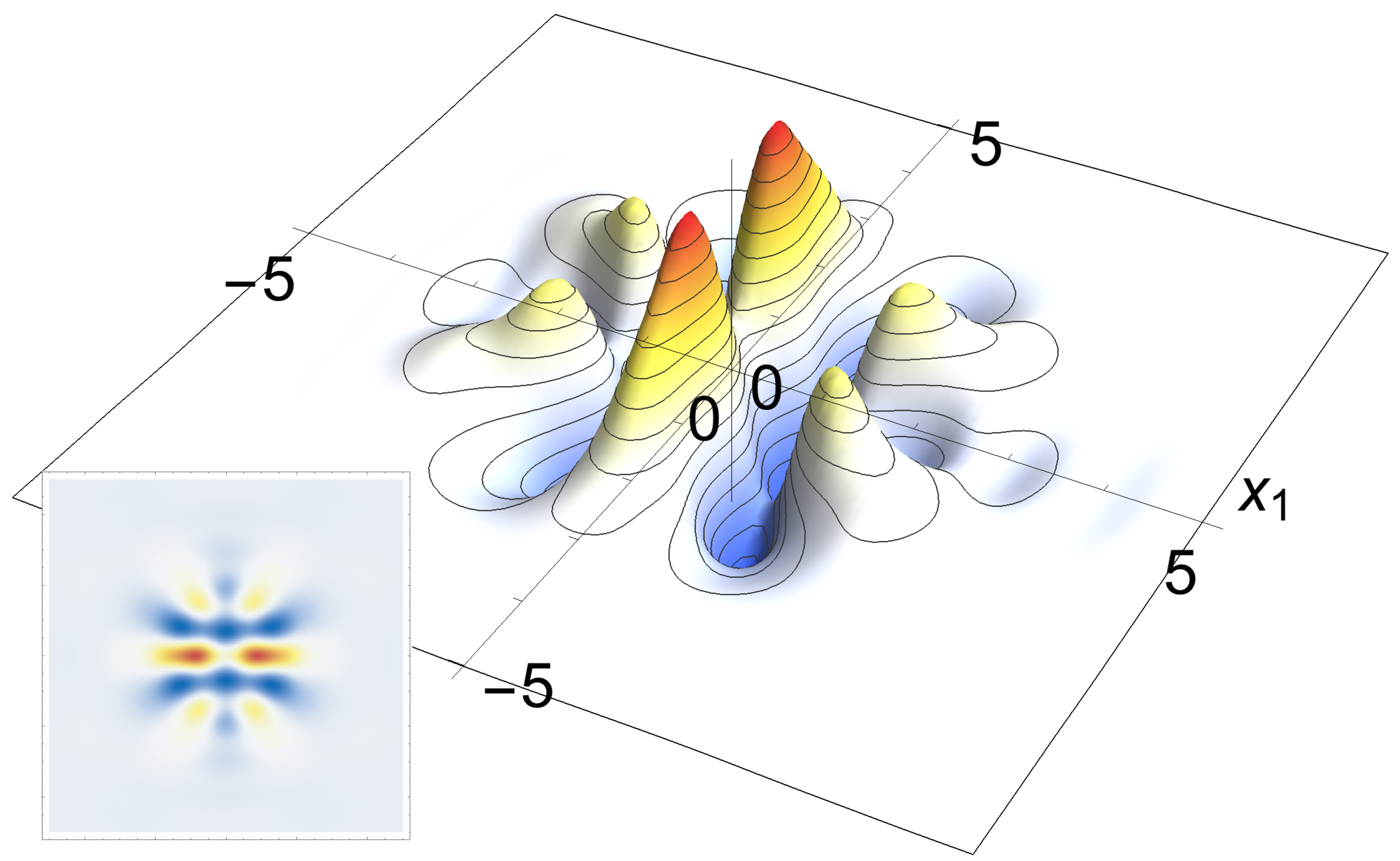
C. Lessig. A local Fourier slice equation. Optics Express, 26(23):29769–29783, nov 2018.

Local Fourier slice photography

$$\psi_j^{(3)}(x_1, x_2, x_3)$$



$$\psi_j^{(2)}(x_1, x_2)$$



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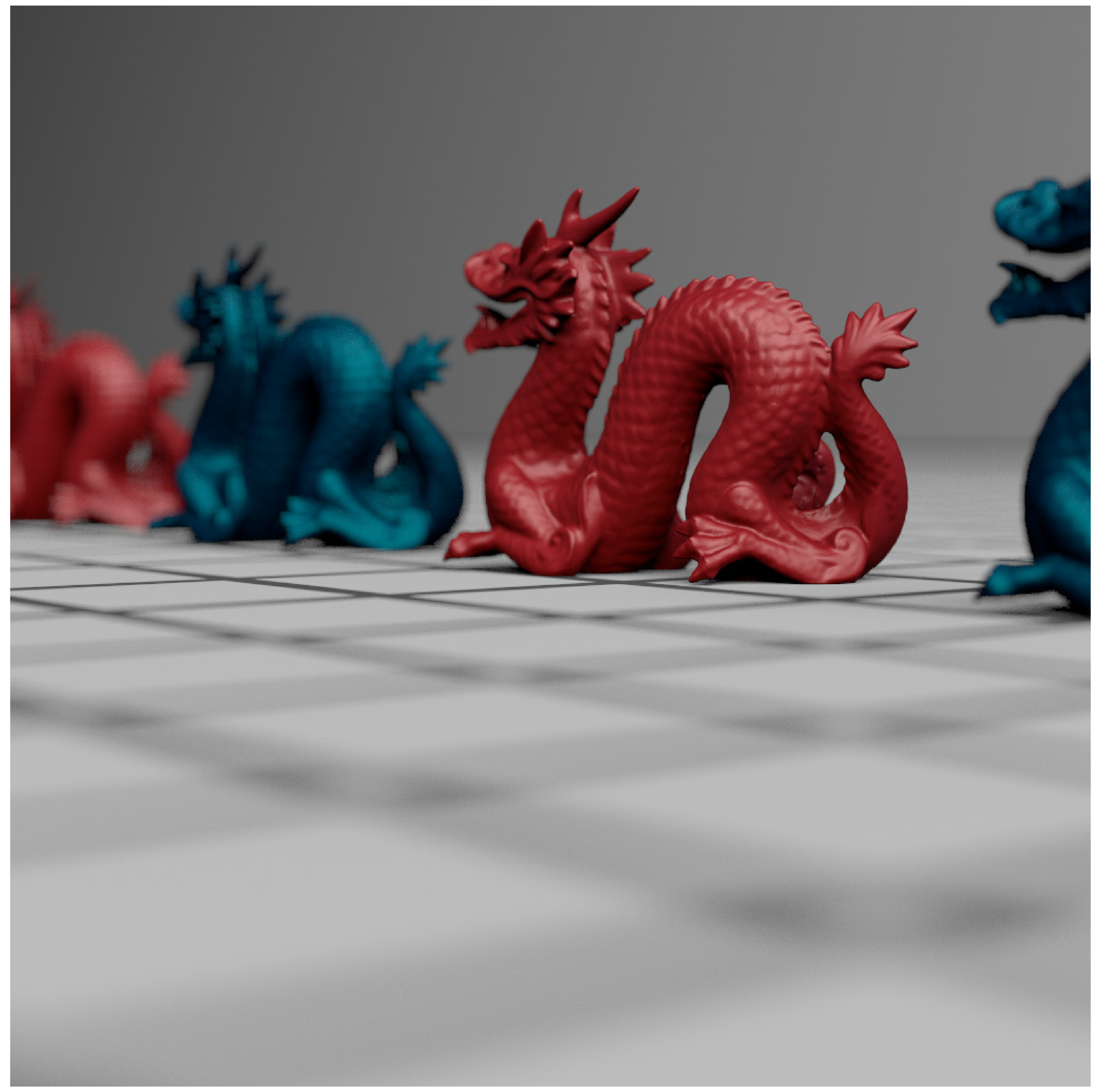
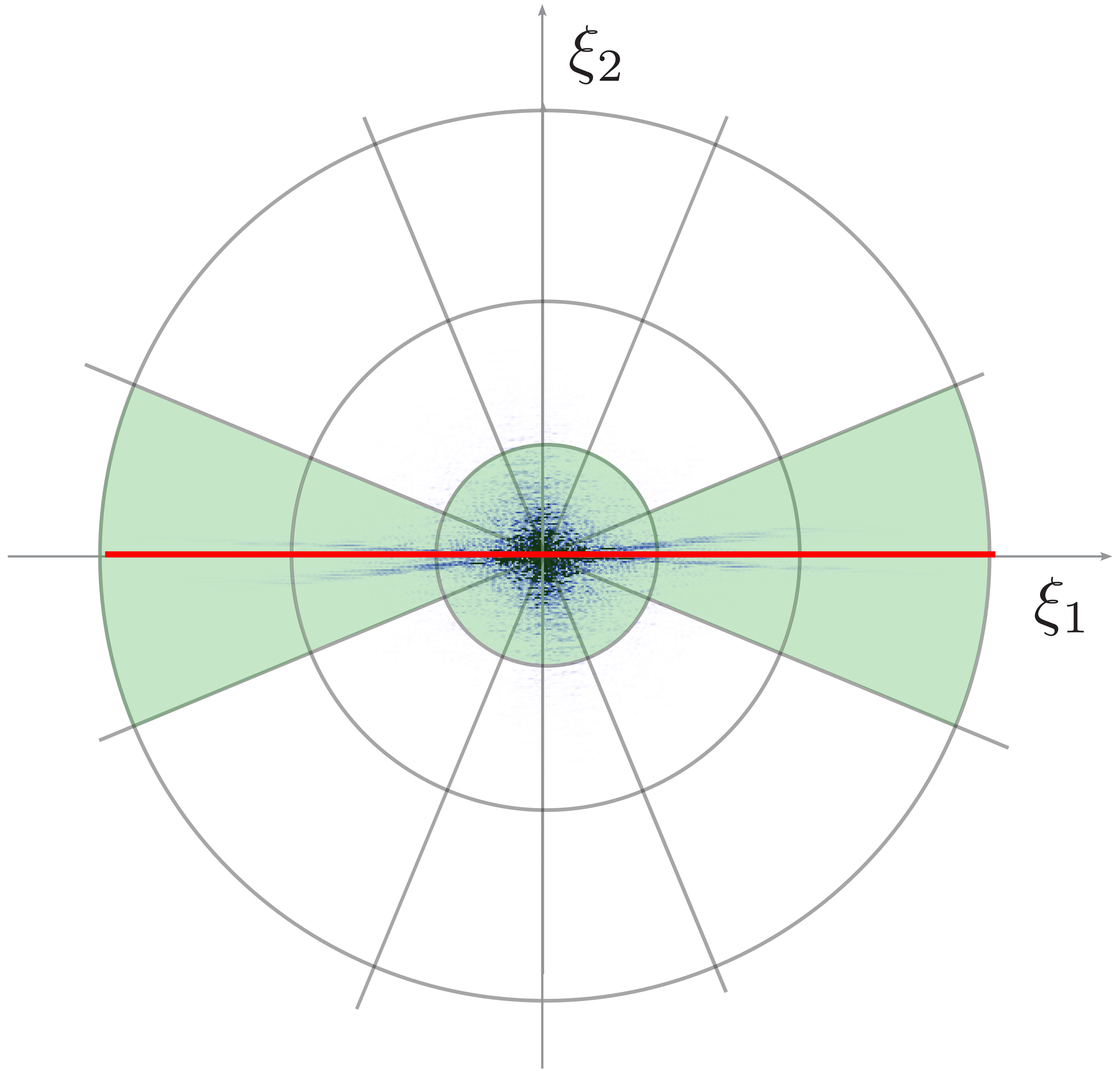
Local Fourier slice photography

- Refocusing is shearing …

$$\hat{\psi}_j^r(\xi_1, 0) = \hat{\psi}_j^{r-1}(\xi_1)$$

$\int$  $\mathrm{d}x_2 \iff$ 

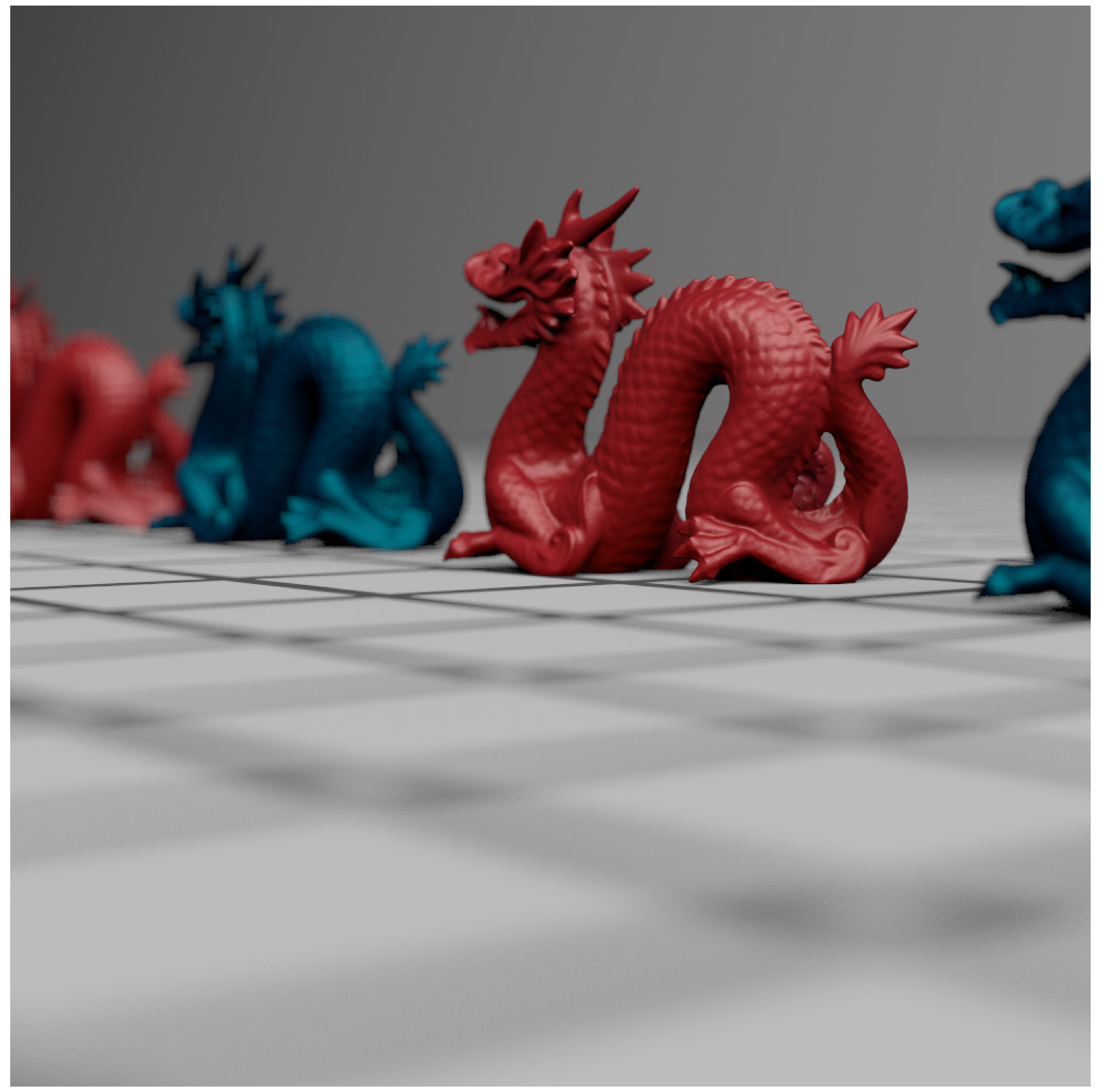
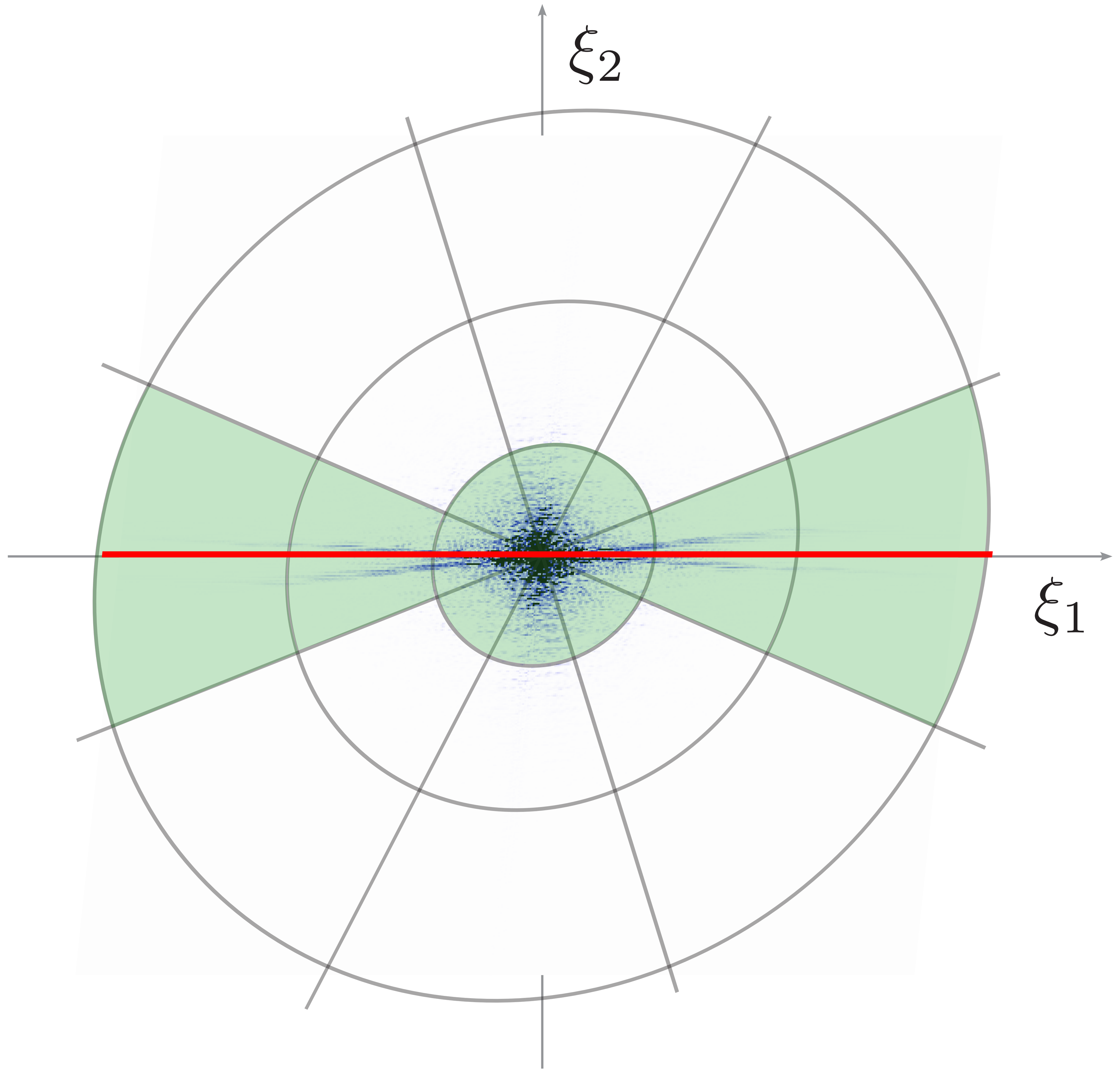
1-form

1-form

Local Fourier slice photography

- Refocusing is shearing …

$$\hat{\psi}_j^r(\xi_1, 0) = \hat{\psi}_j^{r-1}(\xi_1)$$

$\int$  $\mathrm{d}x_2$ $\Longleftrightarrow$ 

1-form

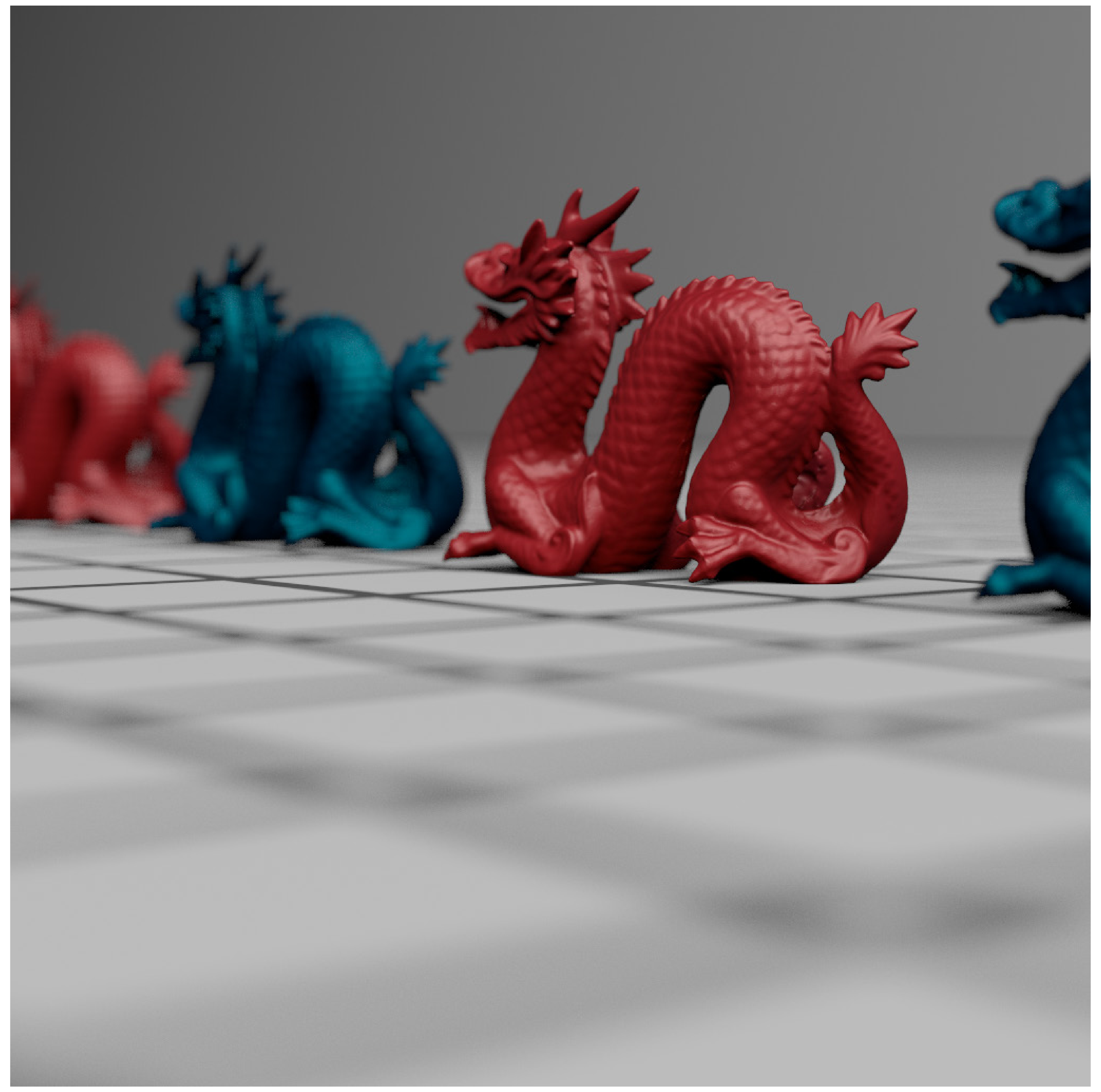
1-form

Local Fourier slice photography

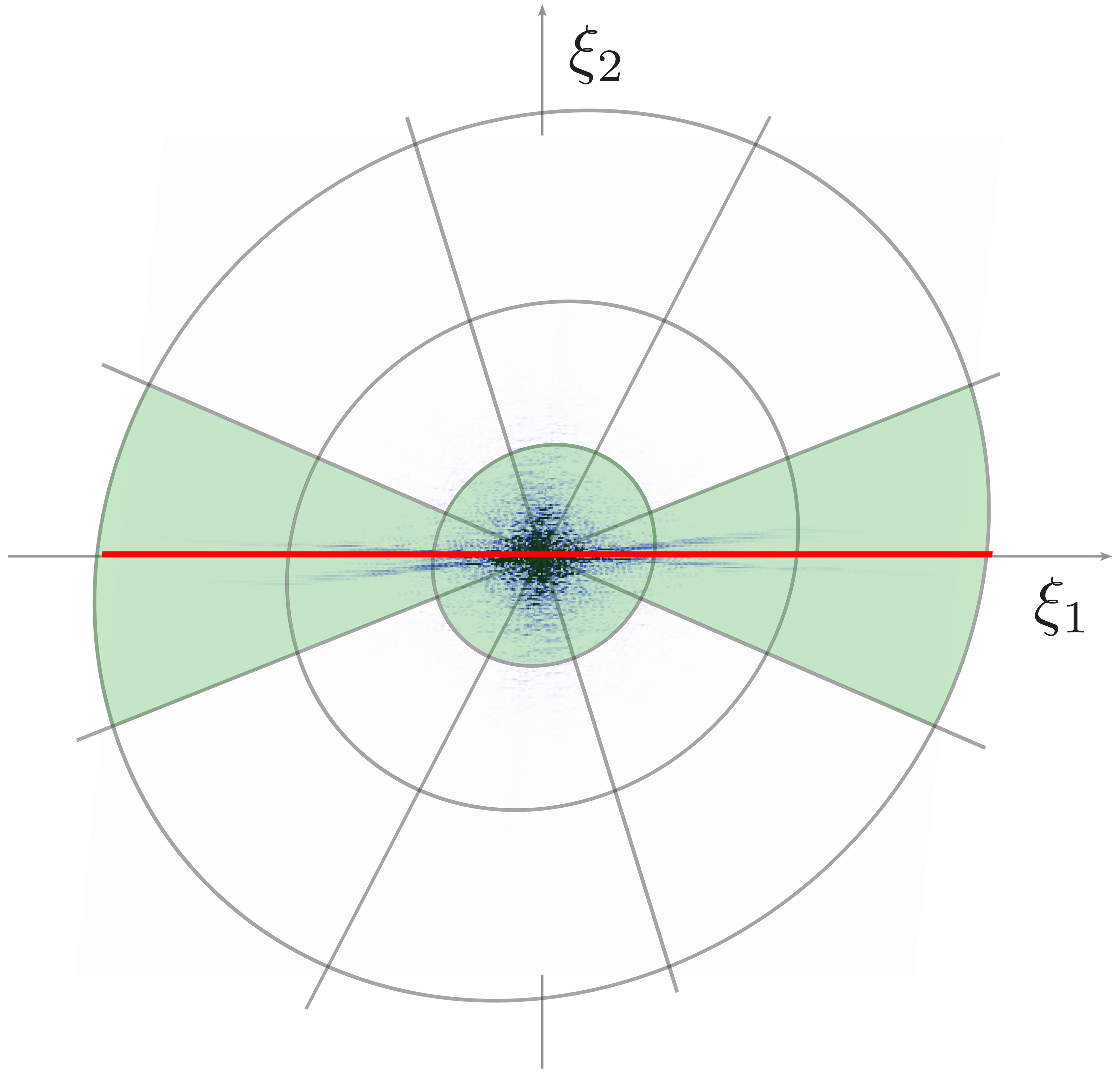
- Refocusing is shearing …

$$\hat{\psi}_j^r(\xi_1, 0) = \hat{\psi}_j^{r-1,\alpha}(\xi_1)$$

$$\int$$



$$\mathrm{d}x_2 \quad \Longleftrightarrow$$



1-form

1-form

Local Fourier slice photography

cr=1.

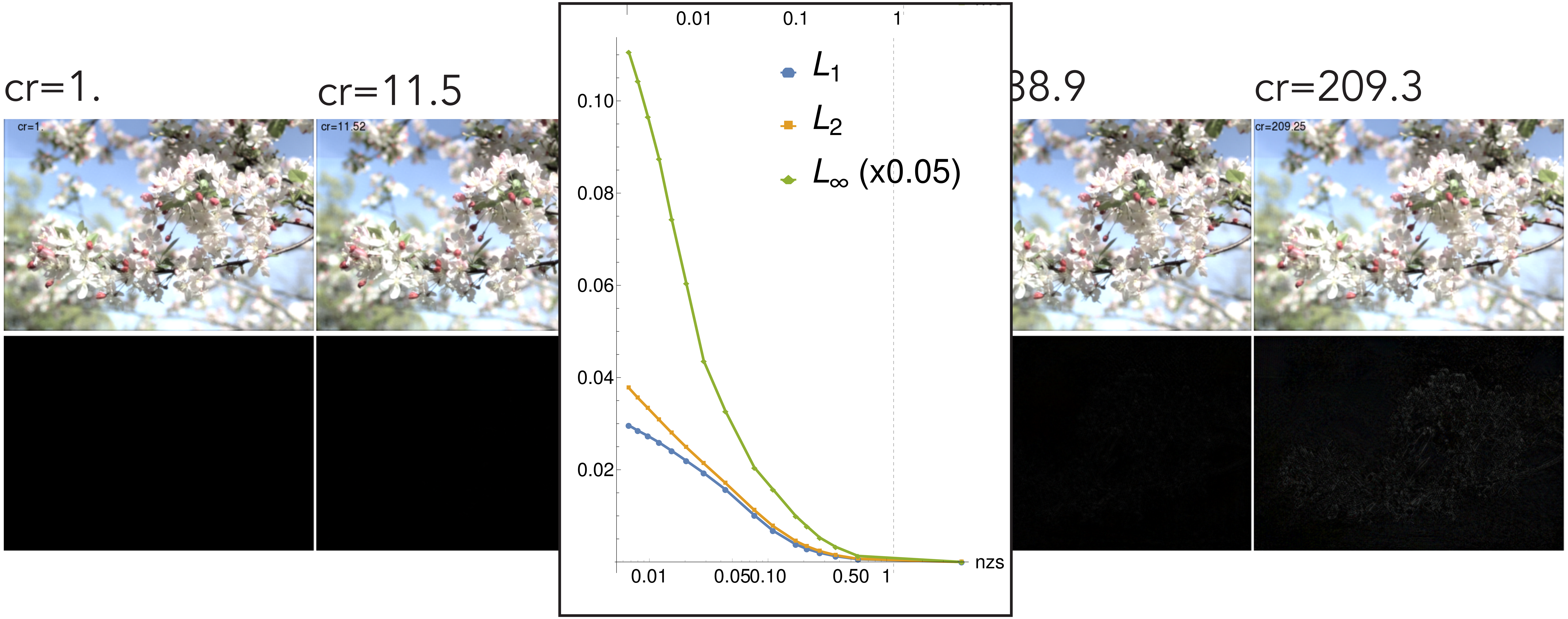
cr=11.5

cr=20.2

cr=38.9

cr=209.3

Local Fourier slice photography



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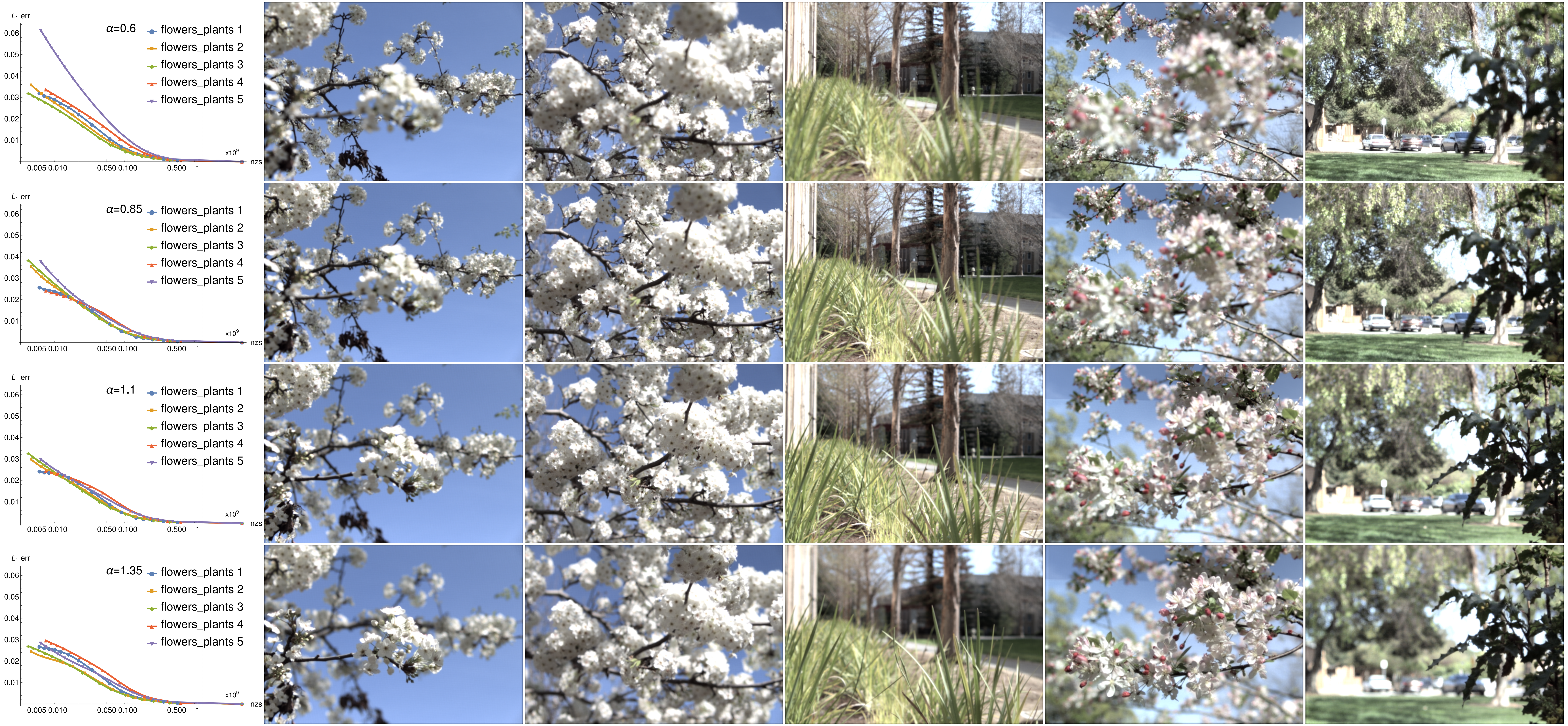

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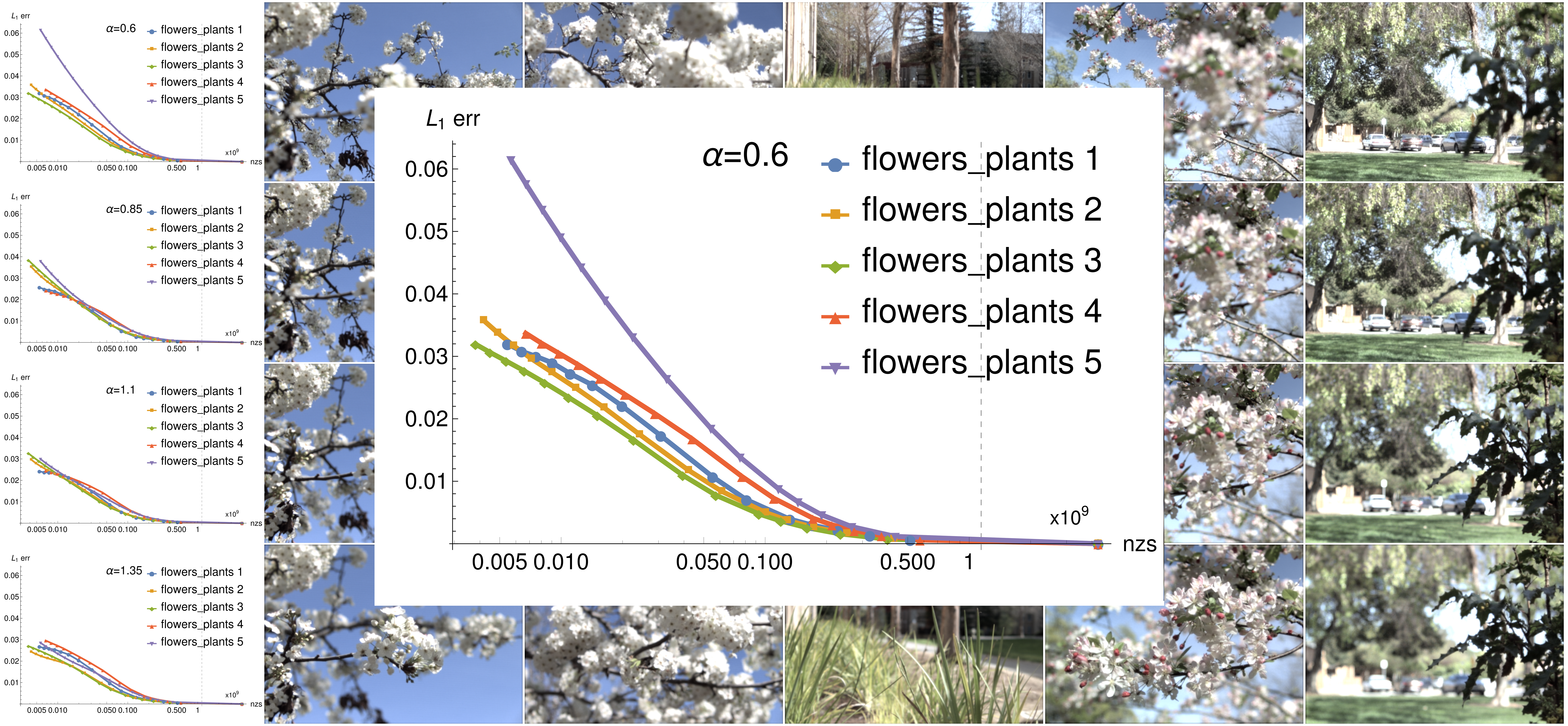

Simulation of shallow water equation

- Simplified 2.5D model for atmosphere

C. C. da Silva, B. Dodov, H. Dijkstra, T. Sapsis, and C. Lessig. A local spectral exterior calculus for the sphere and application to the shallow water equations. Submitted to Journal of Computational Physics, 2020.

Simulation of shallow water equation

- Simplified 2.5D model for atmosphere
- Discretization based on $\Psi\mathrm{ec}(S^2)$ for coarse scales

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Simulation of shallow water equation

- Simplified 2.5D model for atmosphere
- Discretization based on $\Psi\mathrm{ec}(S^2)$ for coarse scales
- Neural network-based correction to account for small scales

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Simulation of shallow water equation

- Shallow water equation:

$$\dot{\zeta} = -\nabla \cdot (\zeta + f)\,\vec{u}$$

$$\dot{\mu} = \nabla \times (\zeta + f)\,\vec{u} - \Delta \frac{|\vec{u}|^2}{2} - \Delta g(h + h_e)$$

$$\dot{h} = -\nabla \cdot (h\,\vec{u})$$

$$\vec{u} = \delta \Delta^{-1} \zeta + \mathrm{d}\Delta^{-1} \star \mu$$

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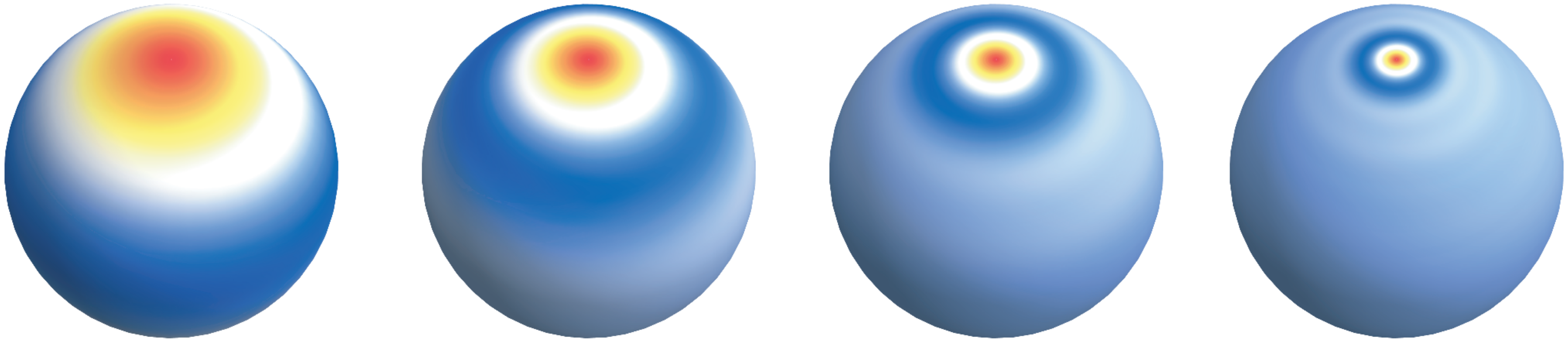
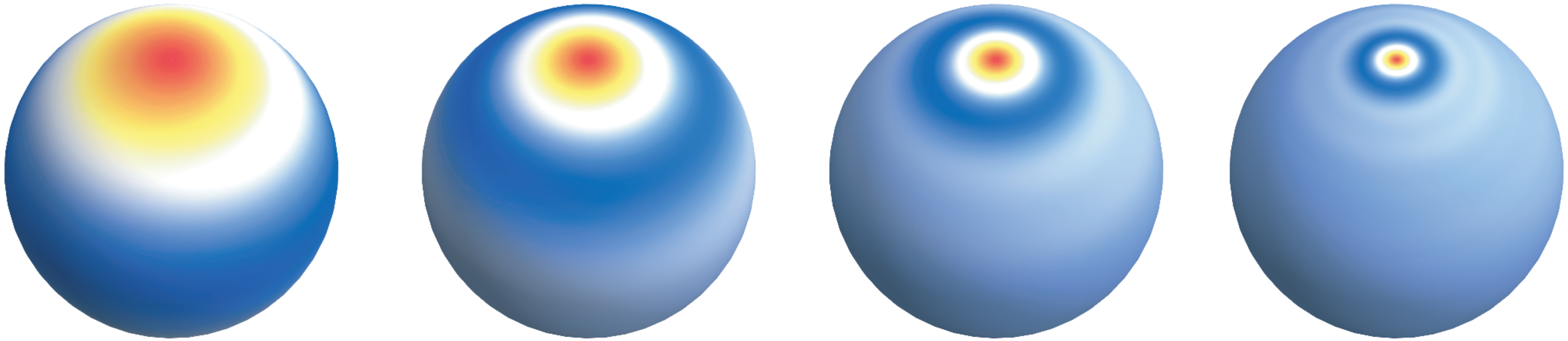
Simulation of shallow water equation



ψ_1 ψ_2 ψ_3 $\dots$

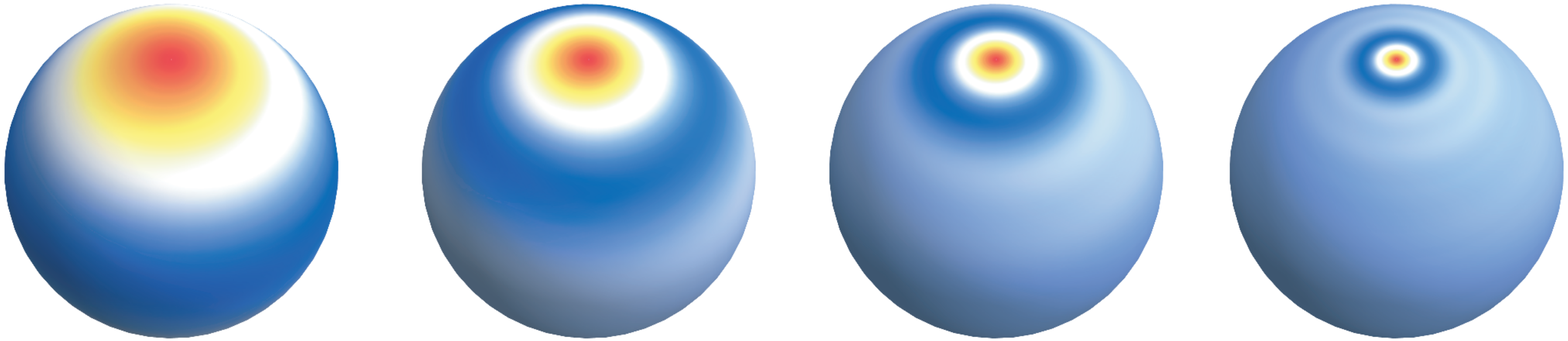
Simulation of shallow water equation



ψ_1 ψ_2 ψ_3 $\cdots$

$$\psi_j(\omega) = \sum_{l=0}^{2^j} \sum_{m=-l}^{l} \psi_{lm}^j \, y_{lm}(\omega)$$

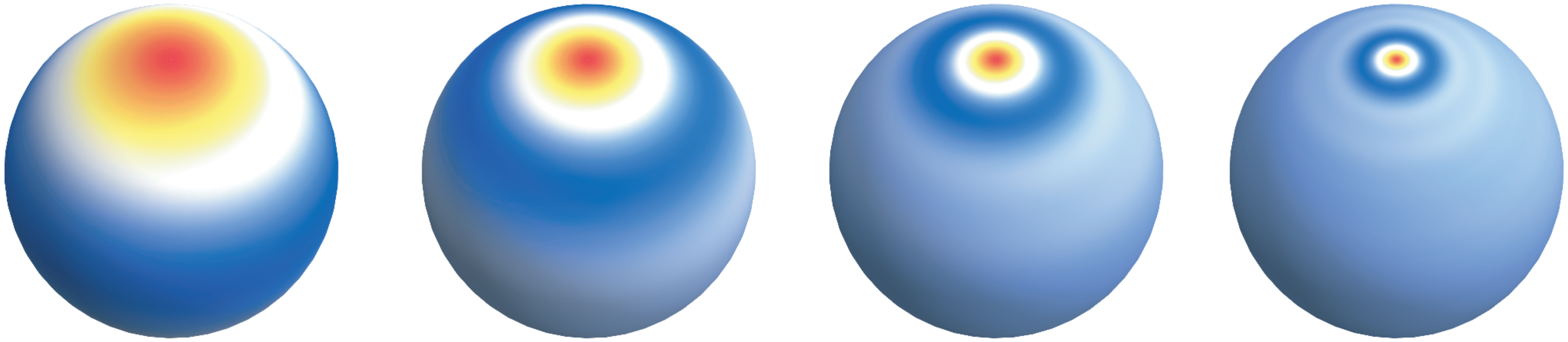
Simulation of shallow water equation



ψ_1 ψ_2 ψ_3 $\dots$

$$\psi_j^{0,\delta}(\omega) = \sum_{l=0}^{2^j} \sum_{m=-l}^{l} \psi_{lm}^j \, y_{lm}(\omega)$$

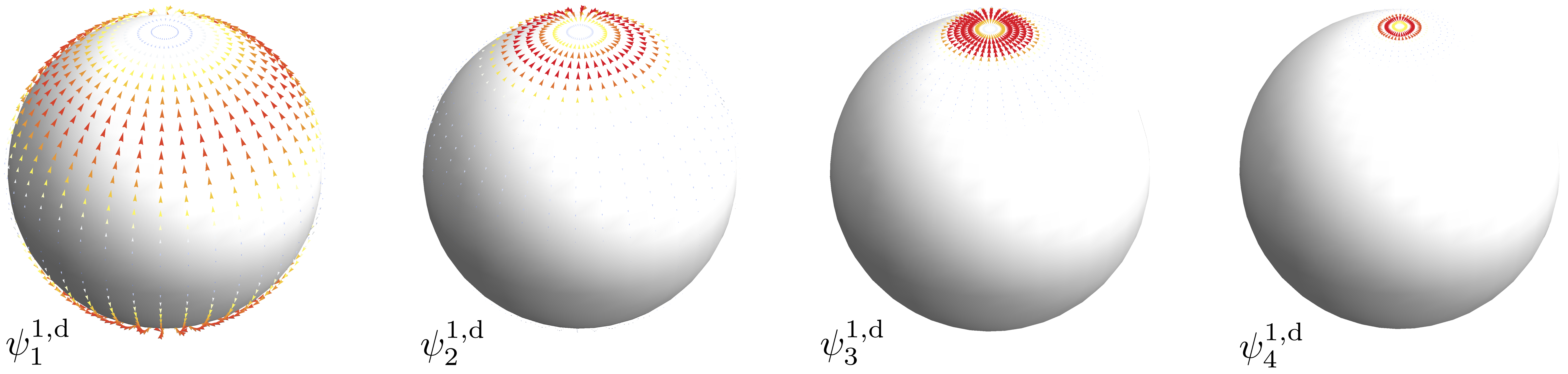
Simulation of shallow water equation



ψ_1 ψ_2 ψ_3 $\cdots$

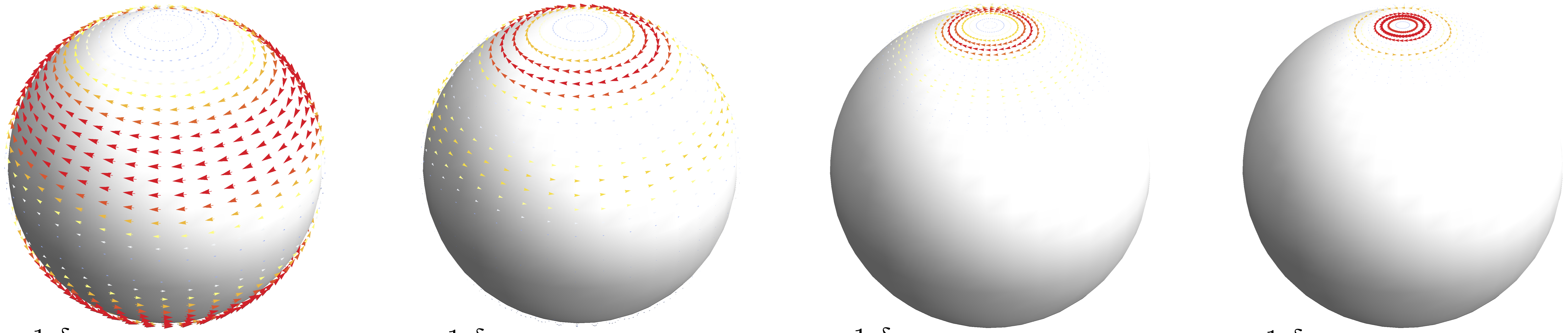
$$\mathrm{d}\psi_j^{0,\delta}(\omega) = \sum_{l=0}^{2^j} \sum_{m=-l}^{l} \psi_{lm}^j \, \mathrm{d}y_{lm}(\omega)$$

Simulation of shallow water equation



$$\psi_j^{1,\mathrm{d}}(\omega) = \sum_{l=0}^{2^j} \sum_{m=-l}^{l} \psi_{lm}^j \, \mathrm{d}y_{lm}(\omega)$$

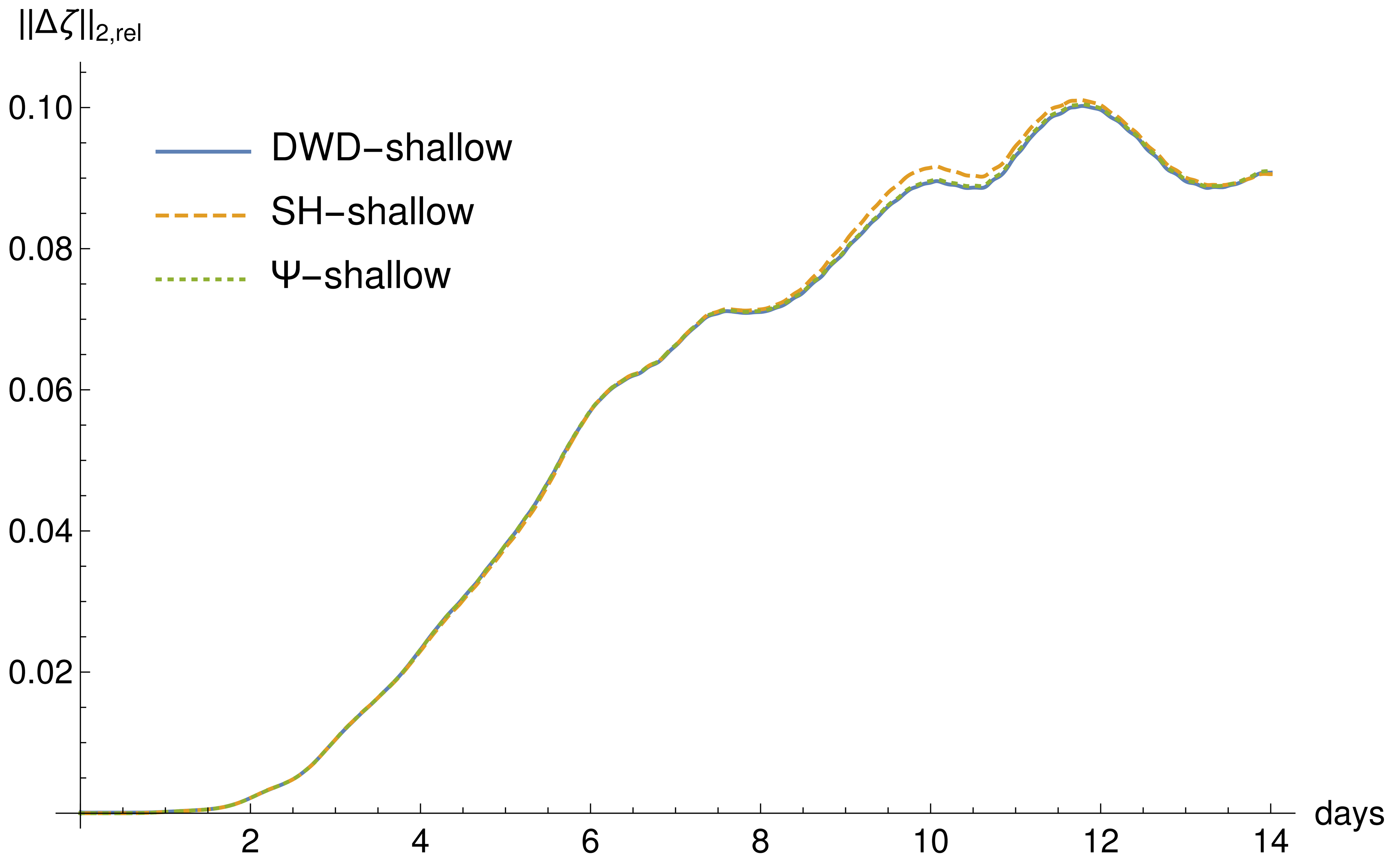
Simulation of shallow water equation

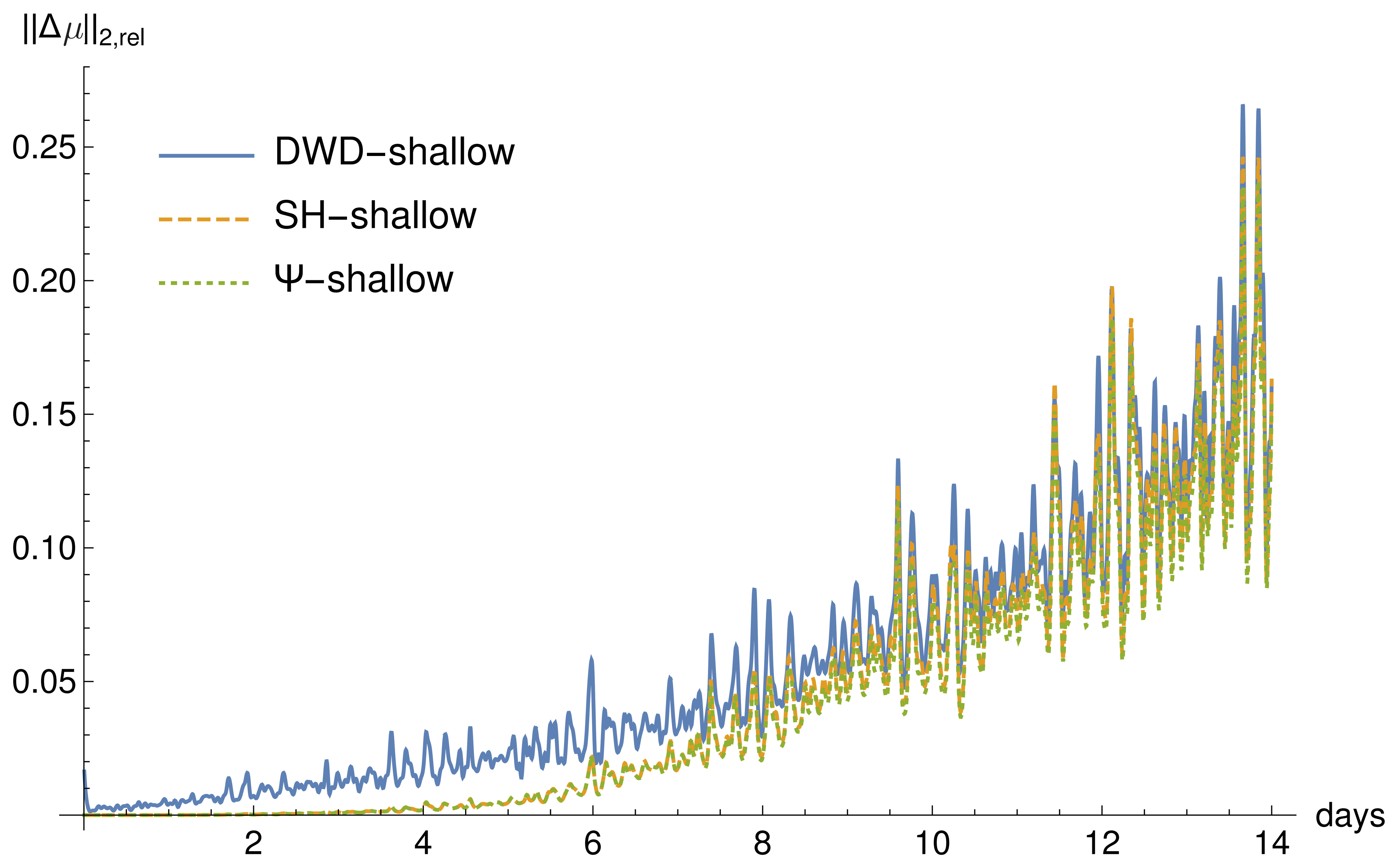


$\psi_1^{1,\delta}$ $\psi_2^{1,\delta}$ $\psi_3^{1,\delta}$ $\psi_4^{1,\delta}$

$$\star\,\psi_j^{1,\mathrm{d}}(\omega) = \sum_{l=0}^{2^j} \sum_{m=-l}^{l} \psi_{lm}^{j} \star \mathrm{d}y_{lm}(\omega)$$

Simulation of shallow water equation





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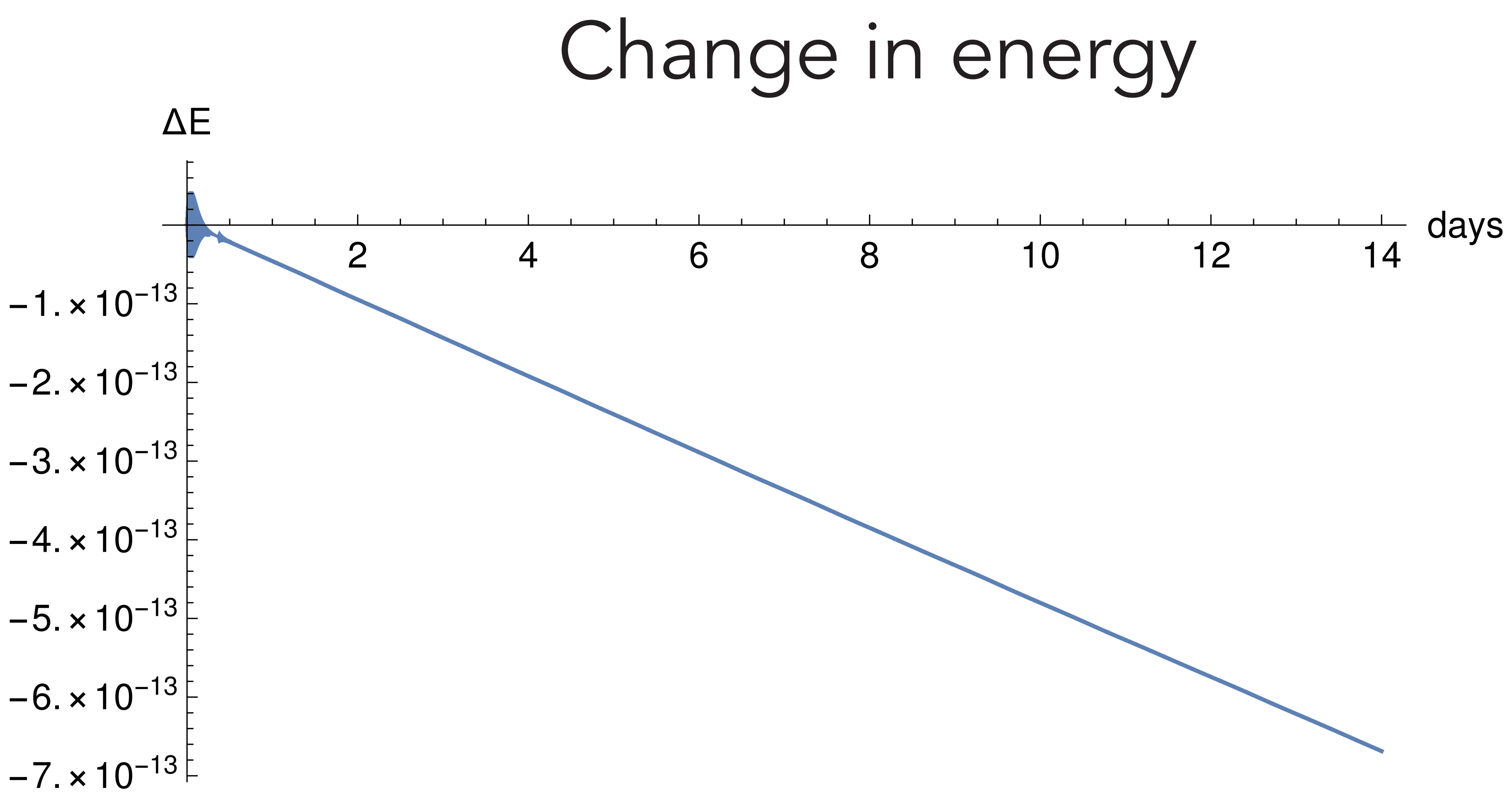

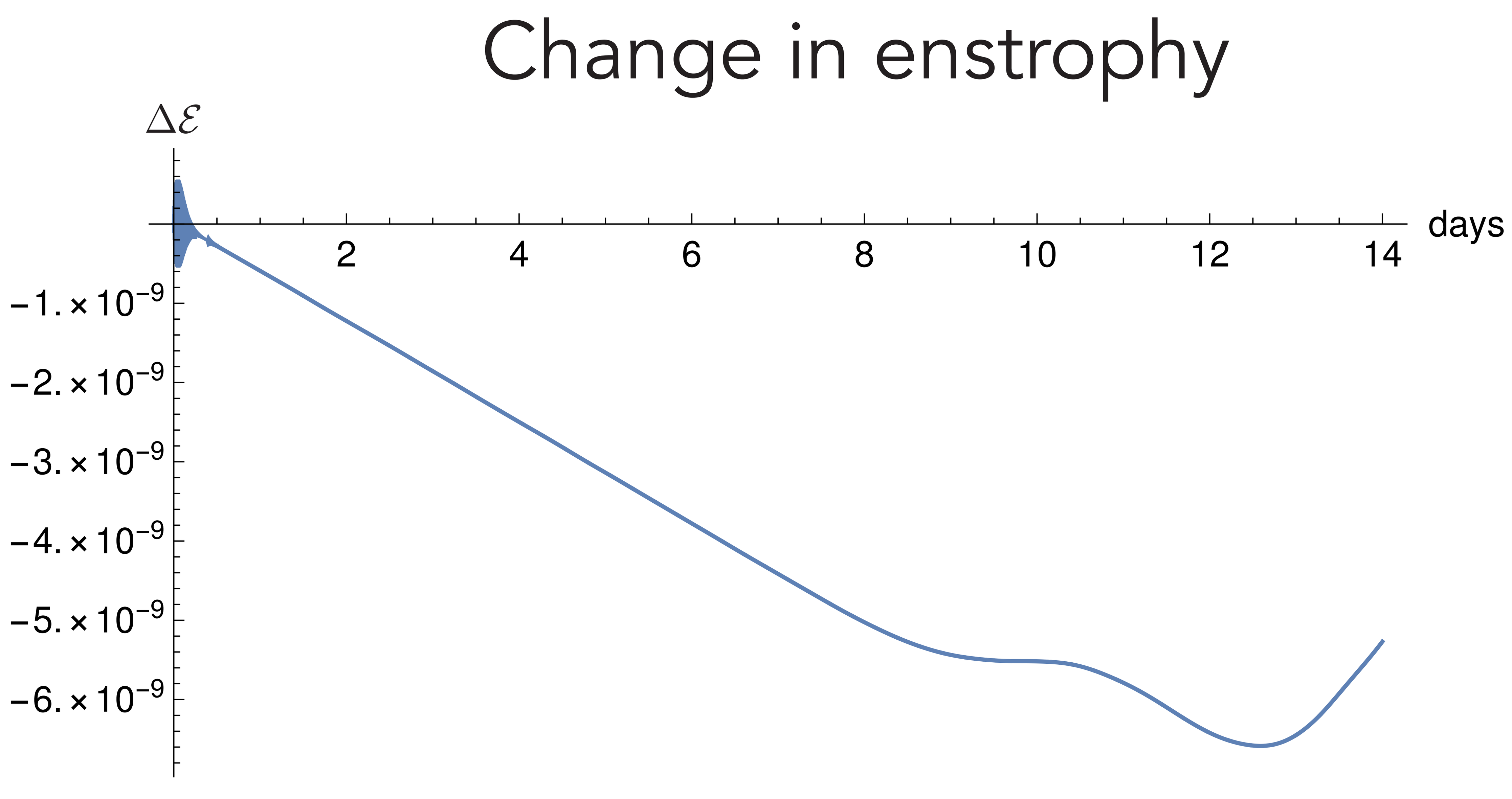

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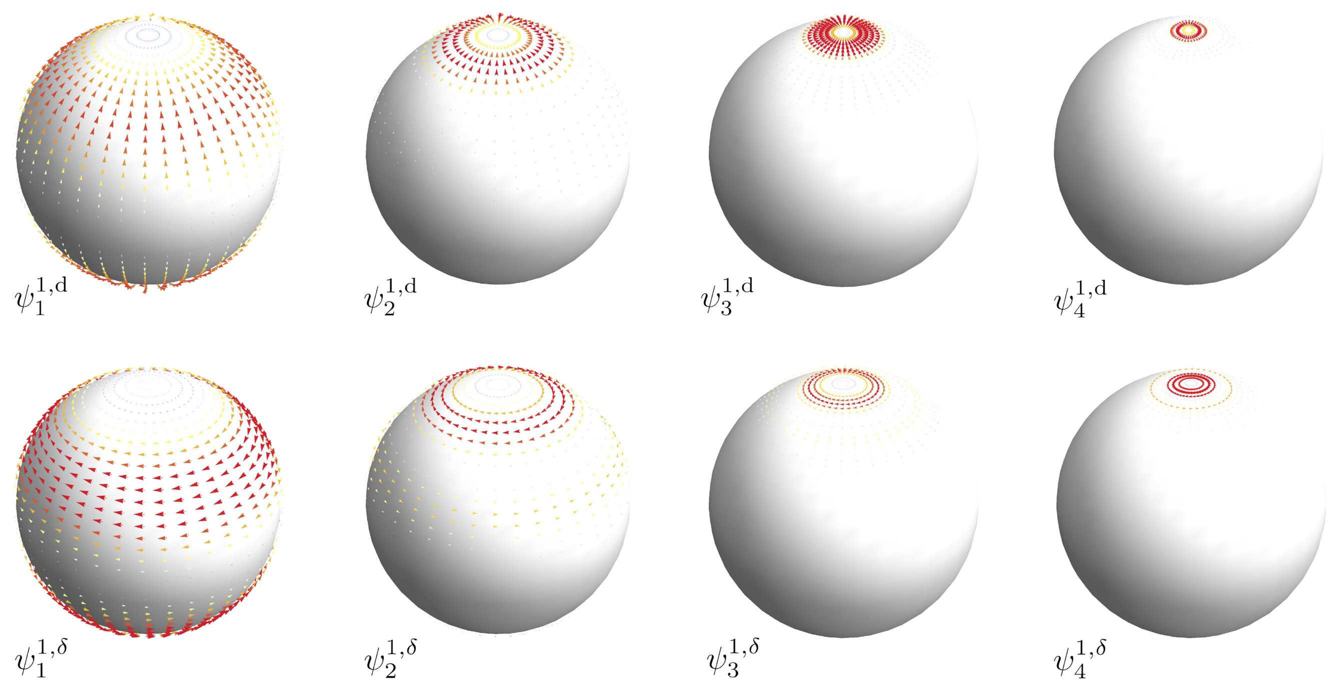
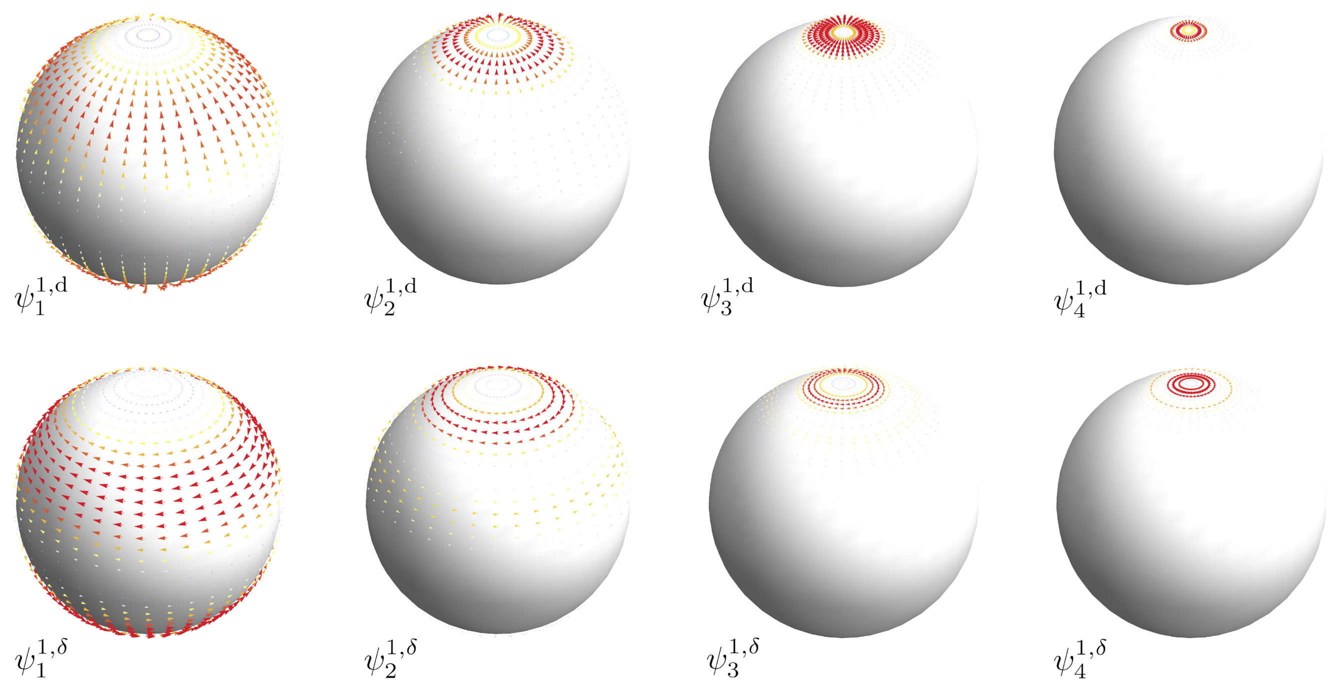
Data-assisted climate simulation

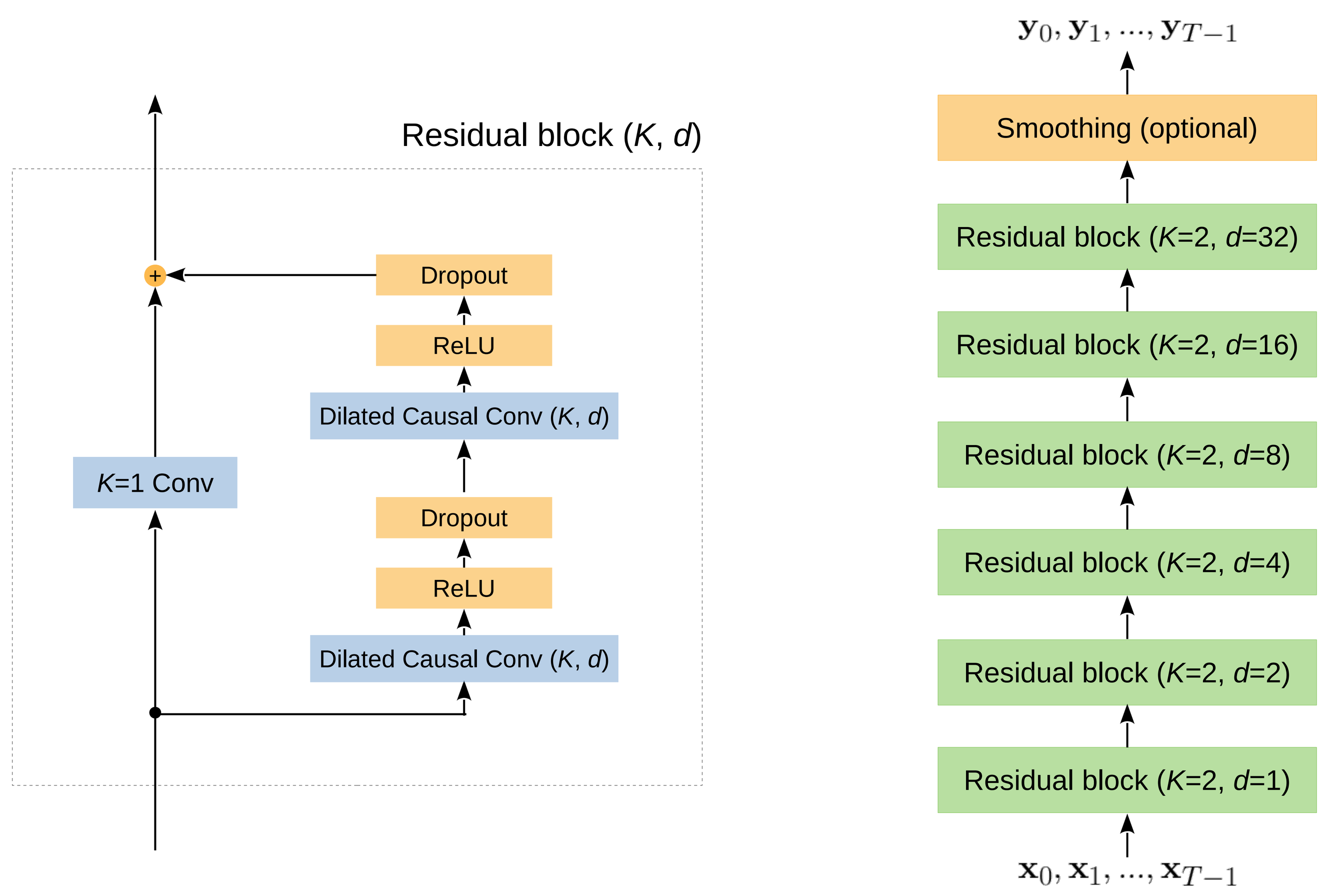


simulation

Data-assisted climate simulation



simulation



ANN-based correction

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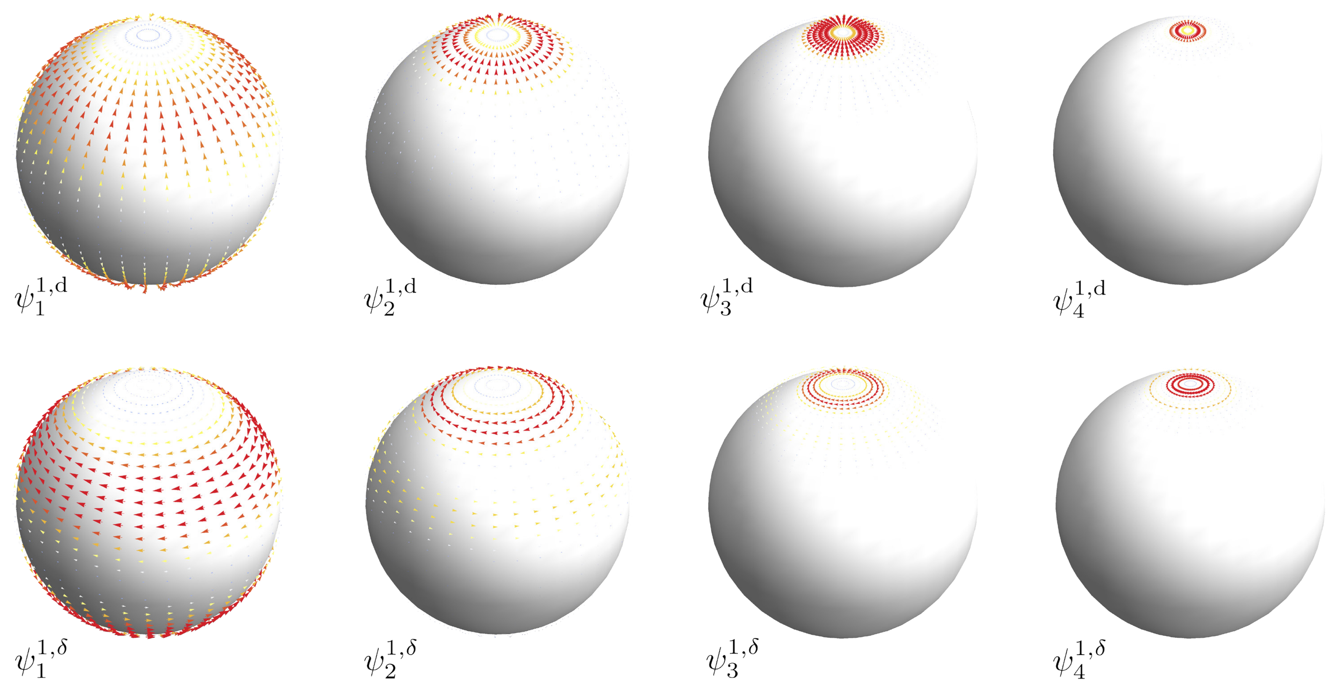
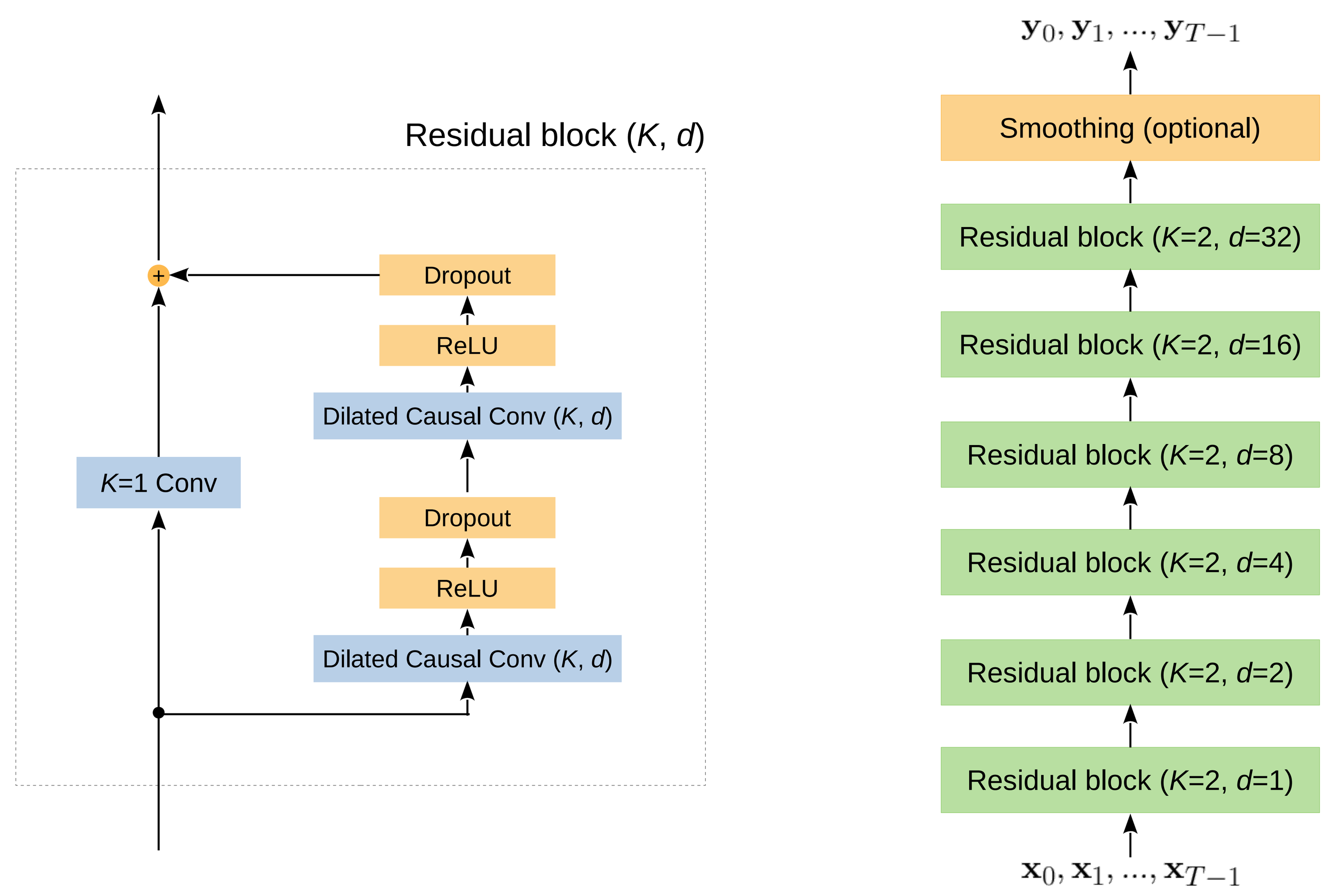


simulation



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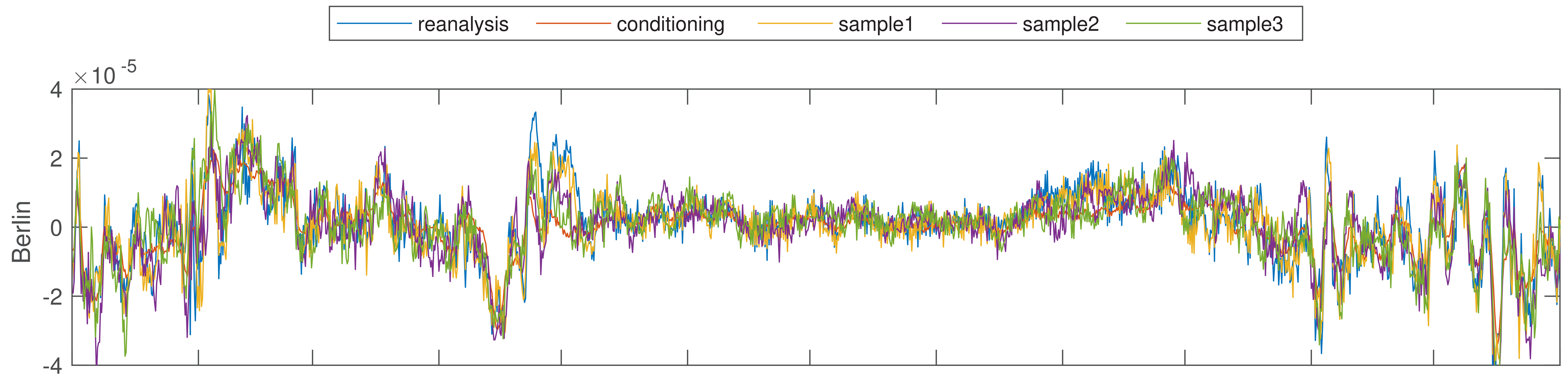

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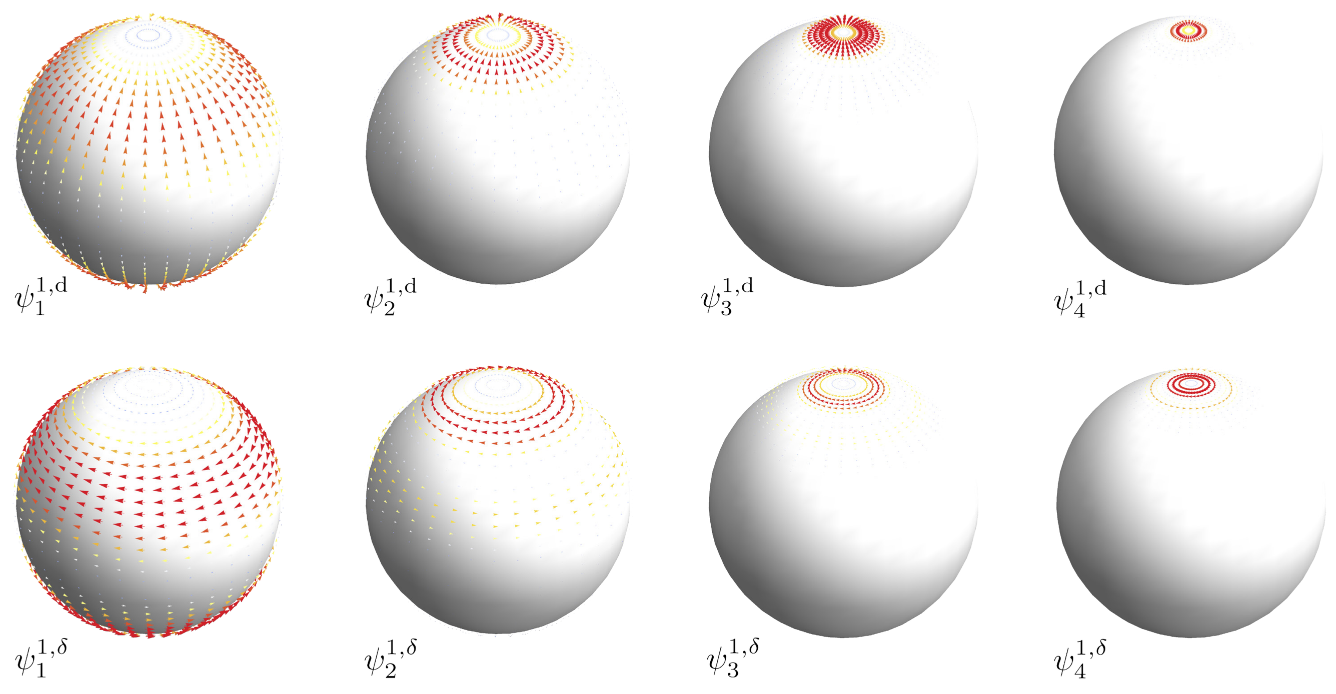

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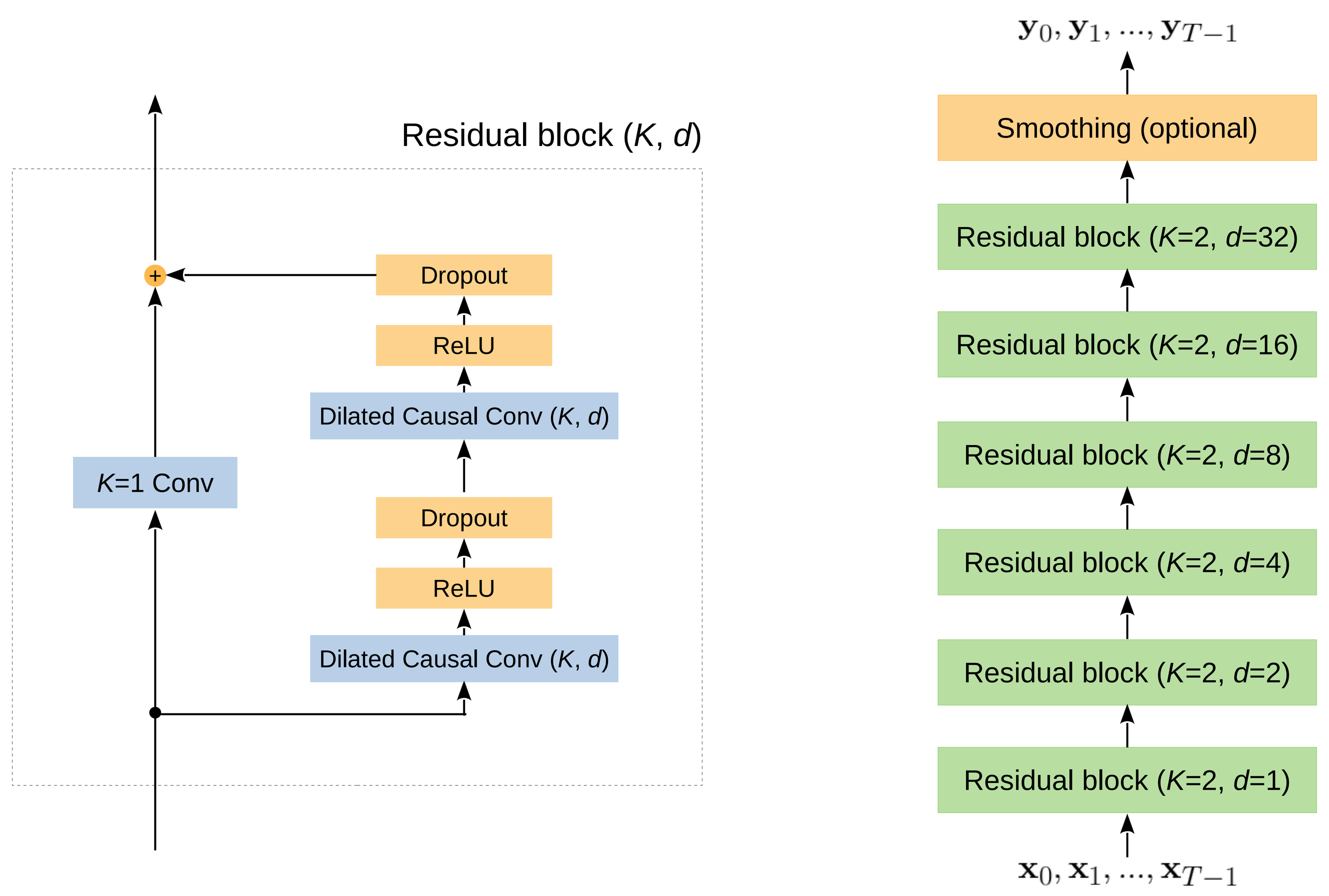

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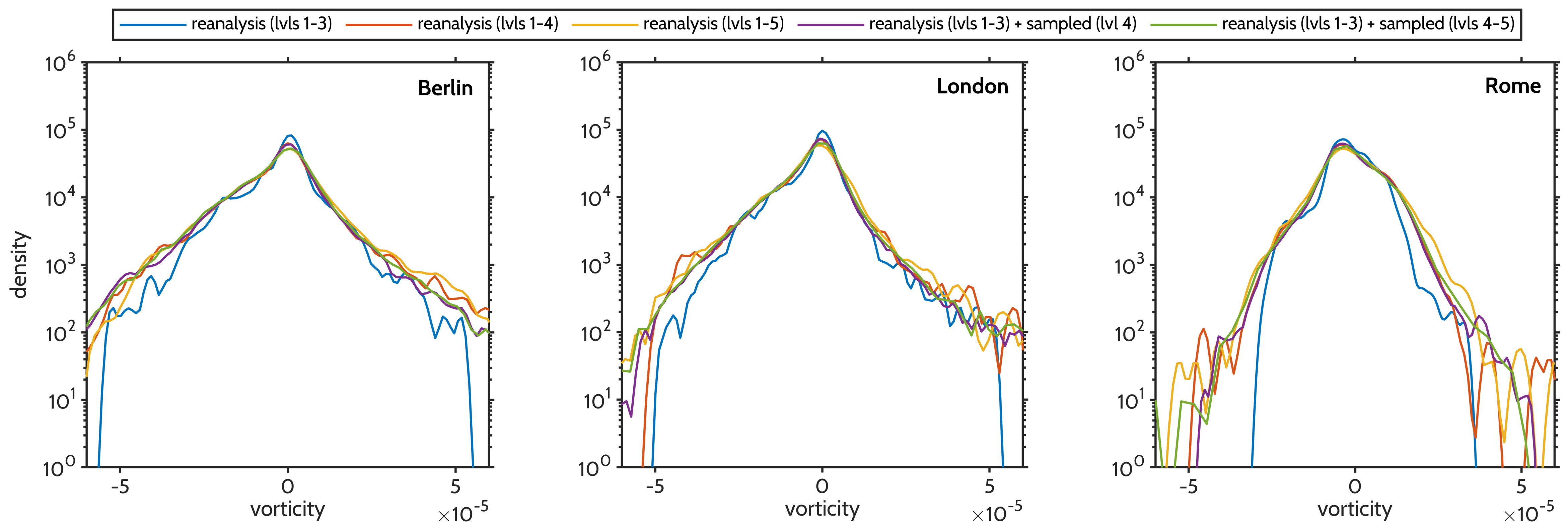

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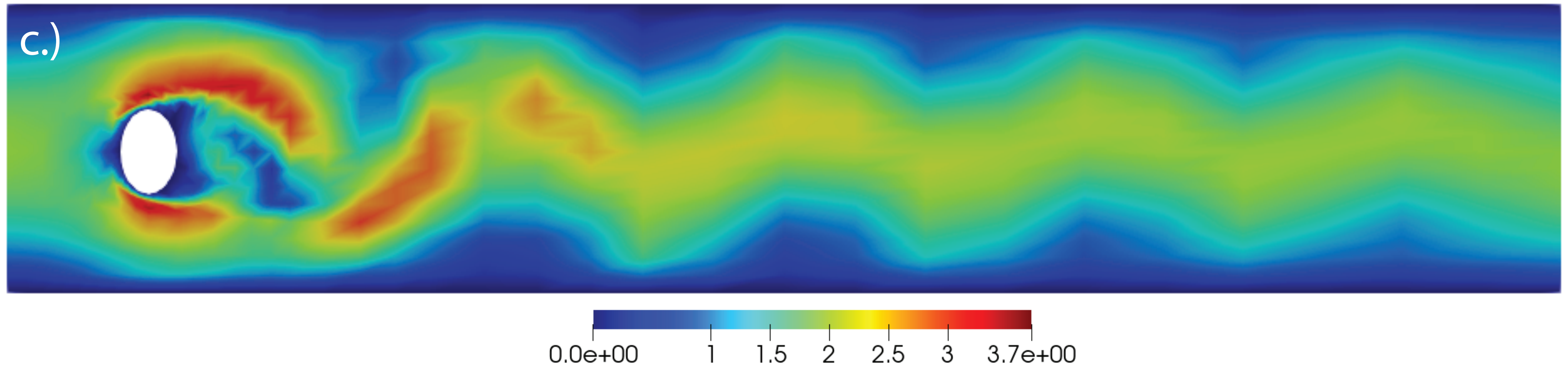
Summary

- $\Psi\mathrm{ec}$: A local spectral exterior calculus
  - Discretization of de Rham complex using differential form wavelets
  - Exterior calculus and sparsity are in a natural way compatible
  - Local support needed for practicality

DNN-MG



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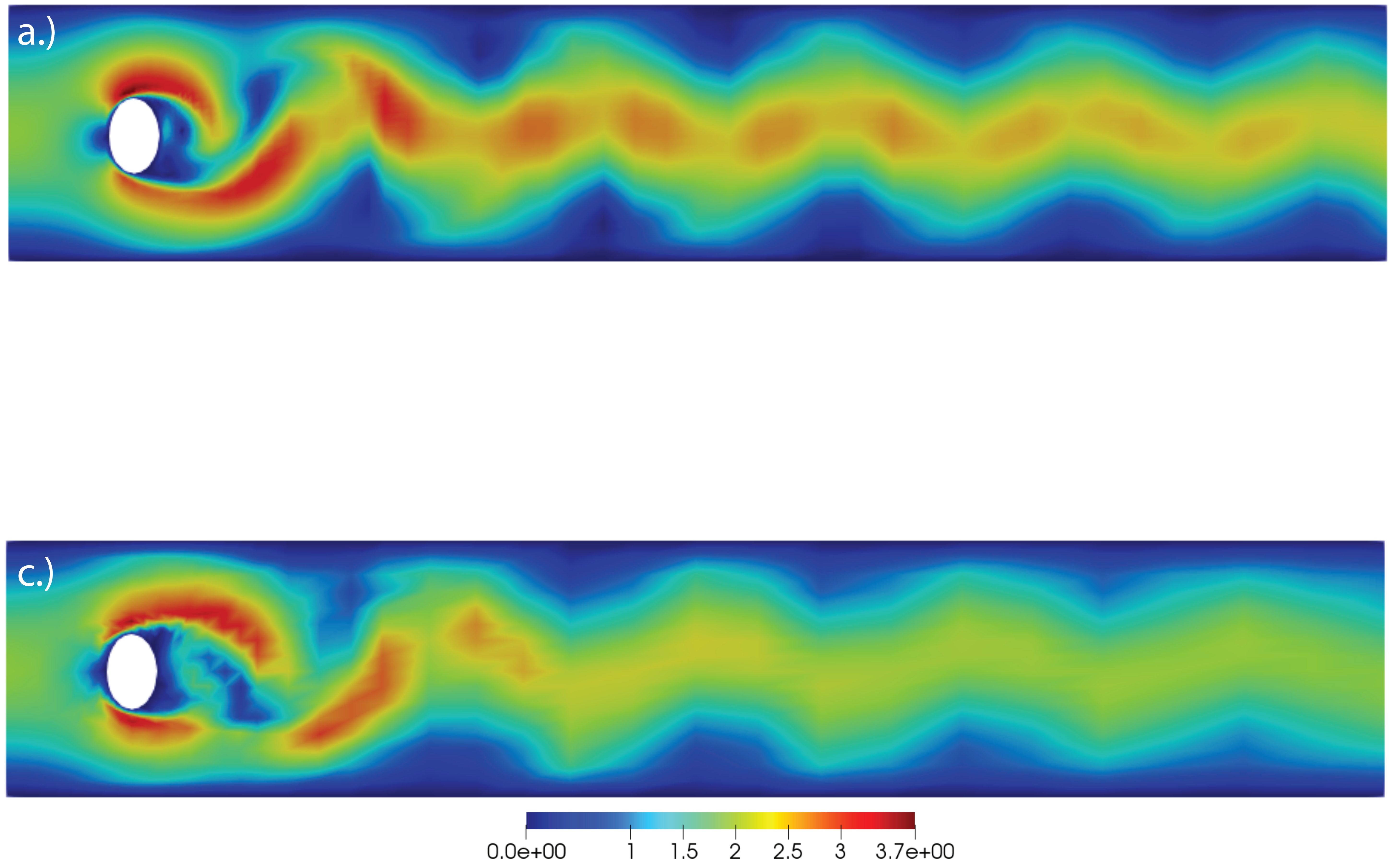

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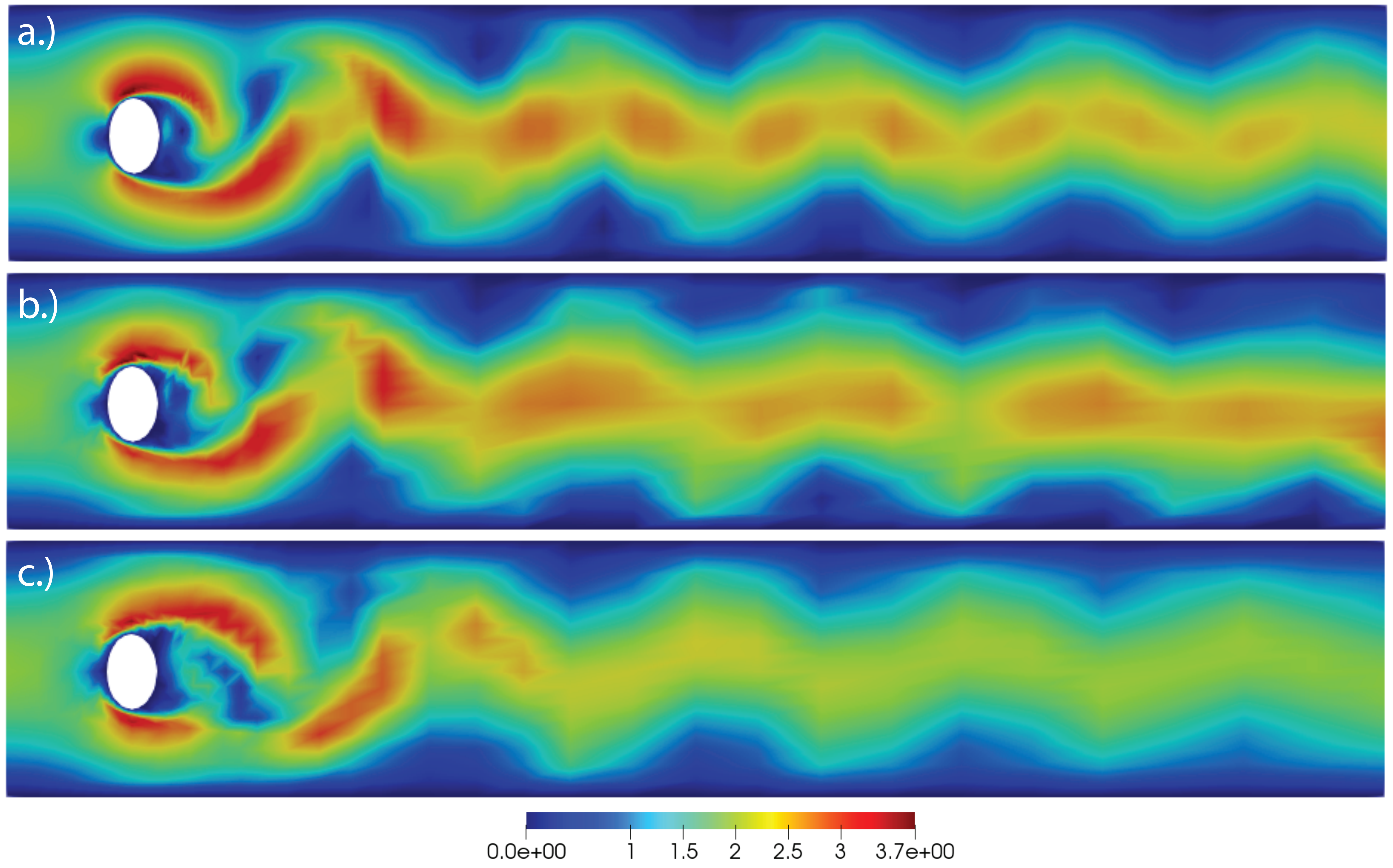

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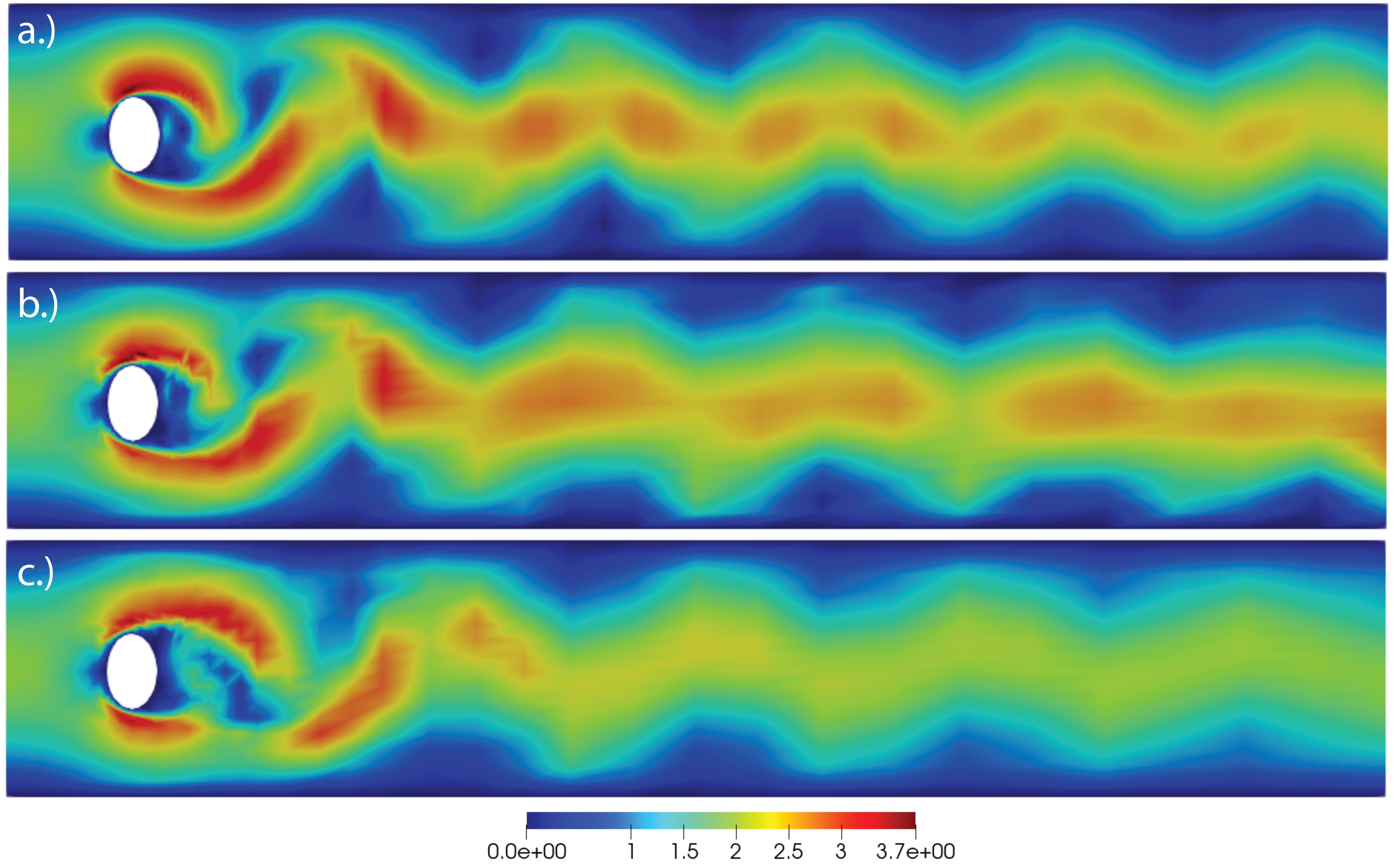

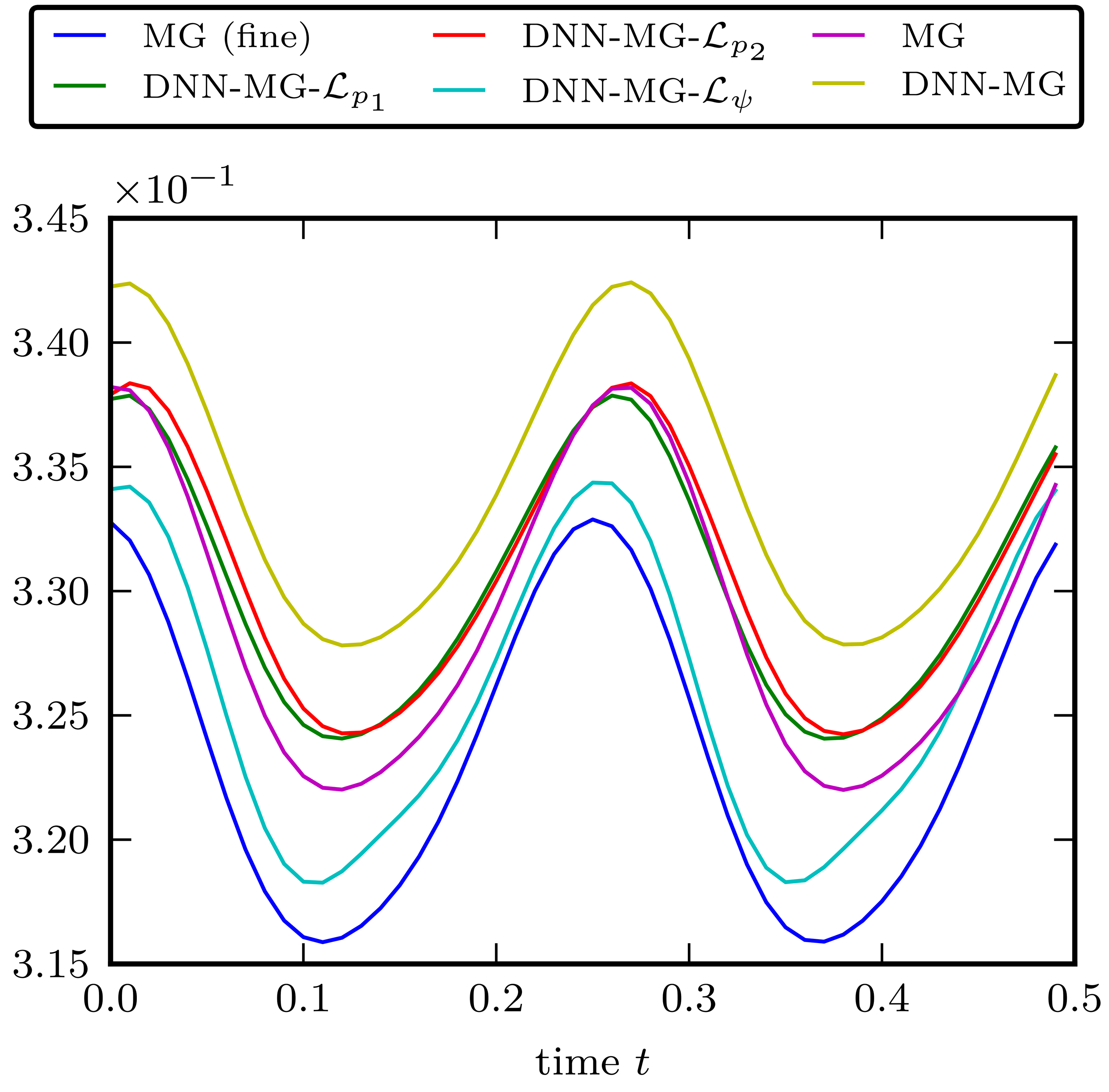

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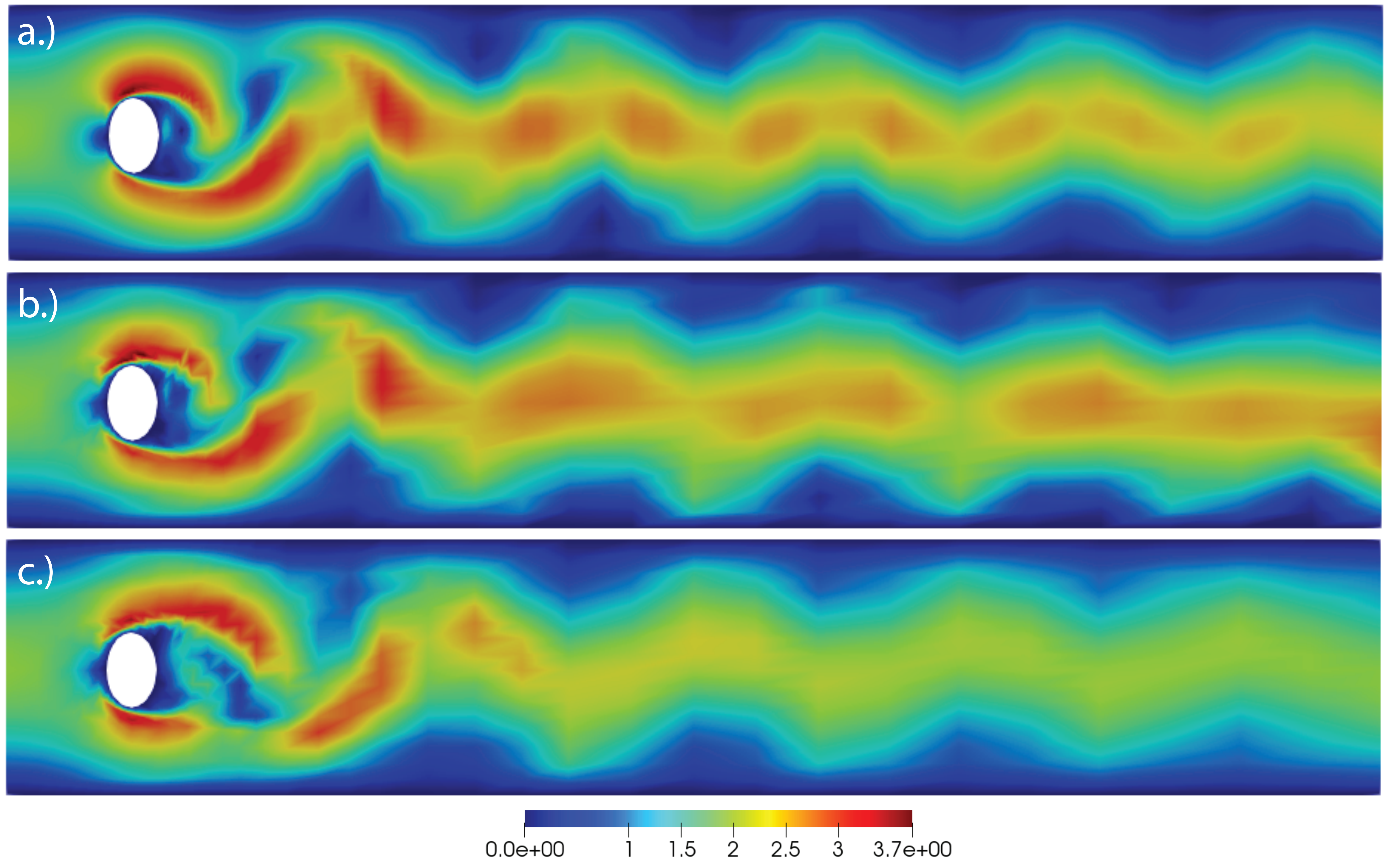

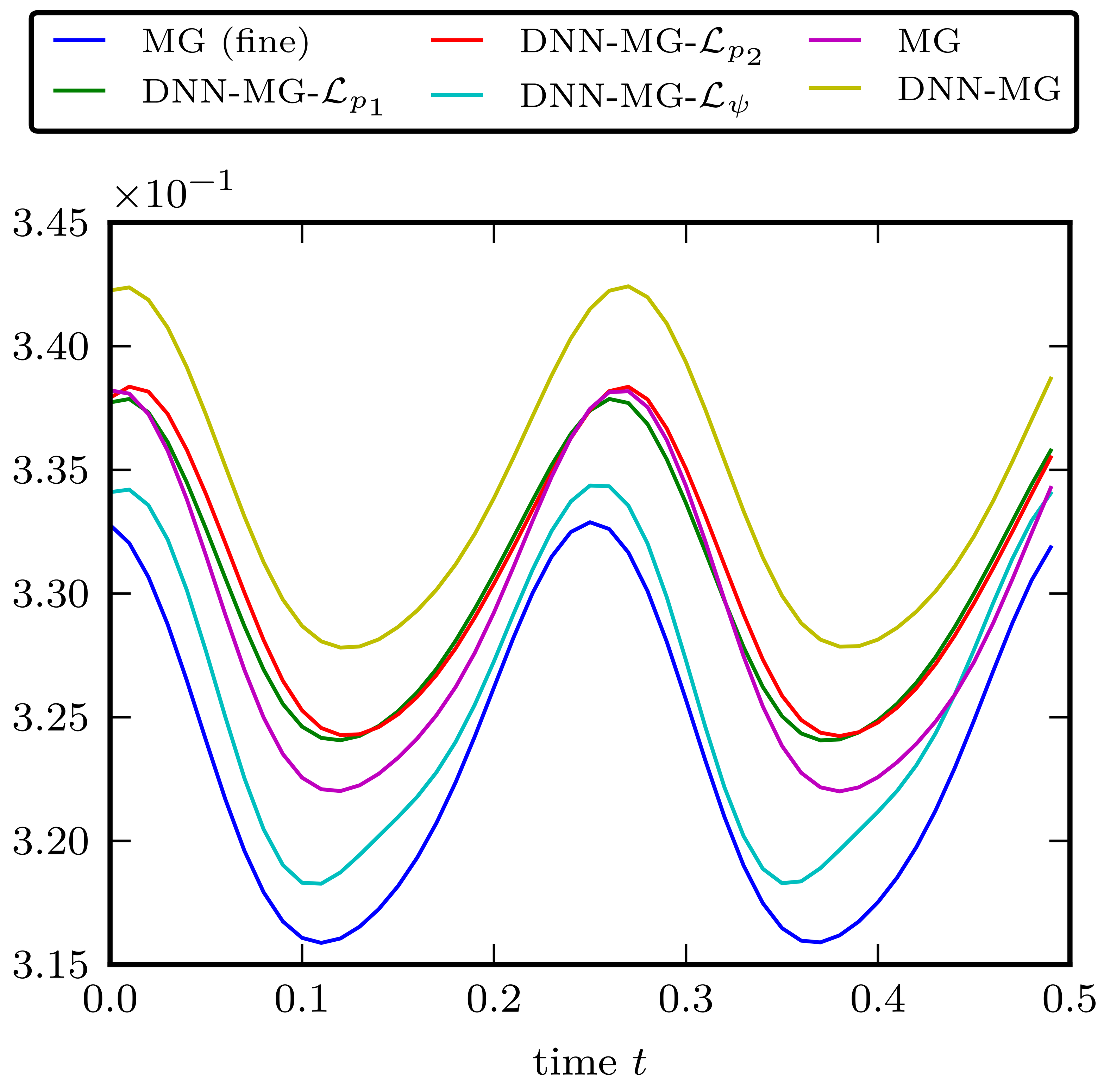

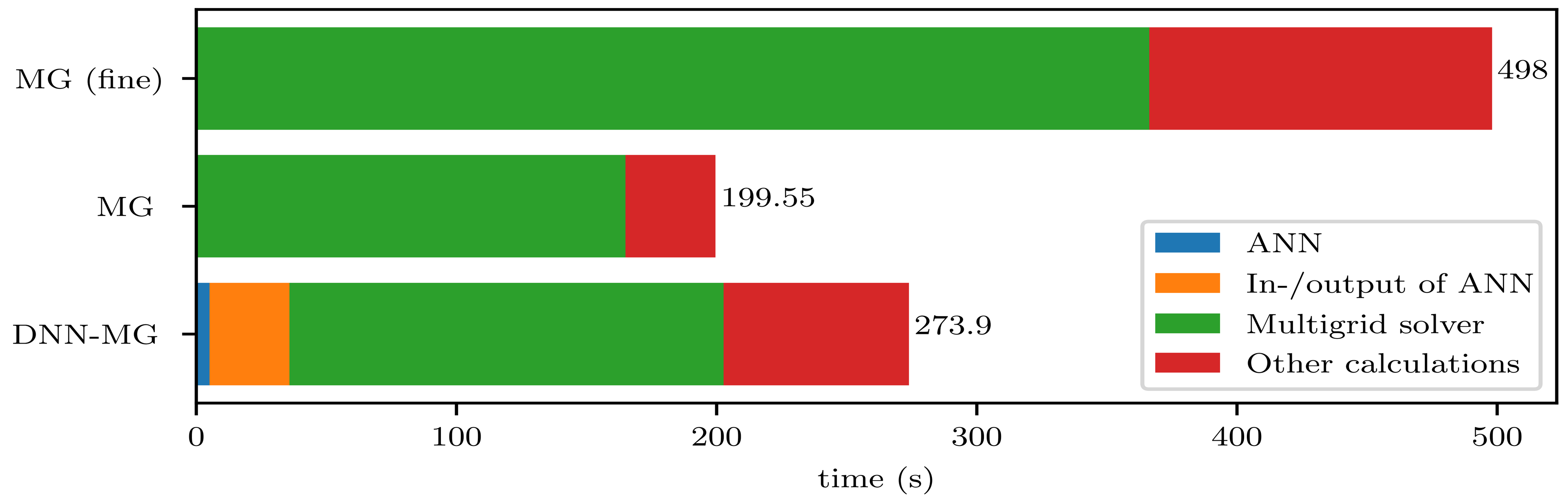

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