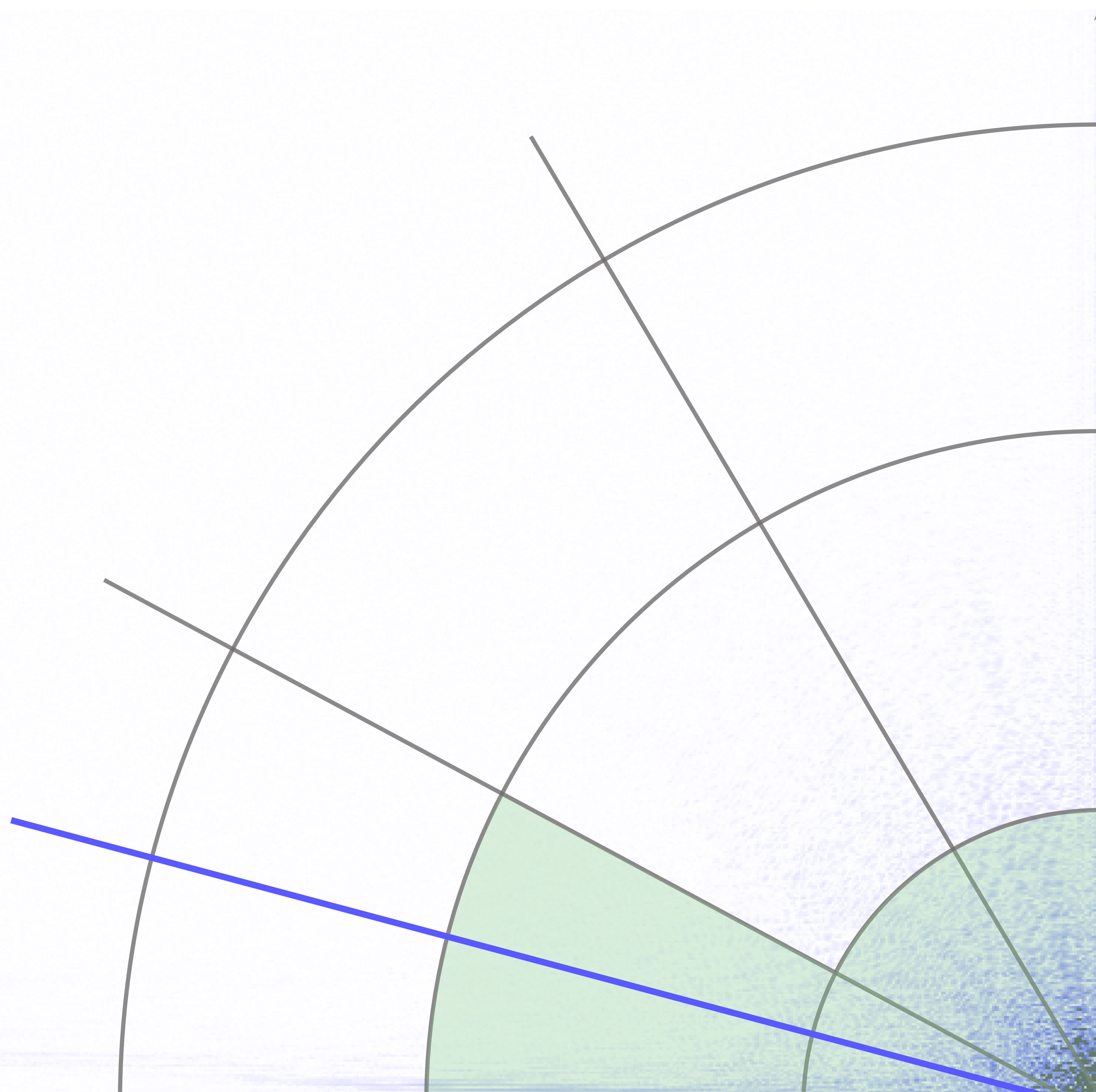
# Divergence Free Wavelets with Flexible Directional Localization

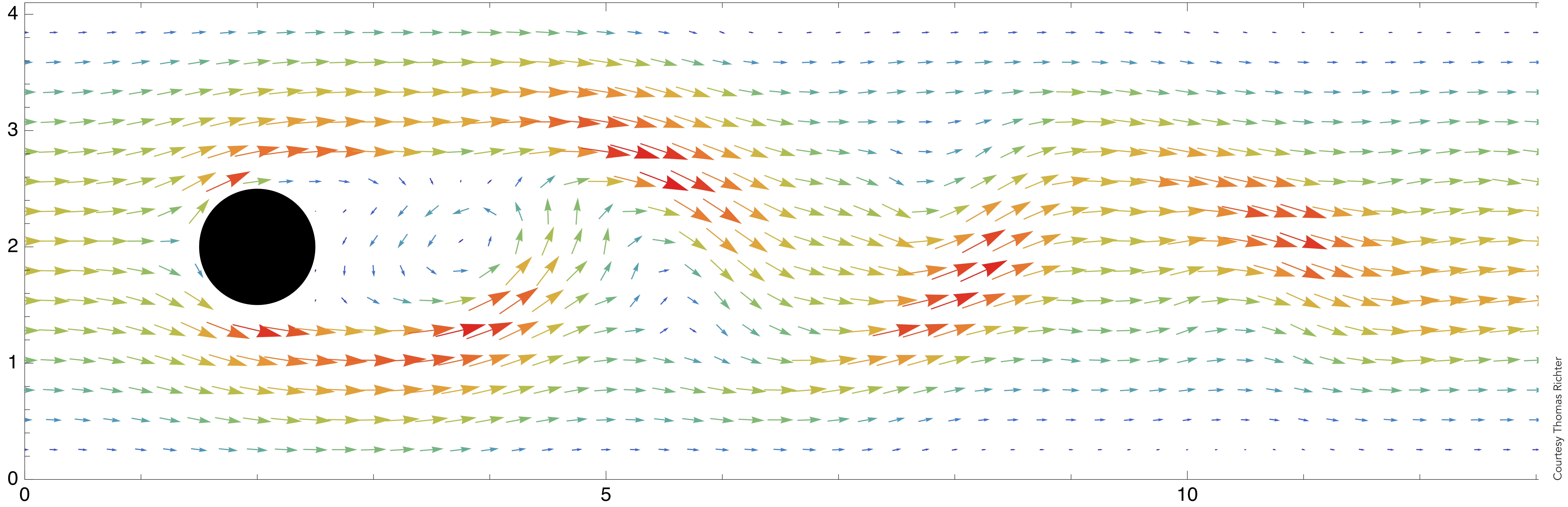
Christian Lessig

Otto-von-Guericke-Universität Magdeburg



# Divergence free wavelets

- Directional selectivity:



Courtesy Thomas Richter

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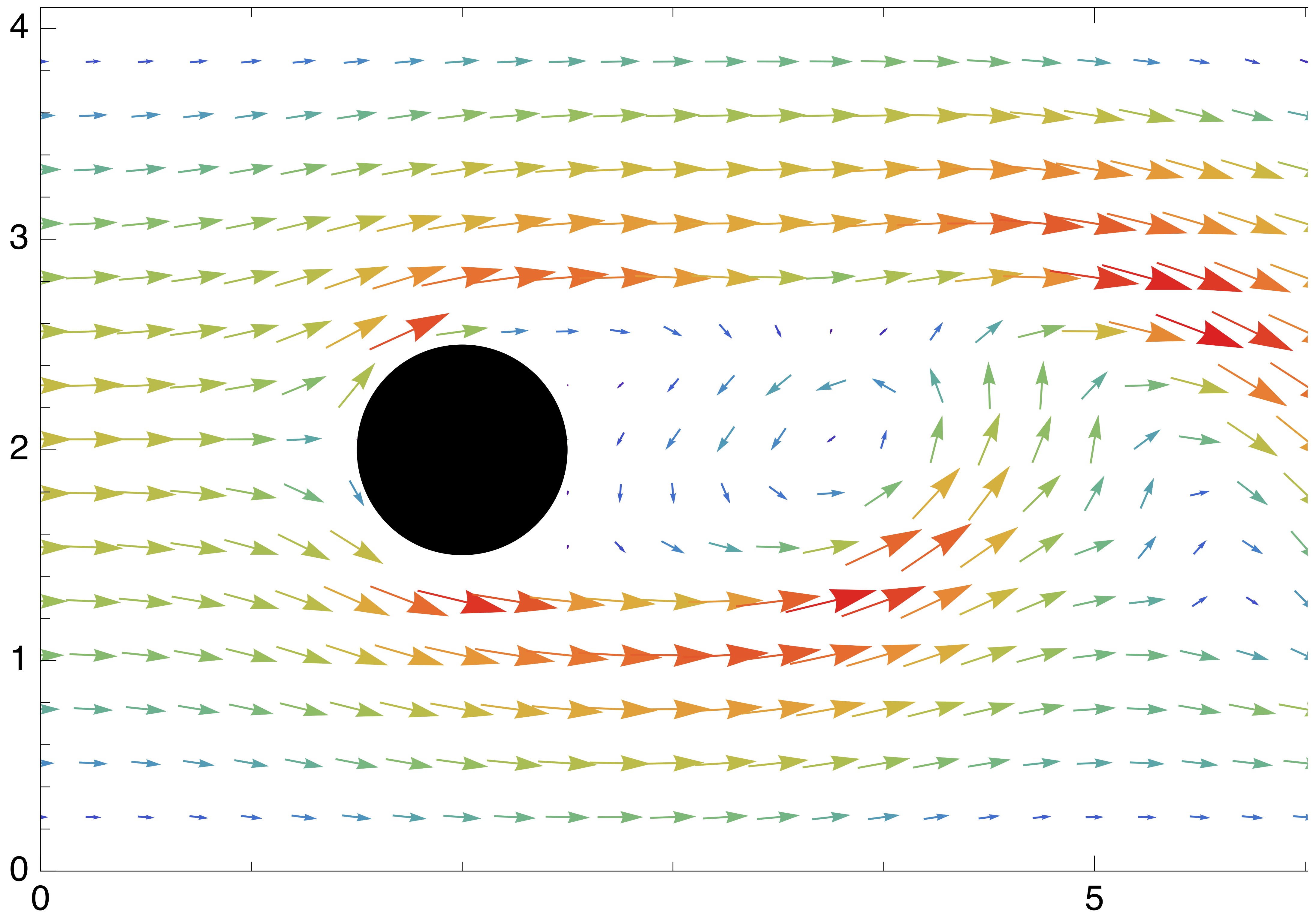

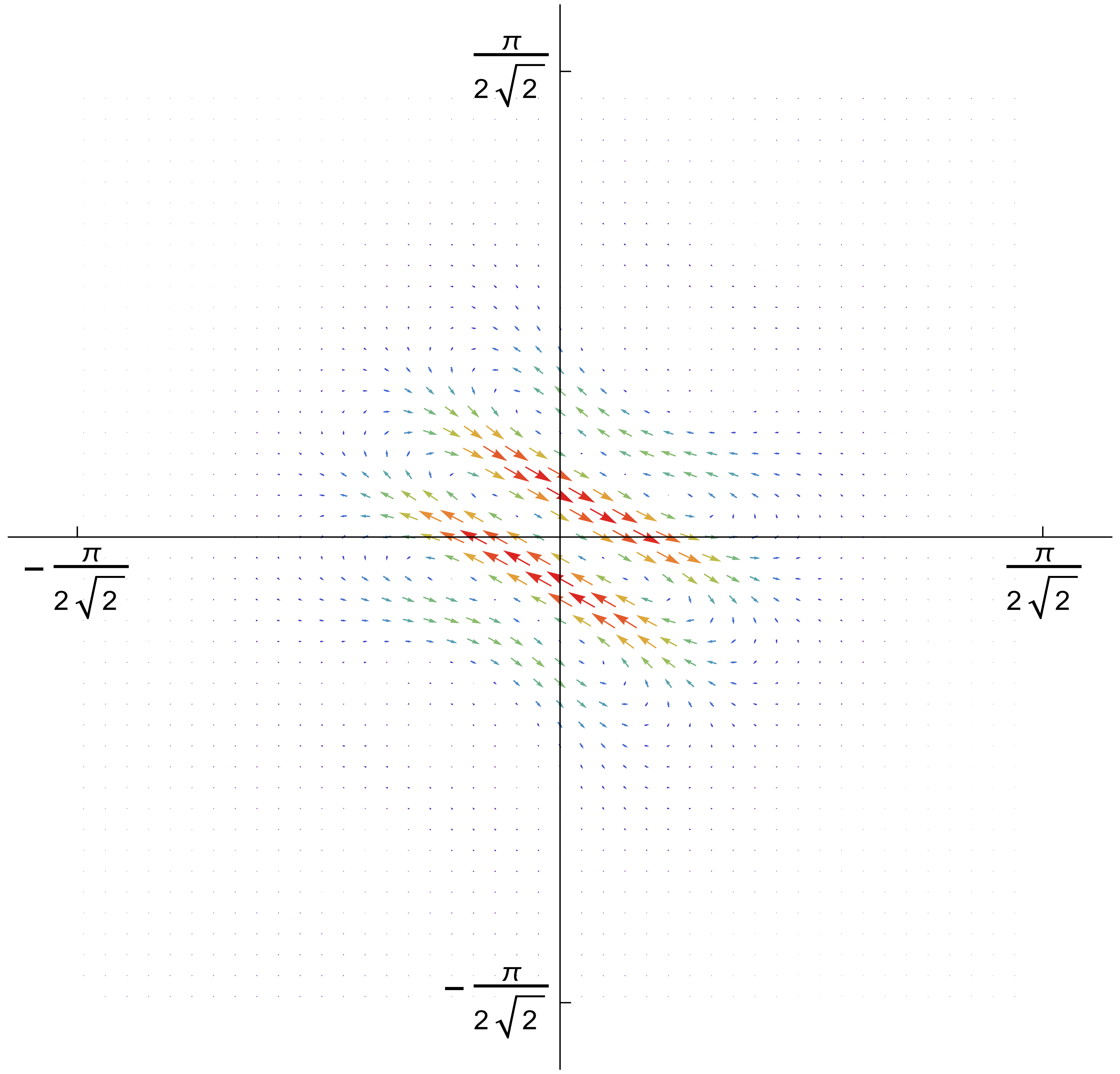

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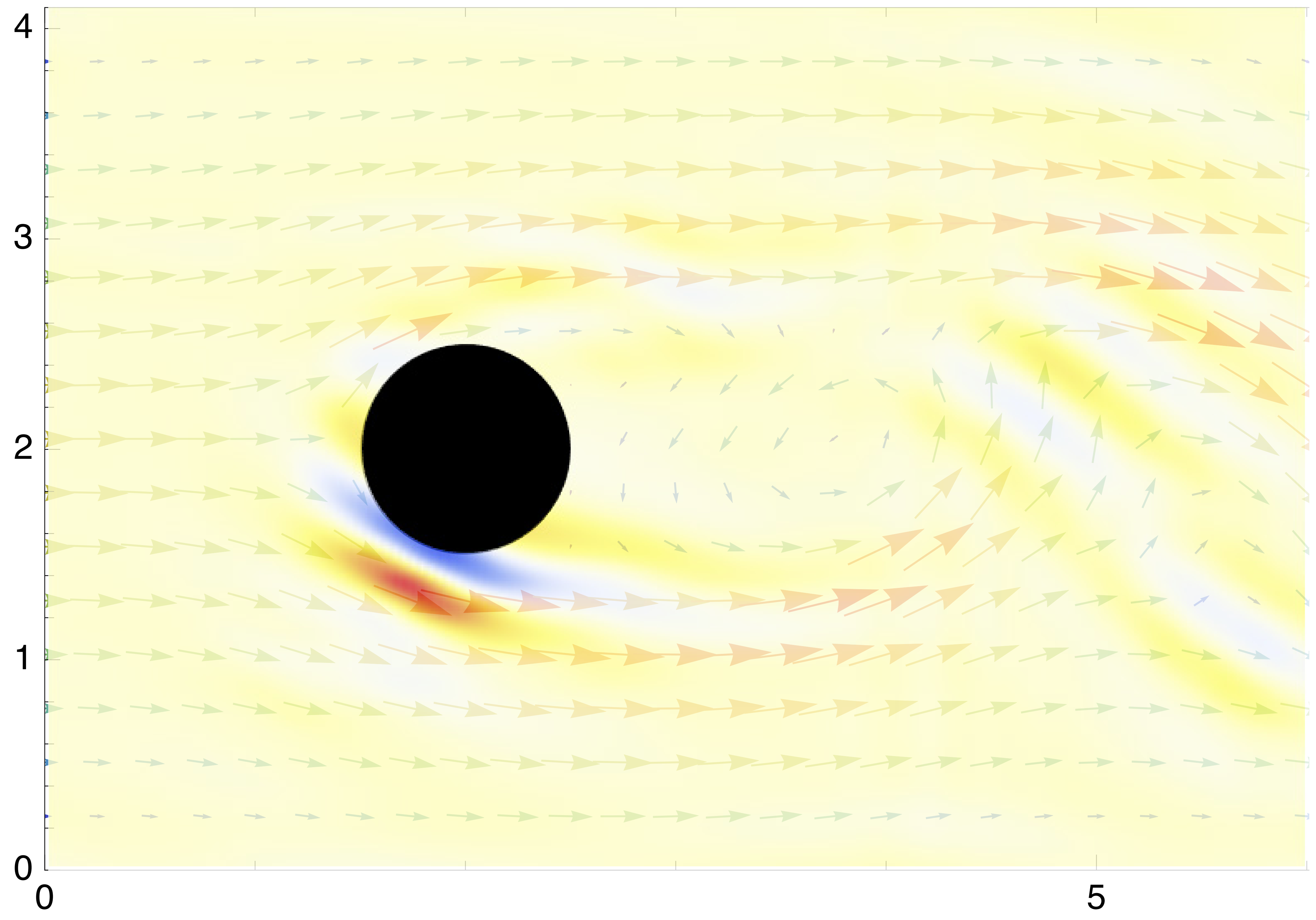

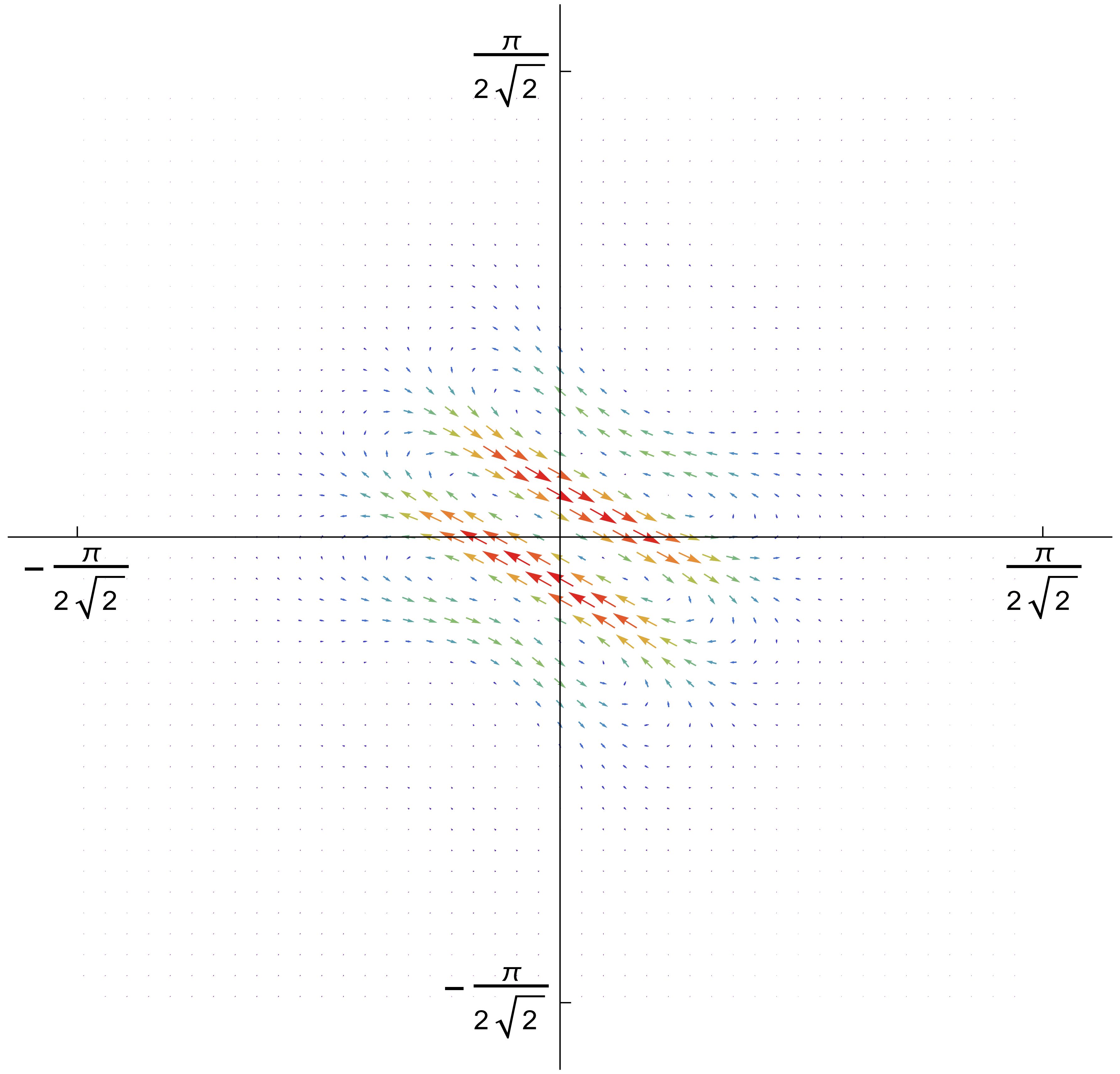

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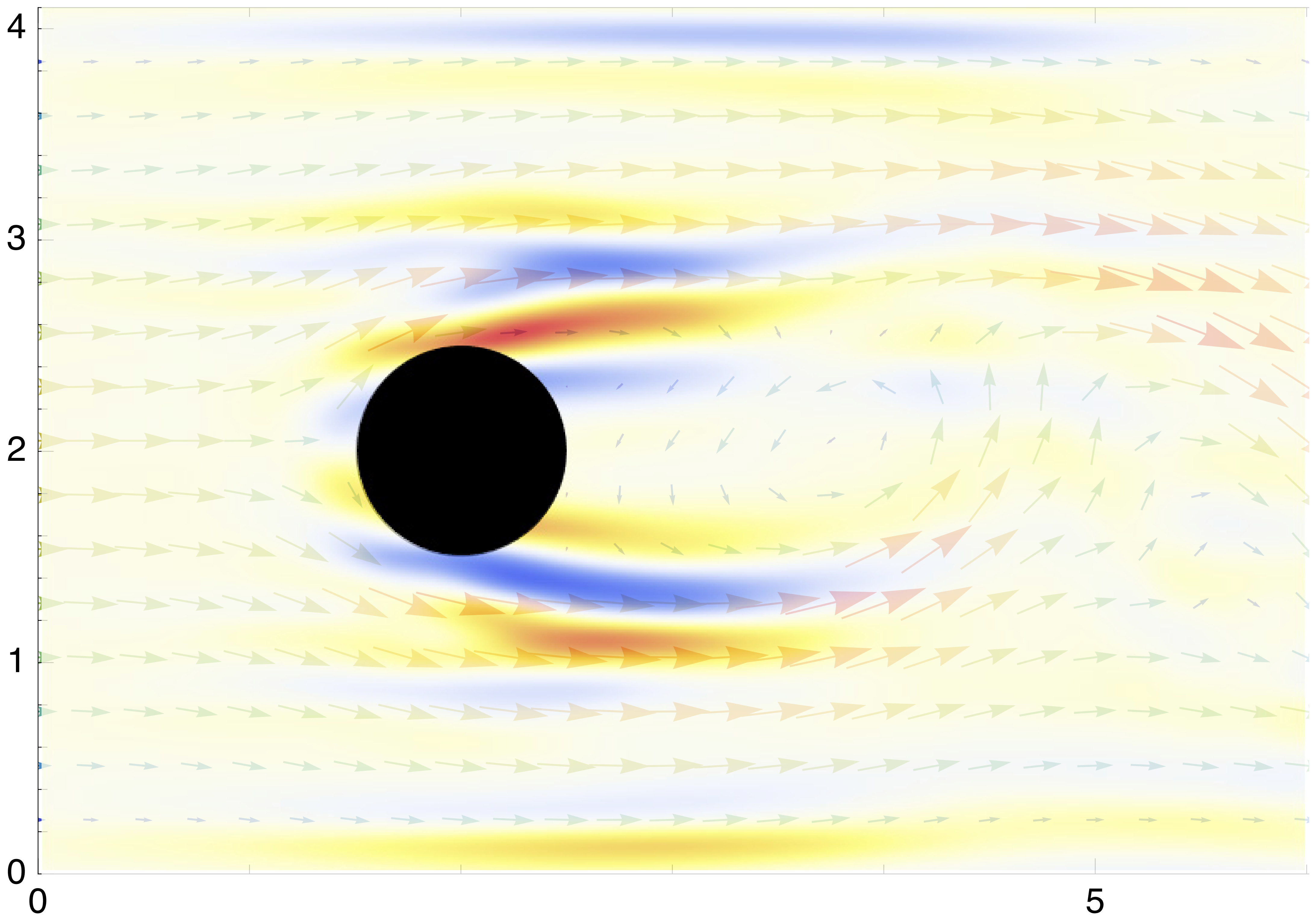

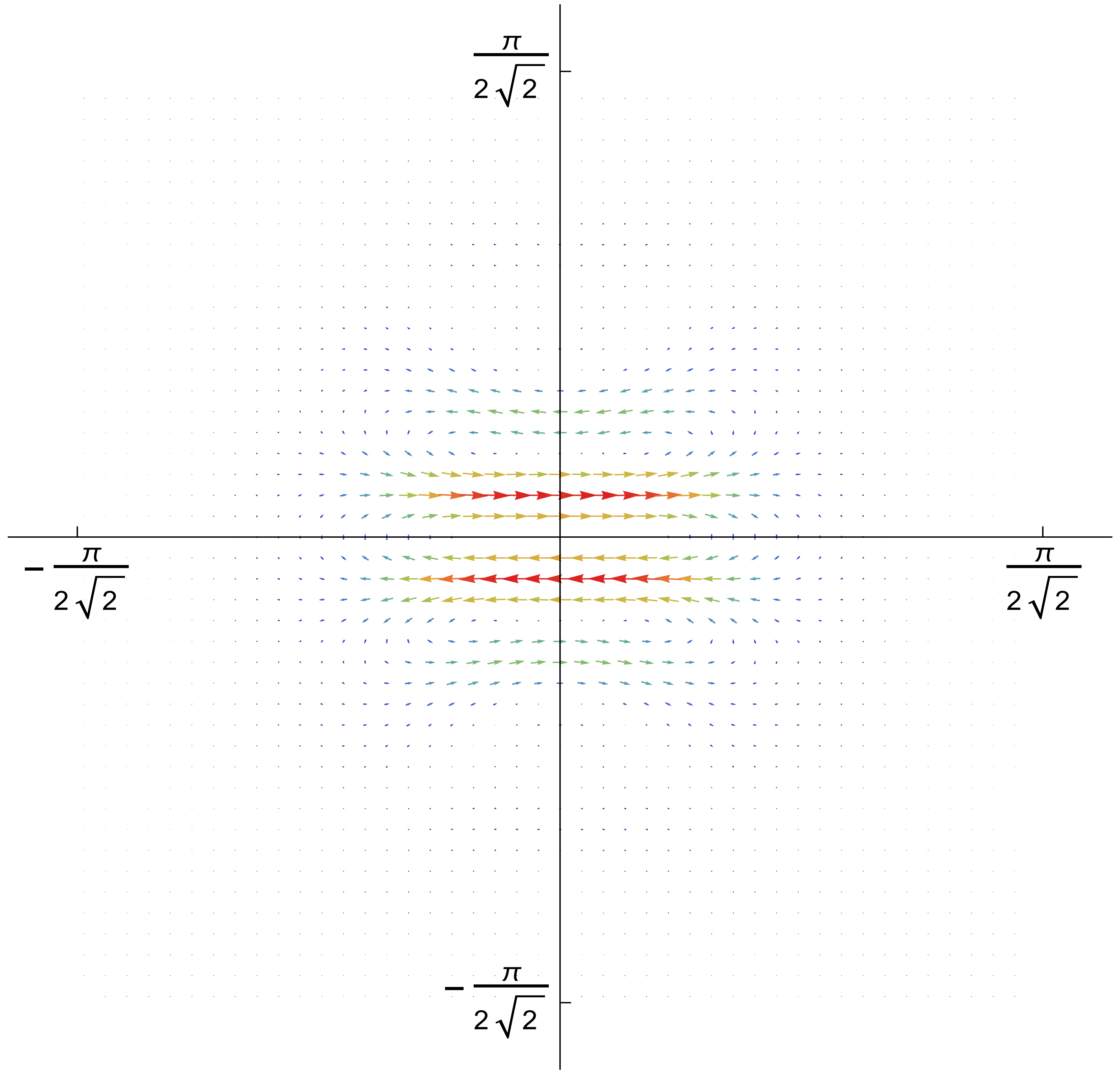

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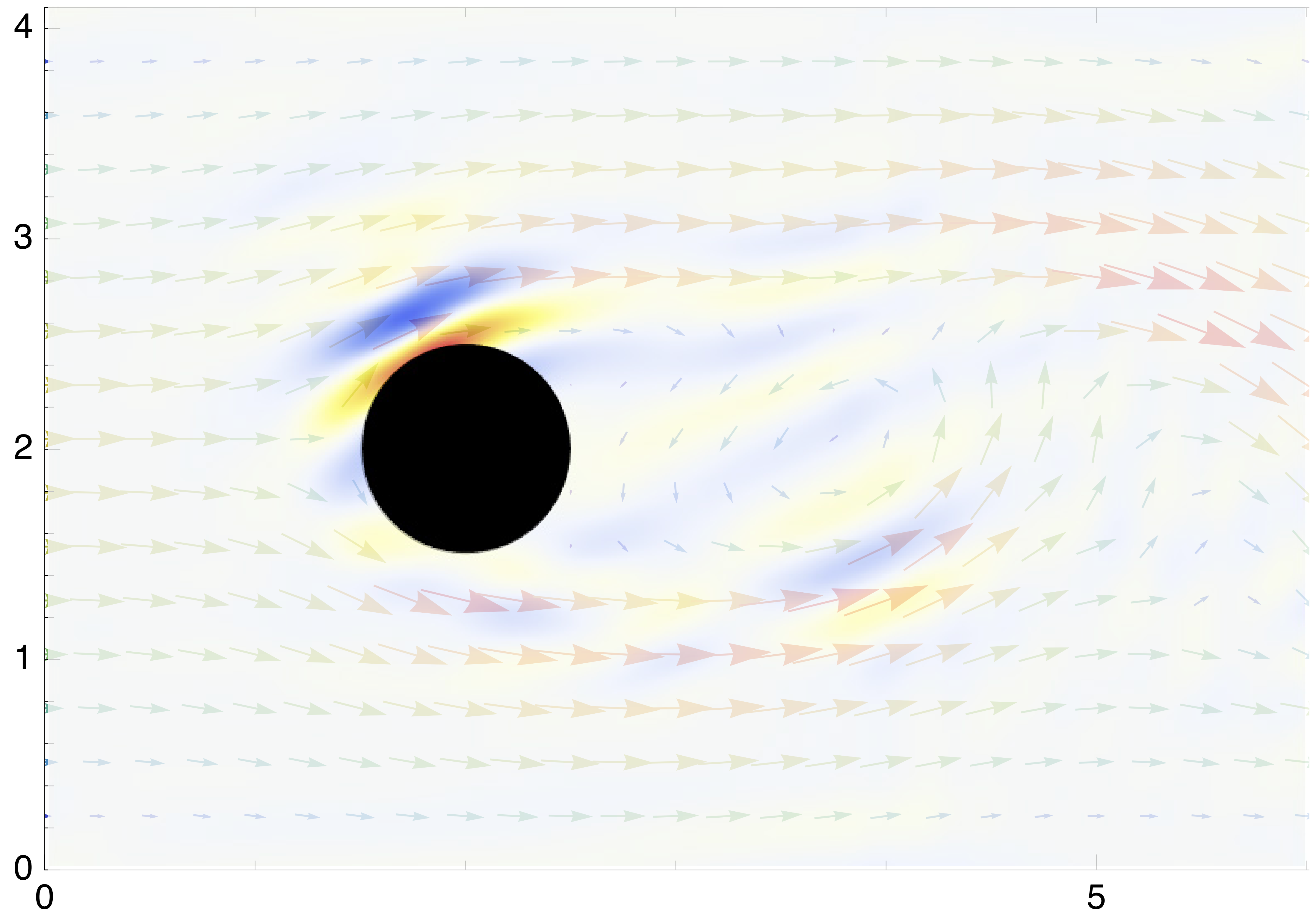
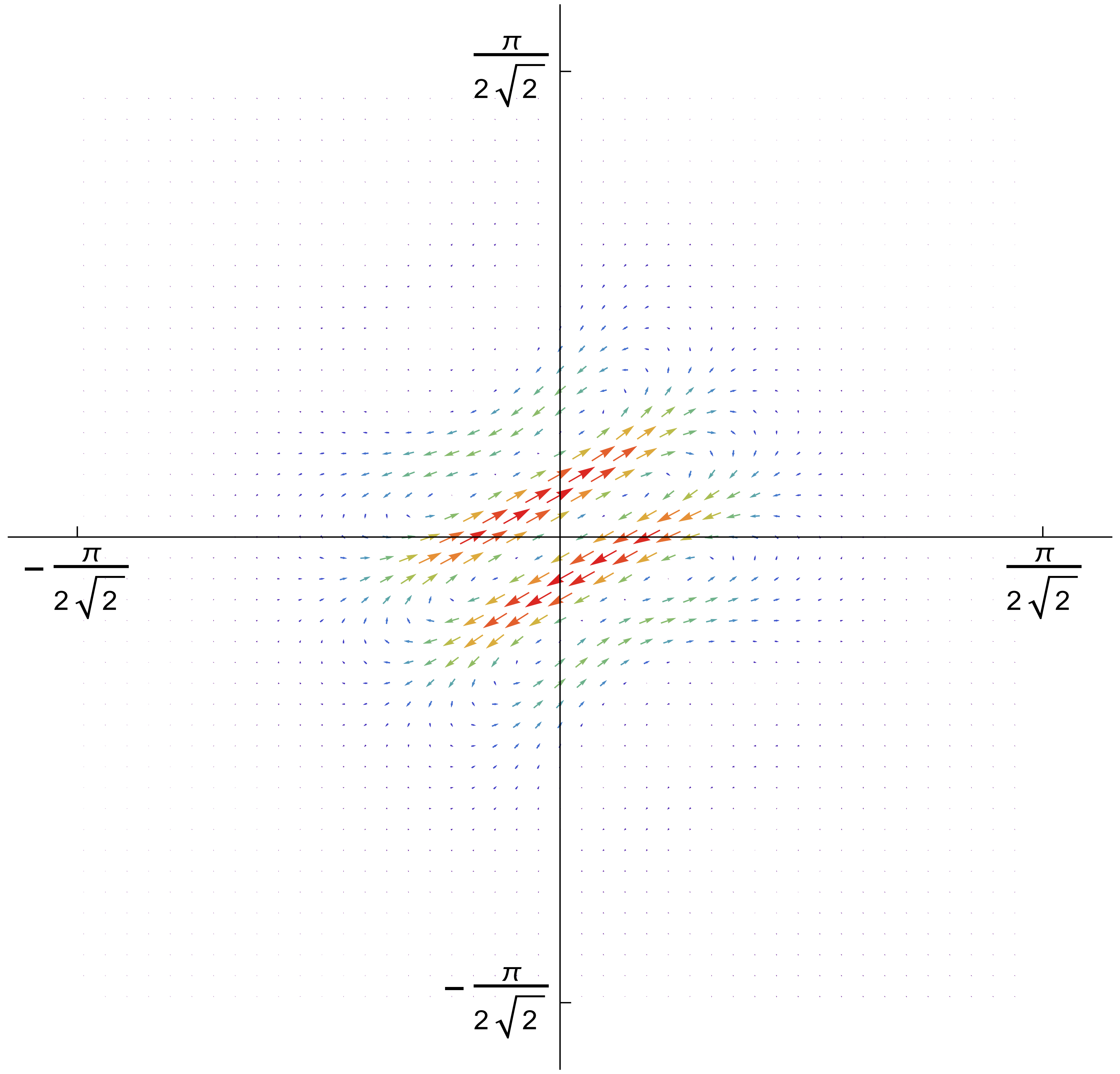
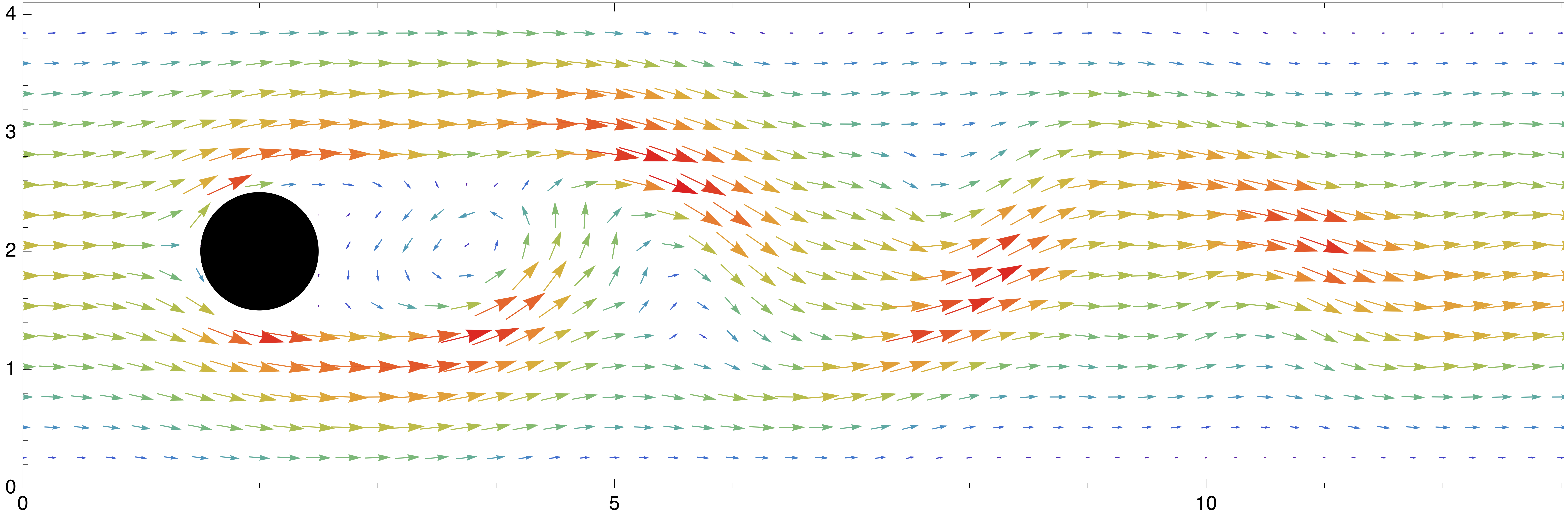




# Divergence free wavelets

- Multi-scale representation:



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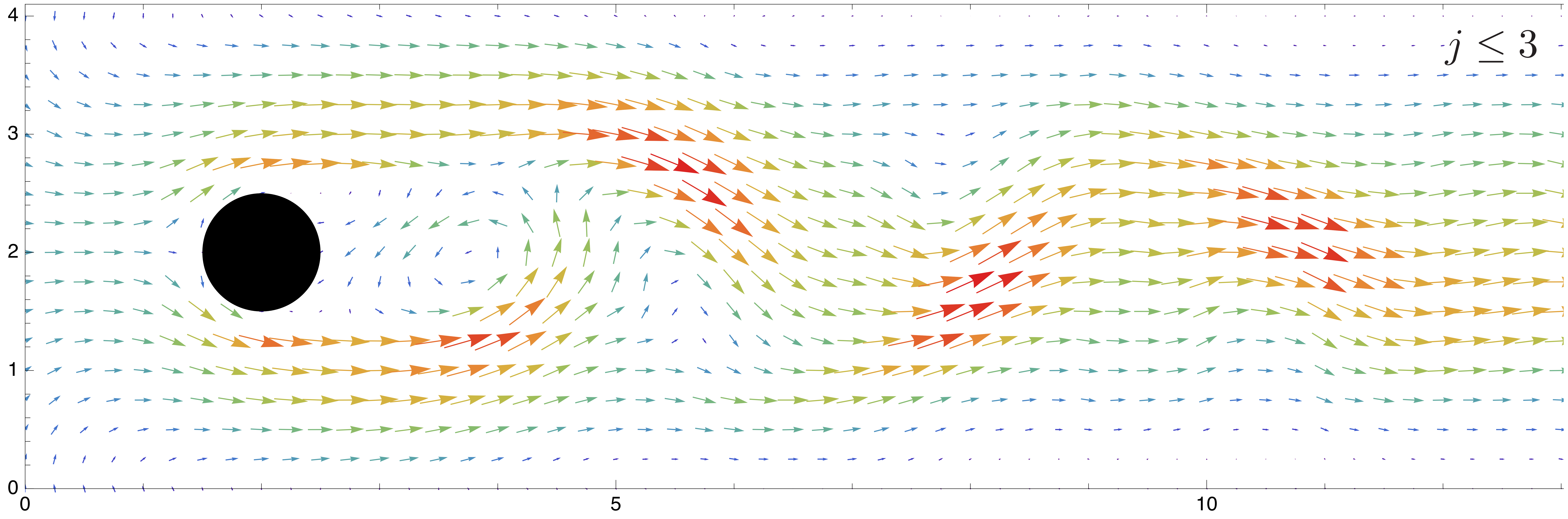

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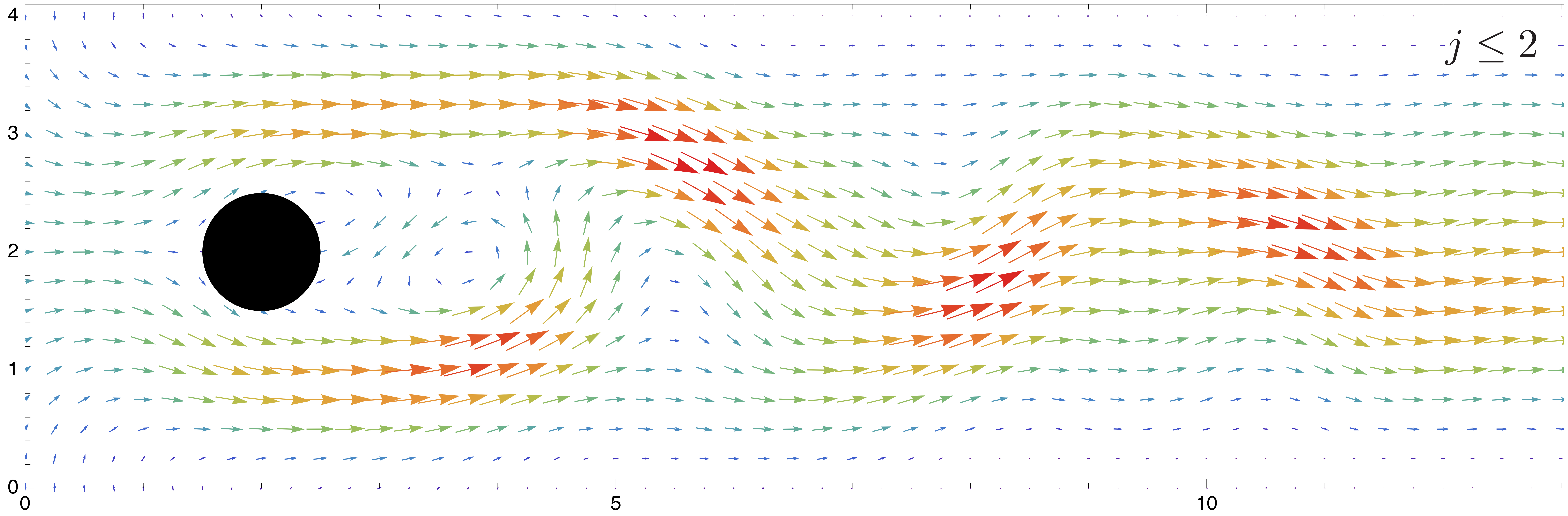

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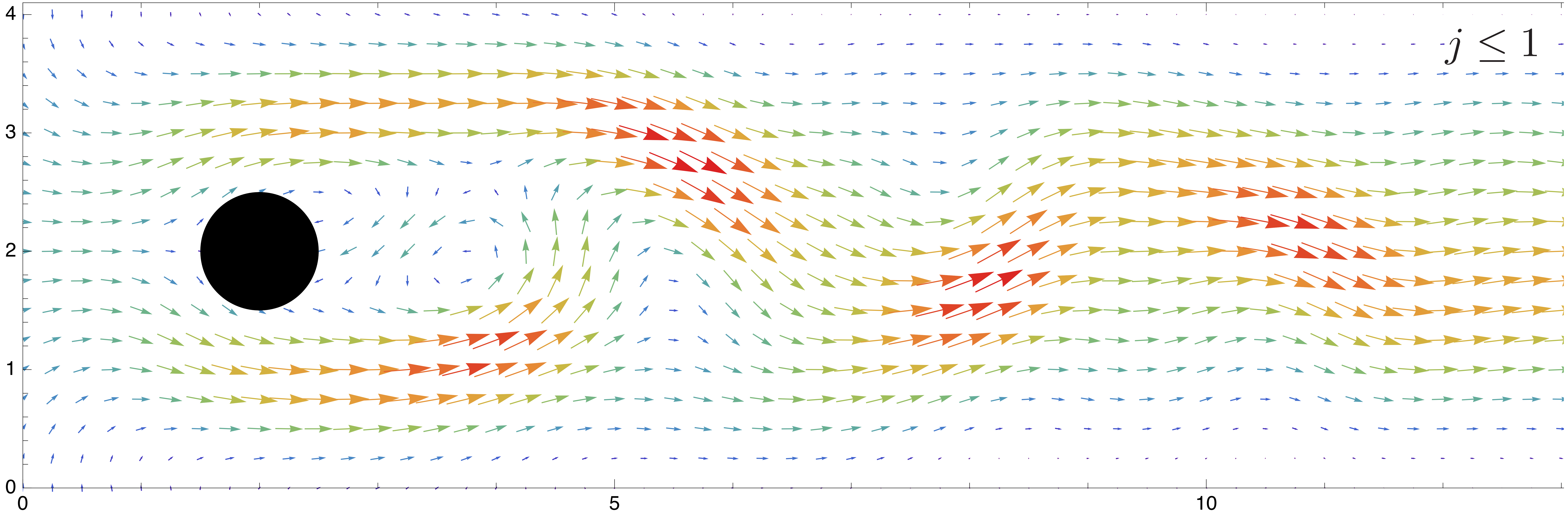

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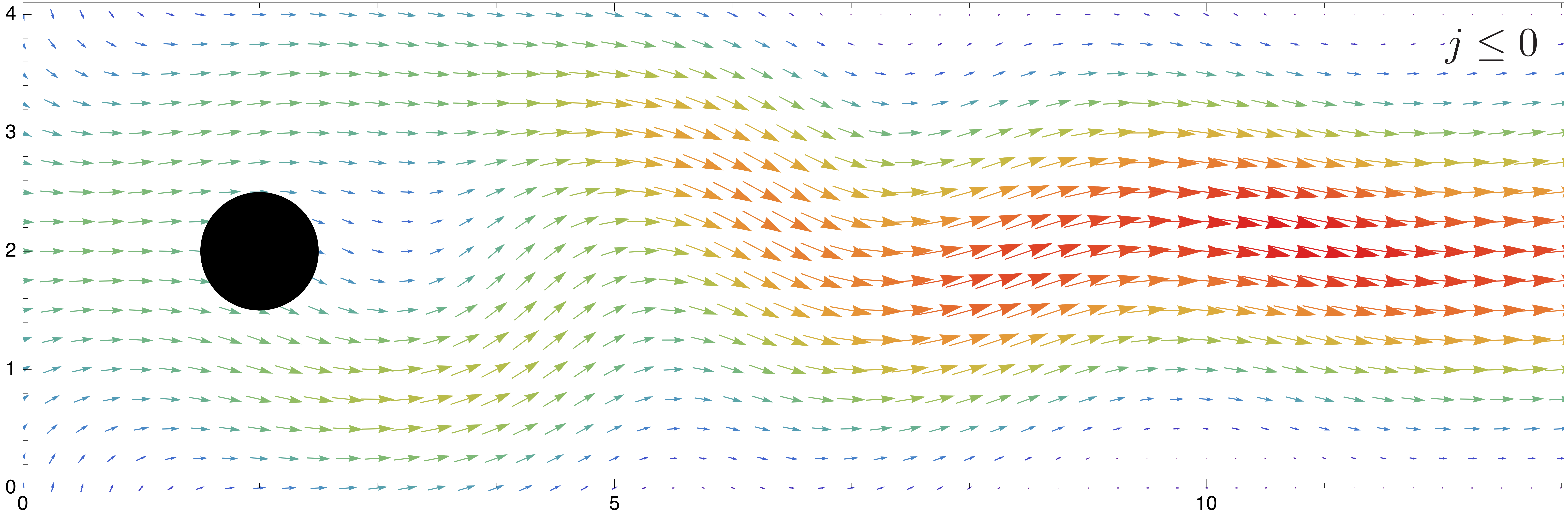

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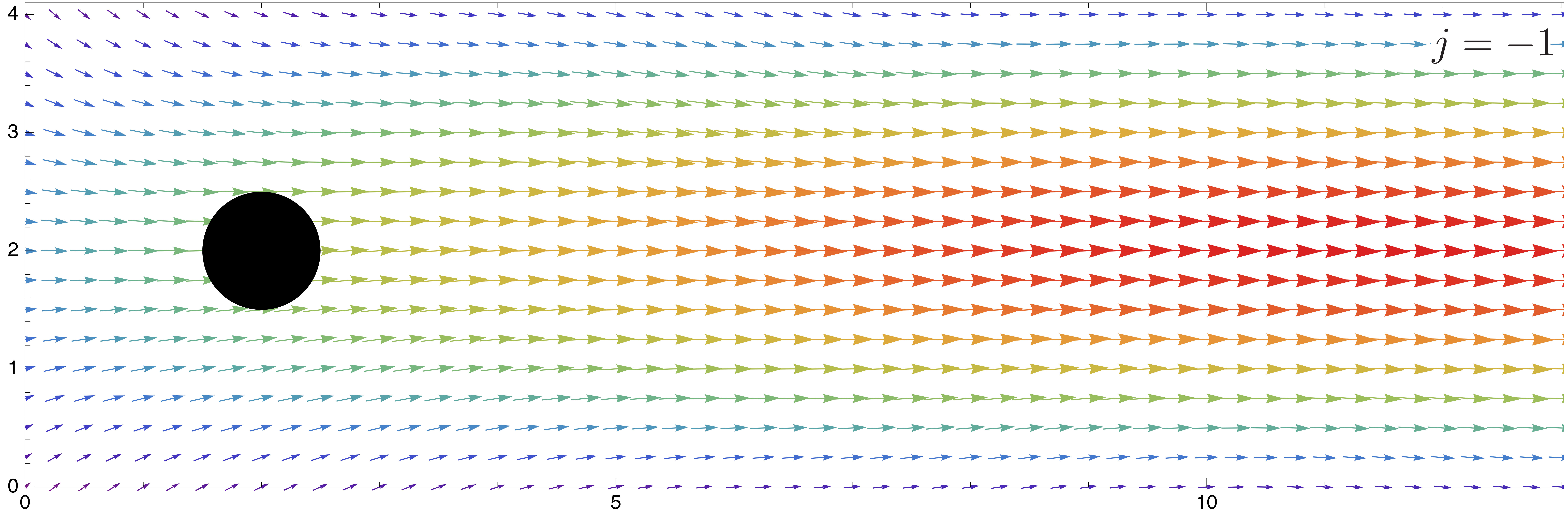

# I. Approximation of functions

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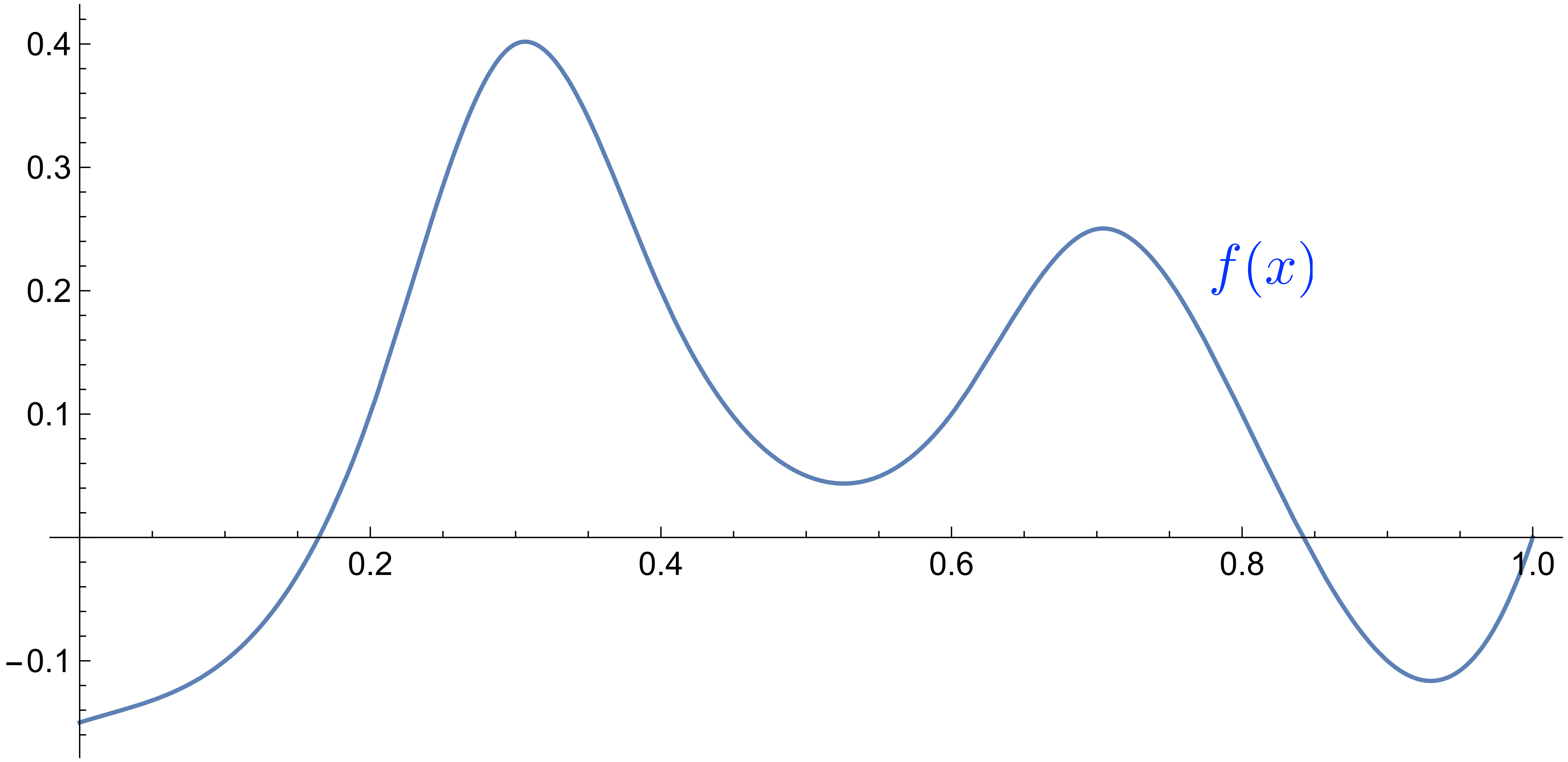

# Approximation of functions

Let $f \in C^k([0,1]) \cap L_2([0,1])$. Then the linear N-term Fourier series approximation $\tilde{f}_N$ satisfies:

$$\| f - \tilde{f}_N \| \leq C \, 2^{-2k}$$

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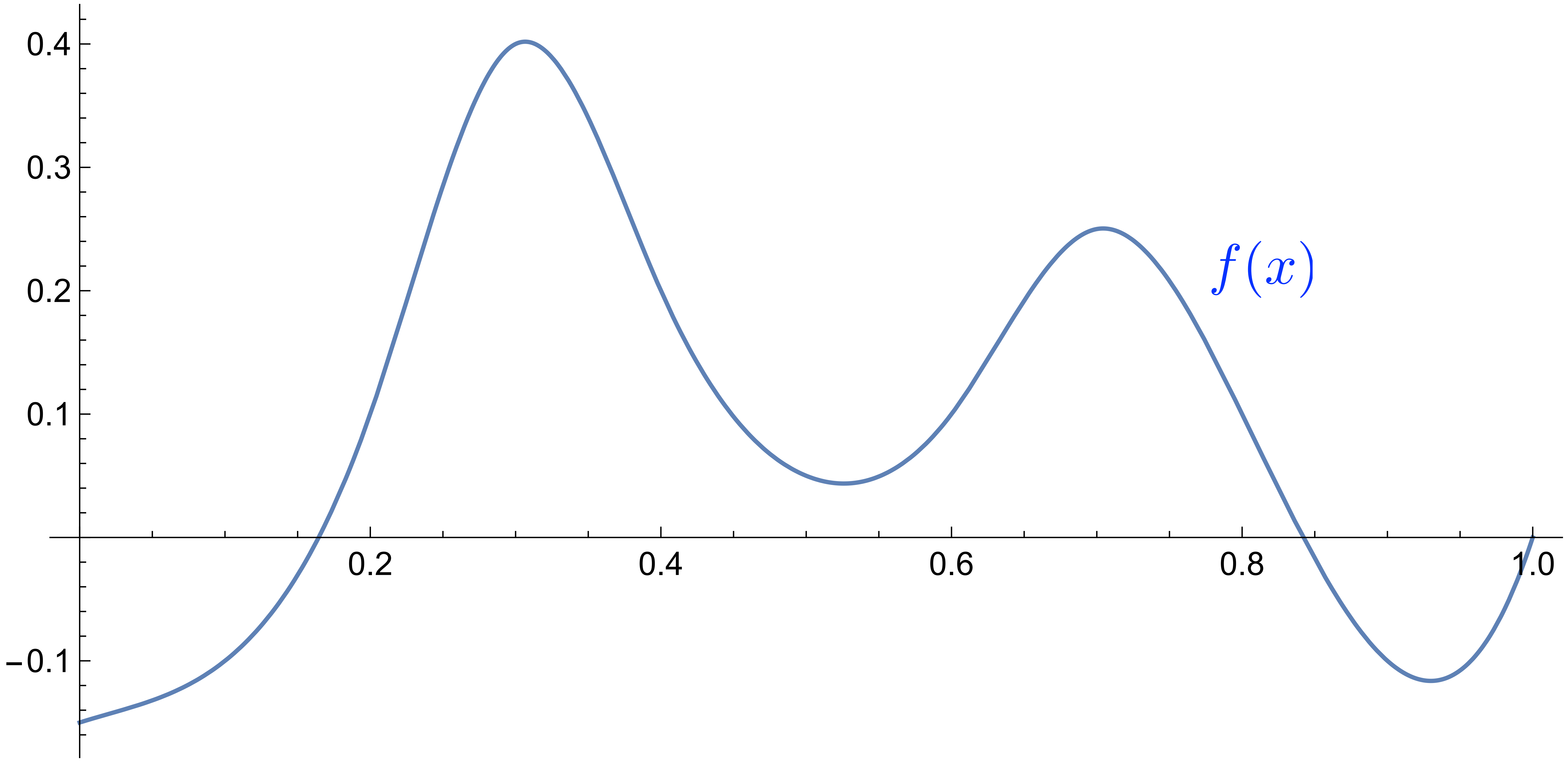

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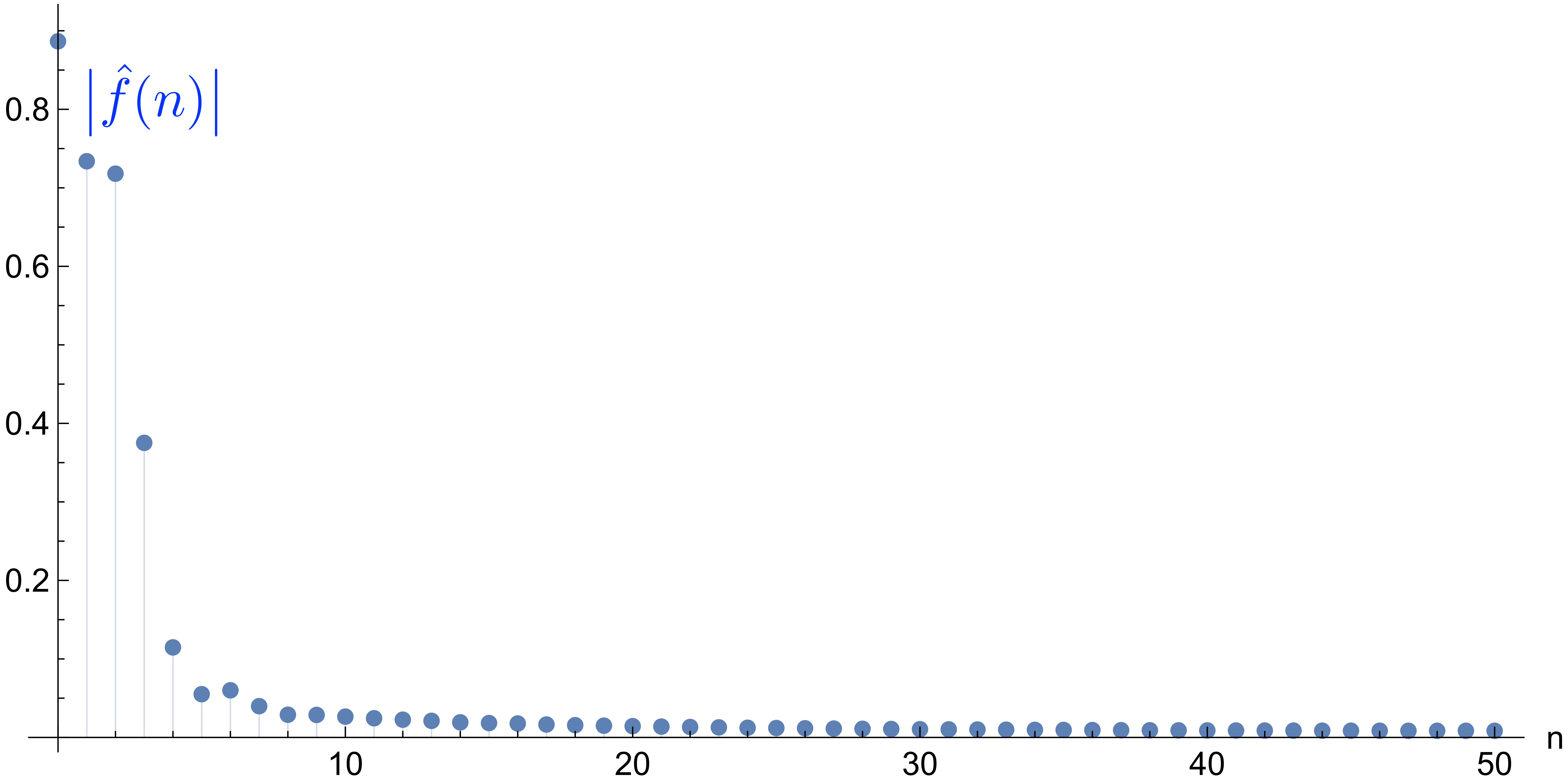

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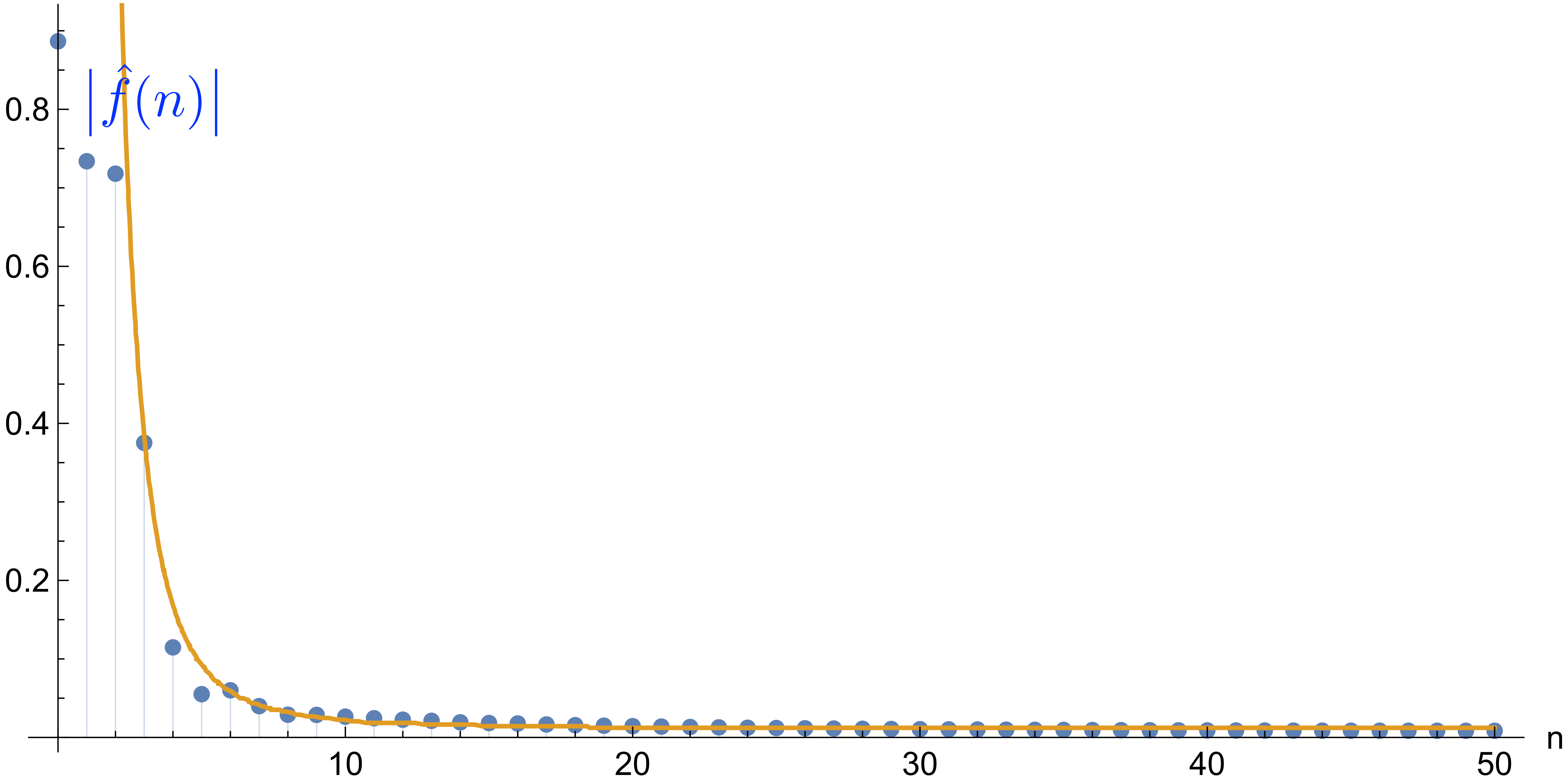

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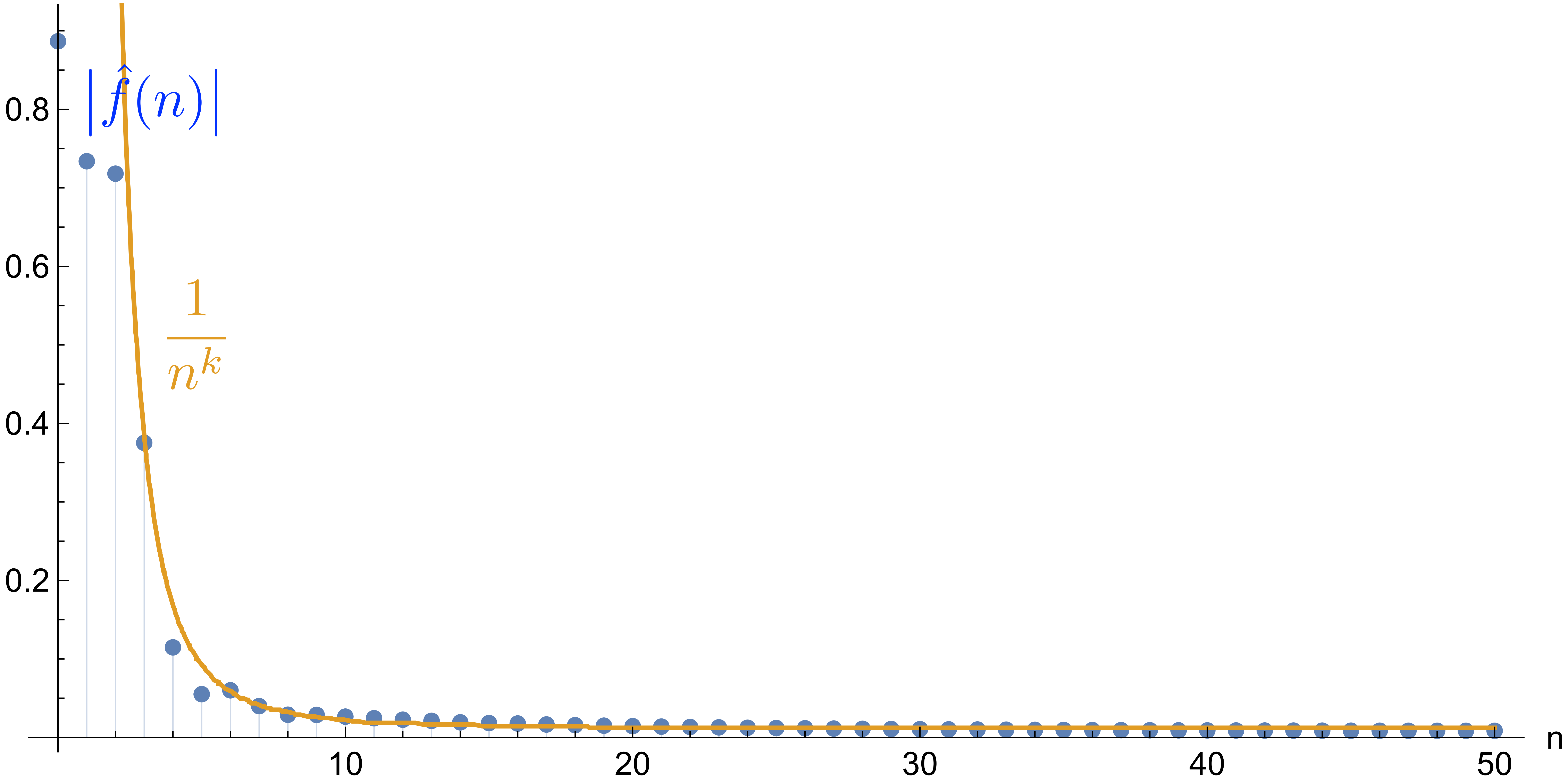

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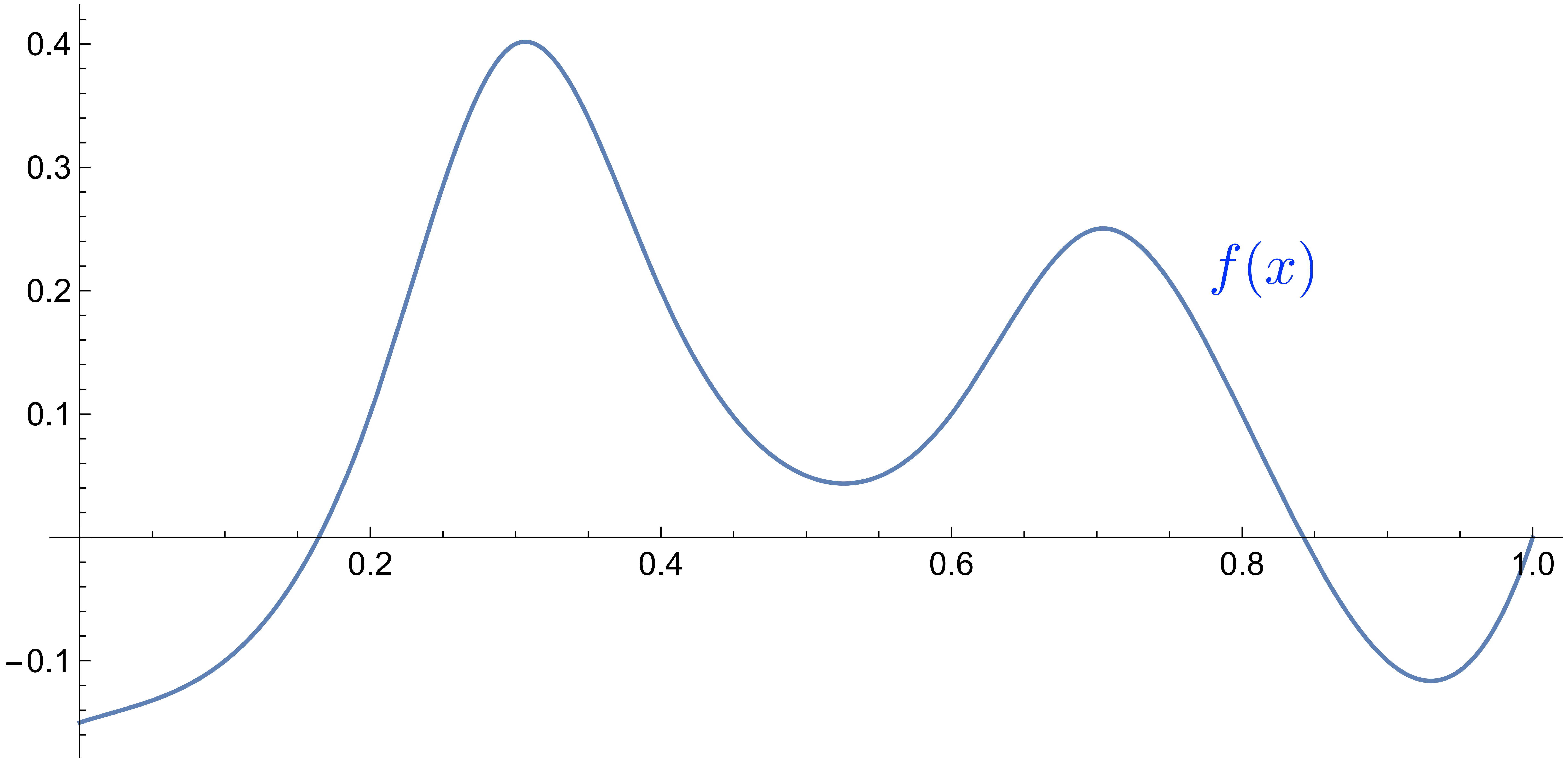

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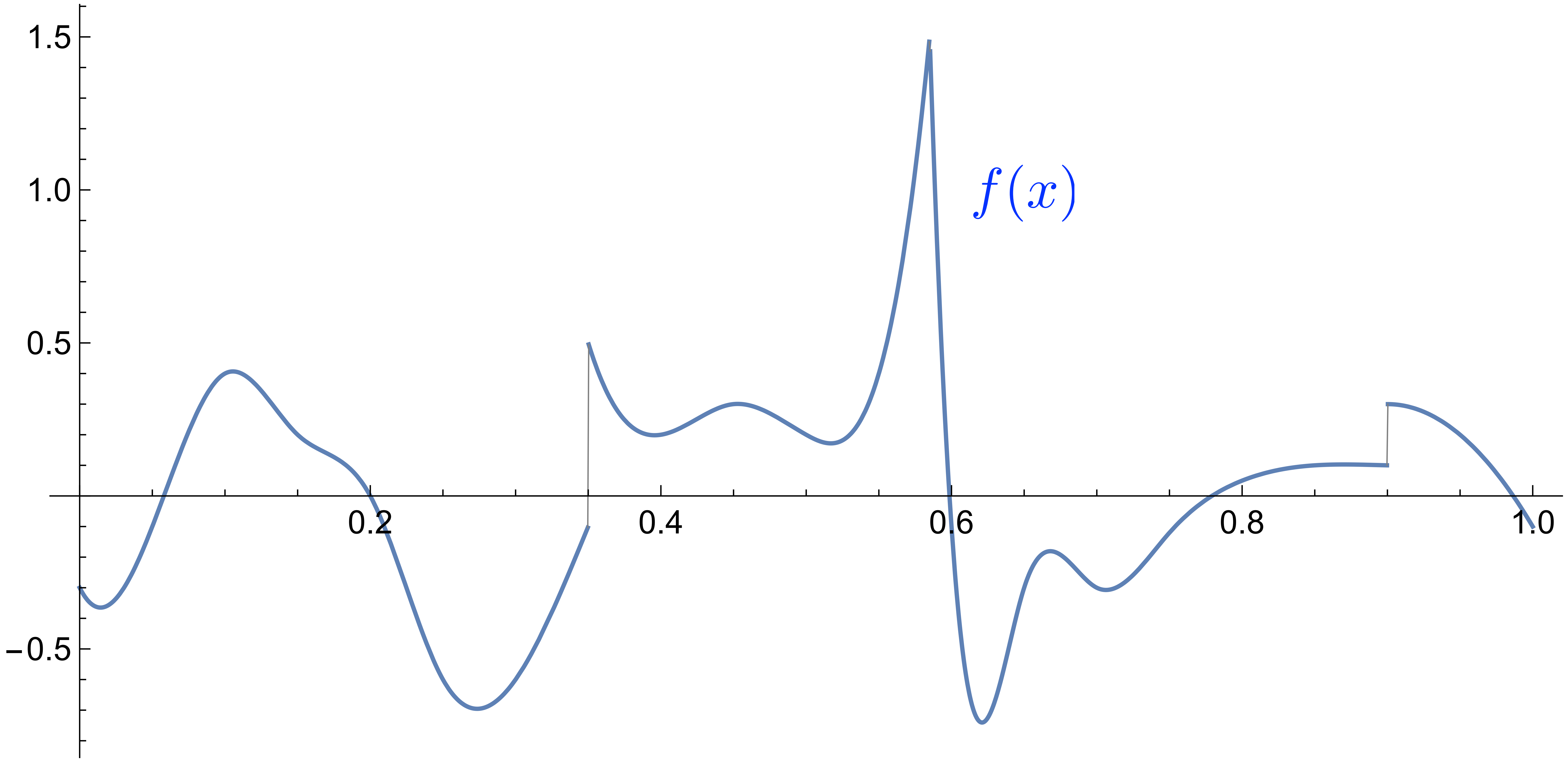

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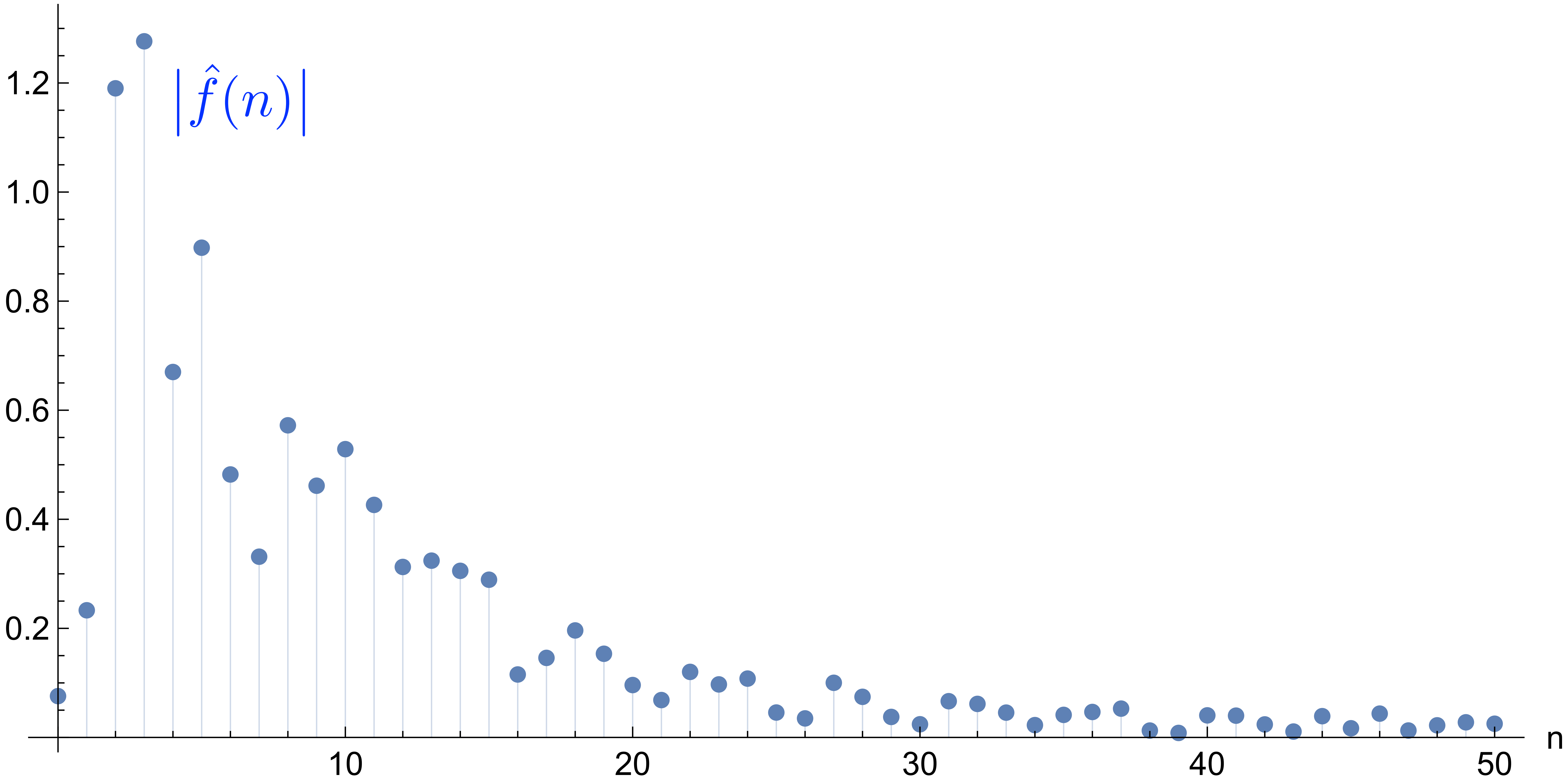

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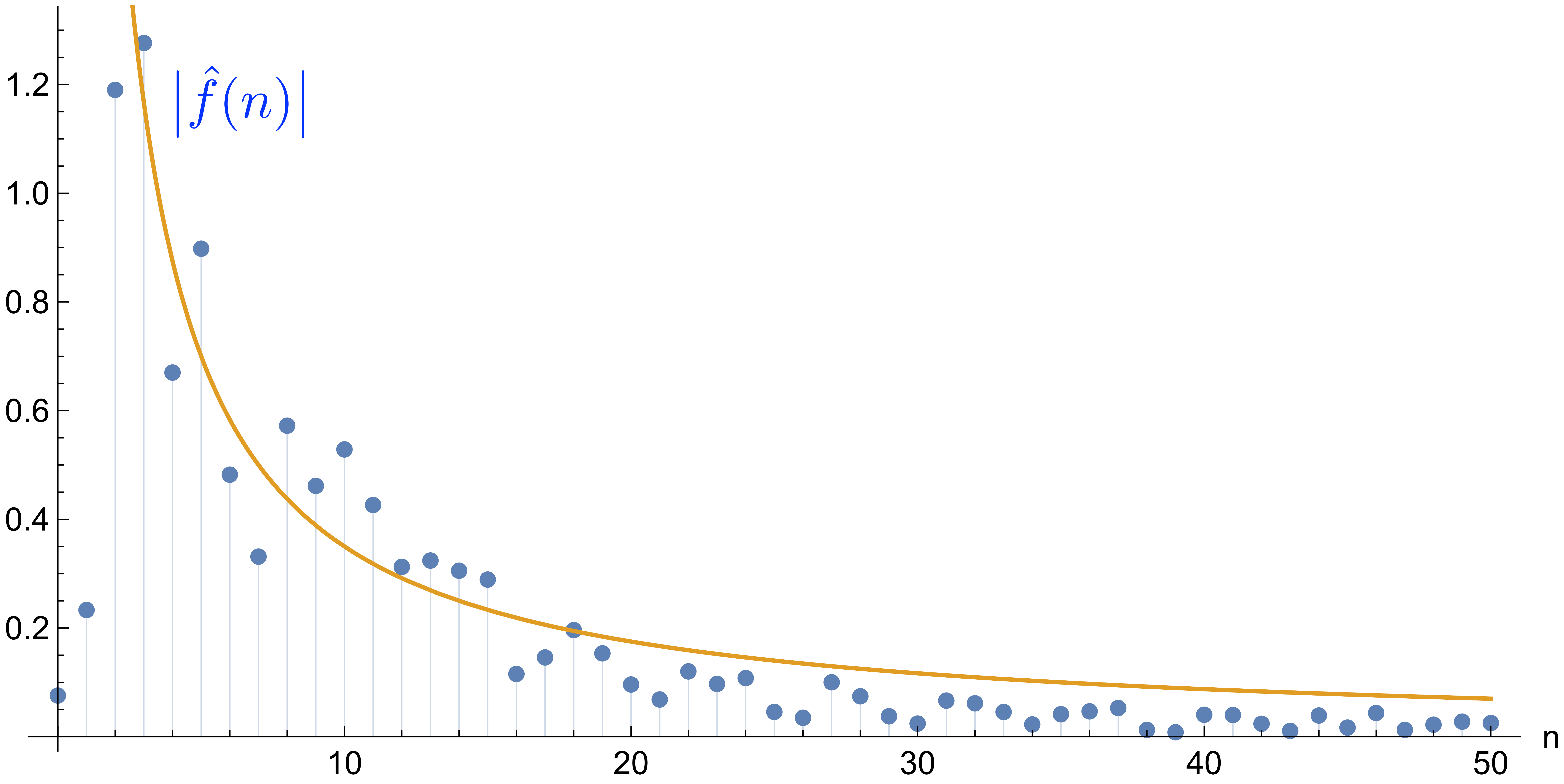

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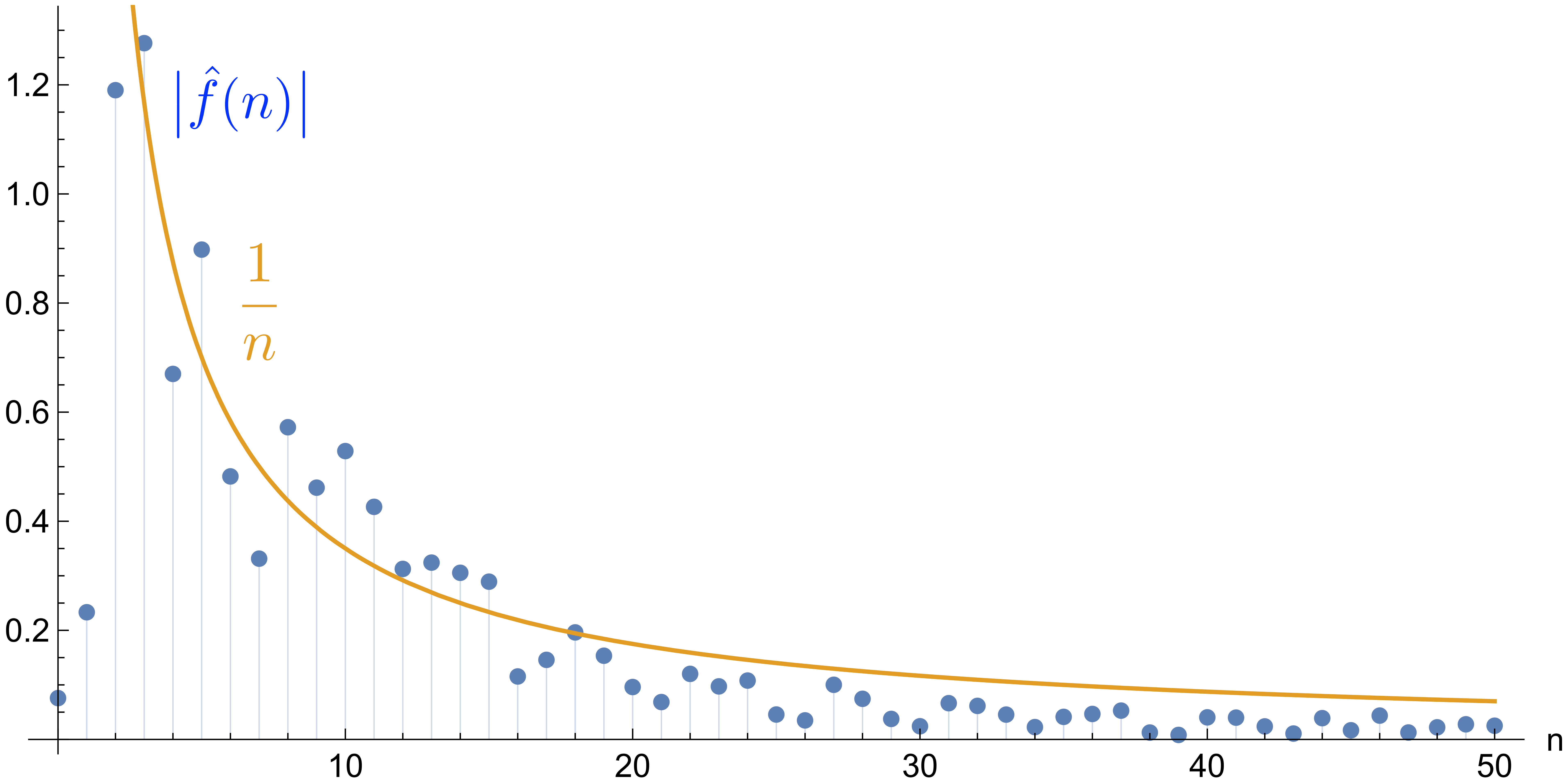

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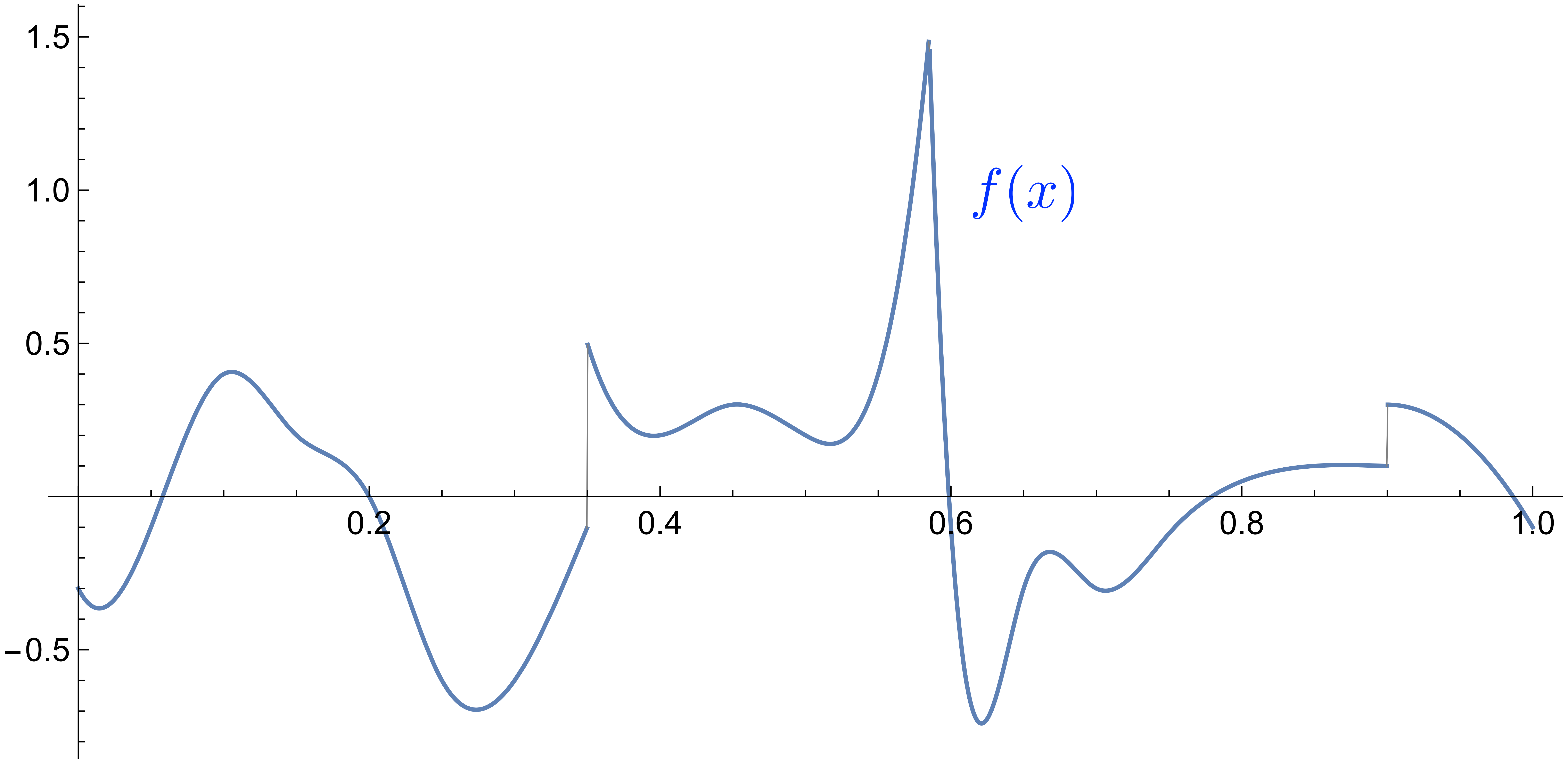

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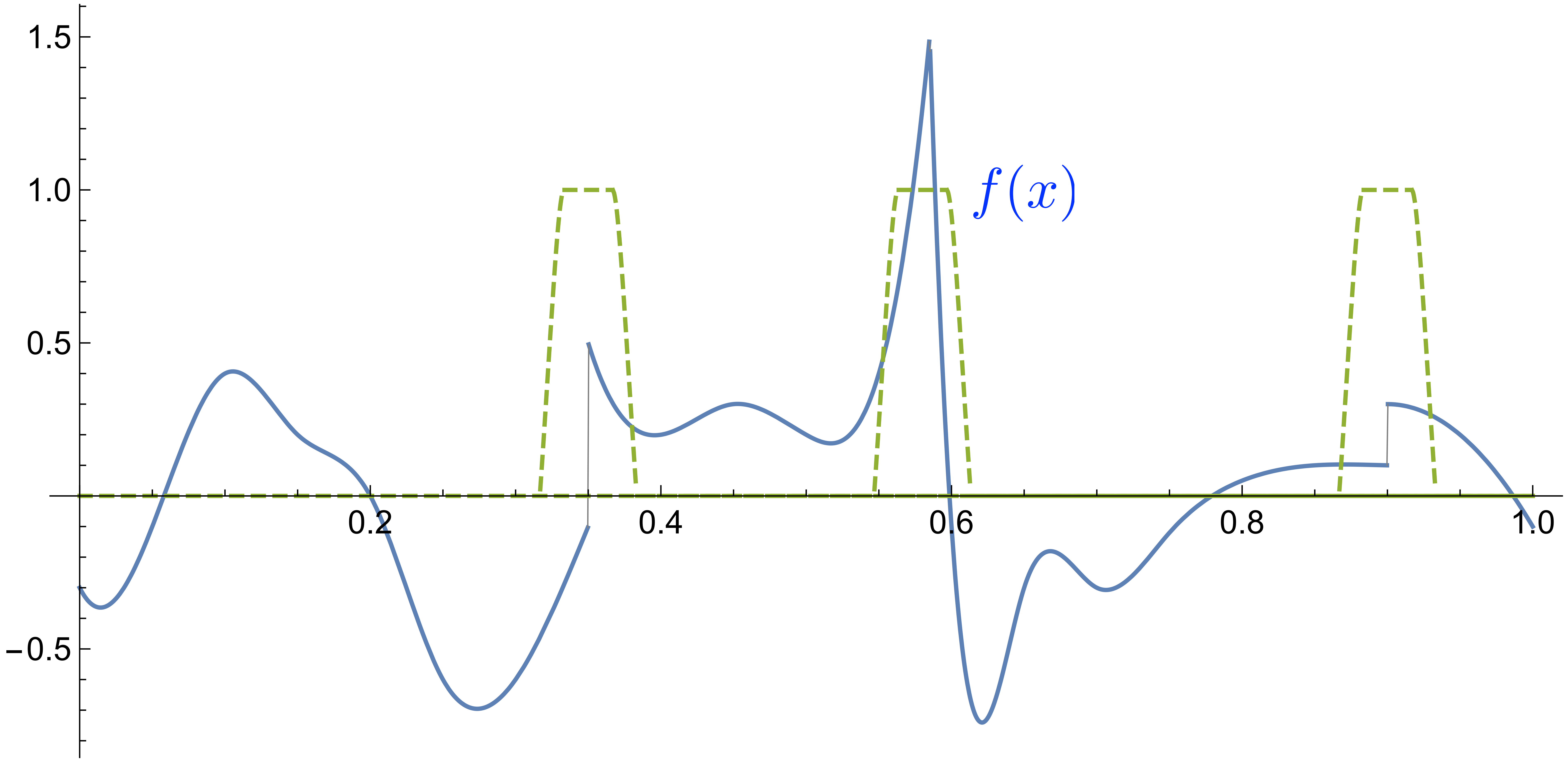

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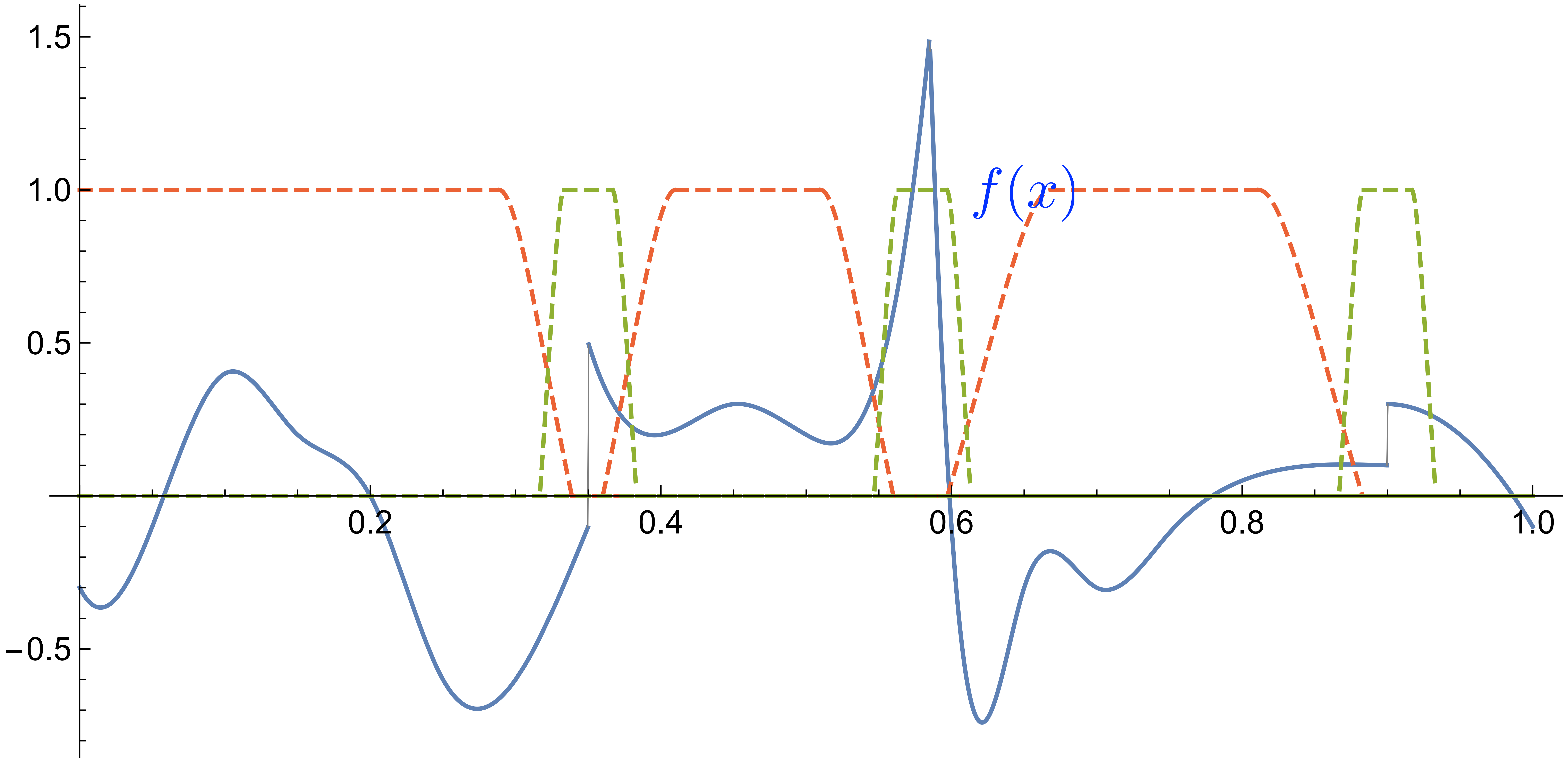

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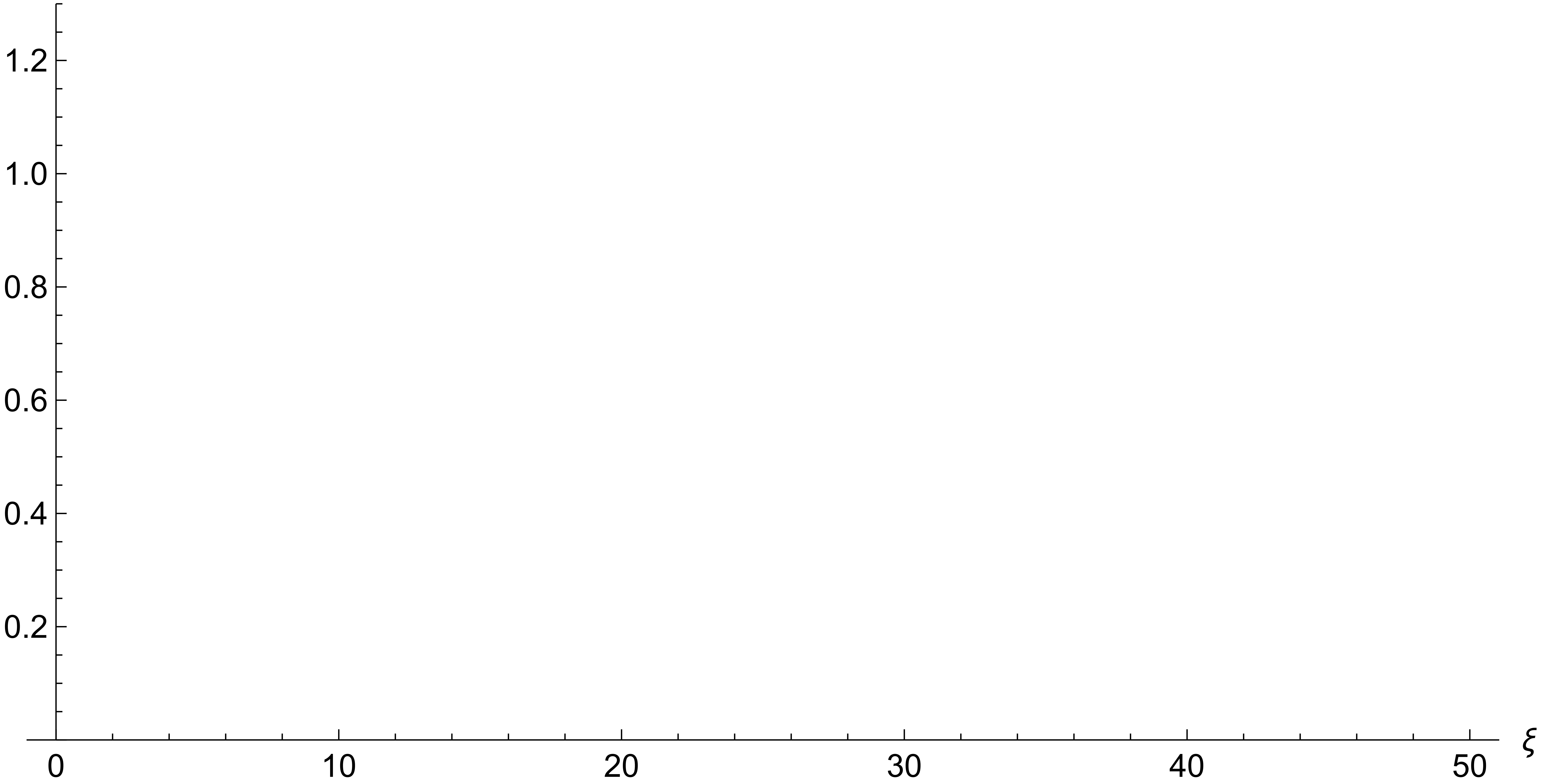

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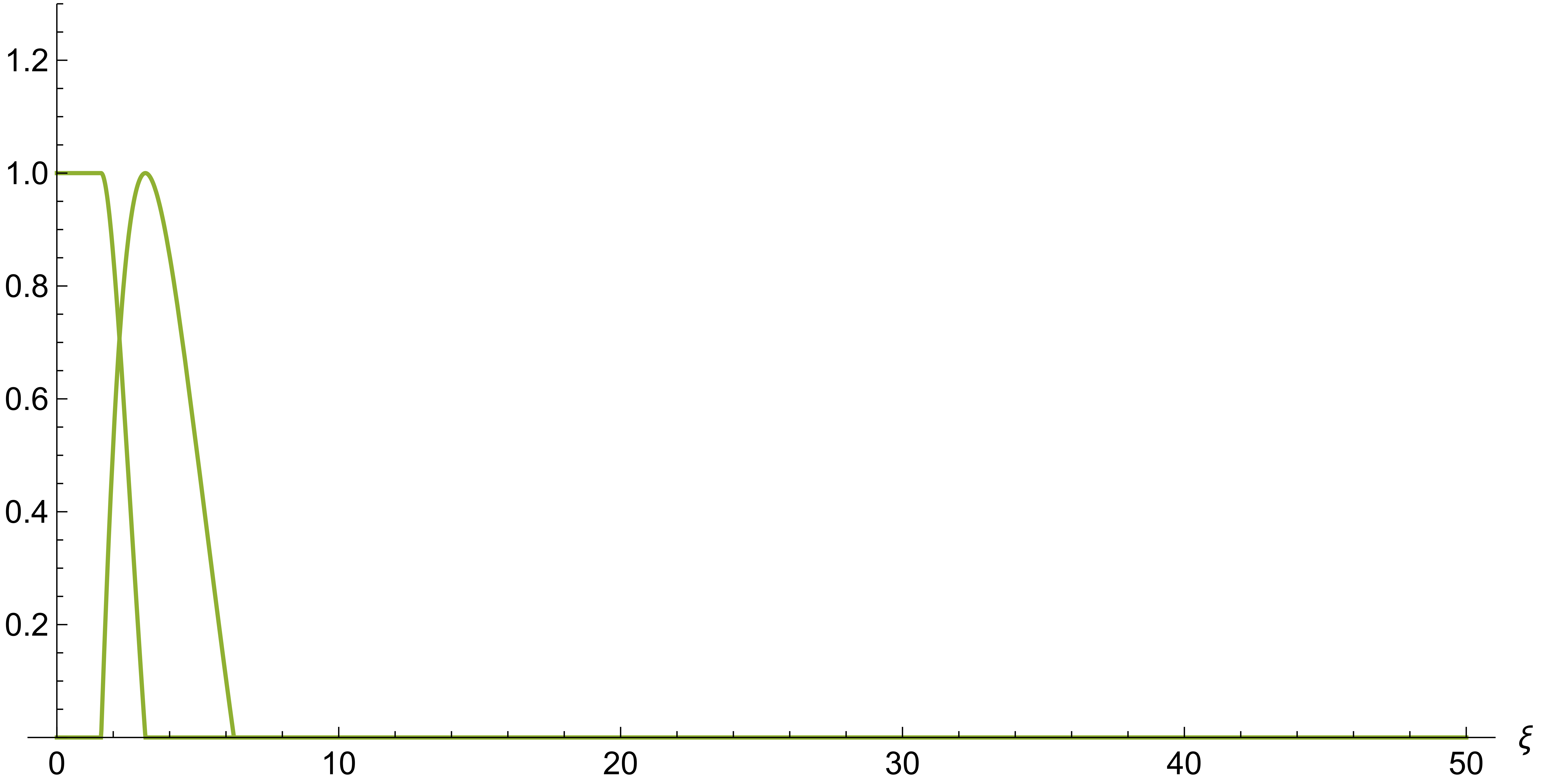

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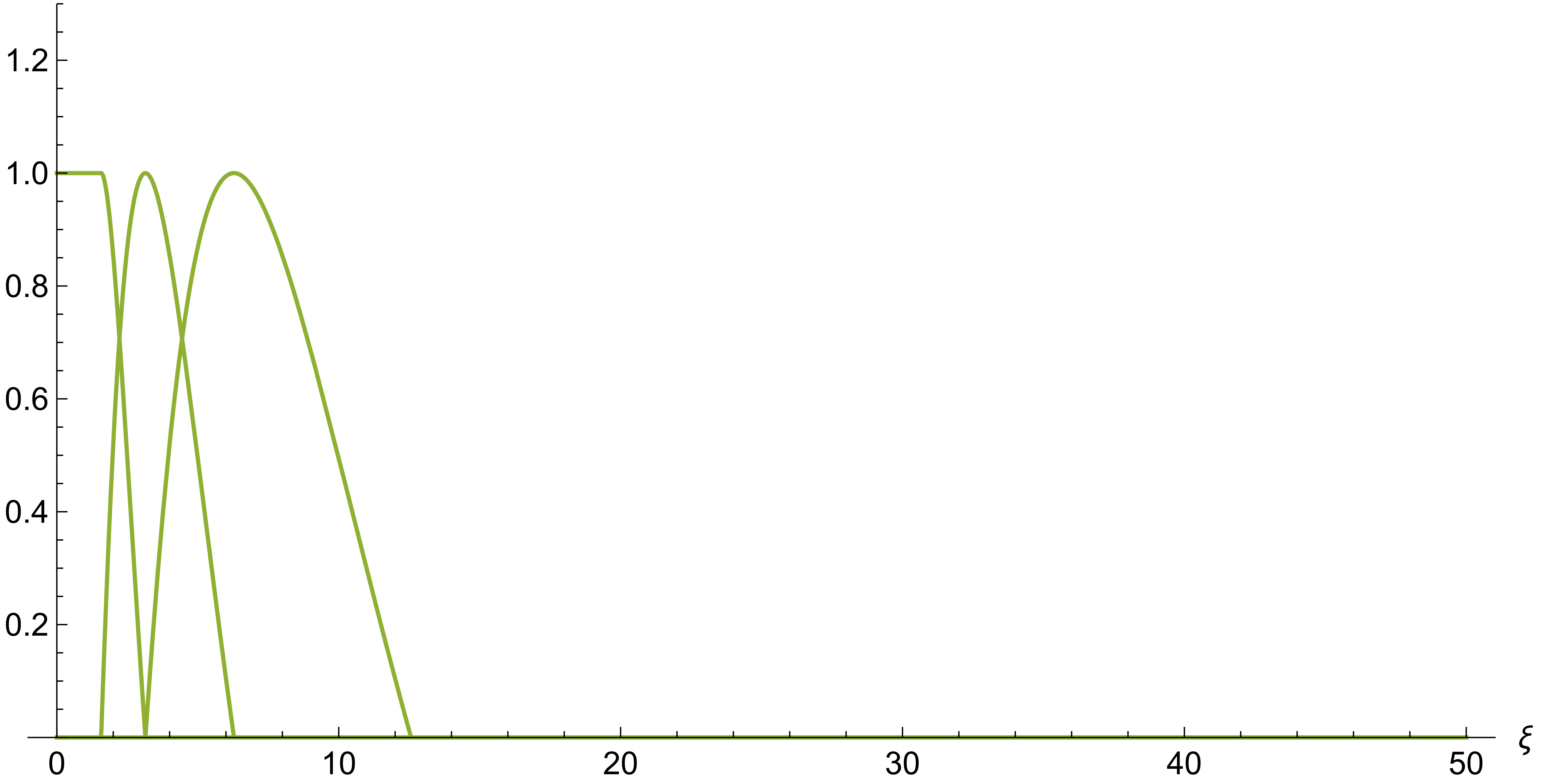

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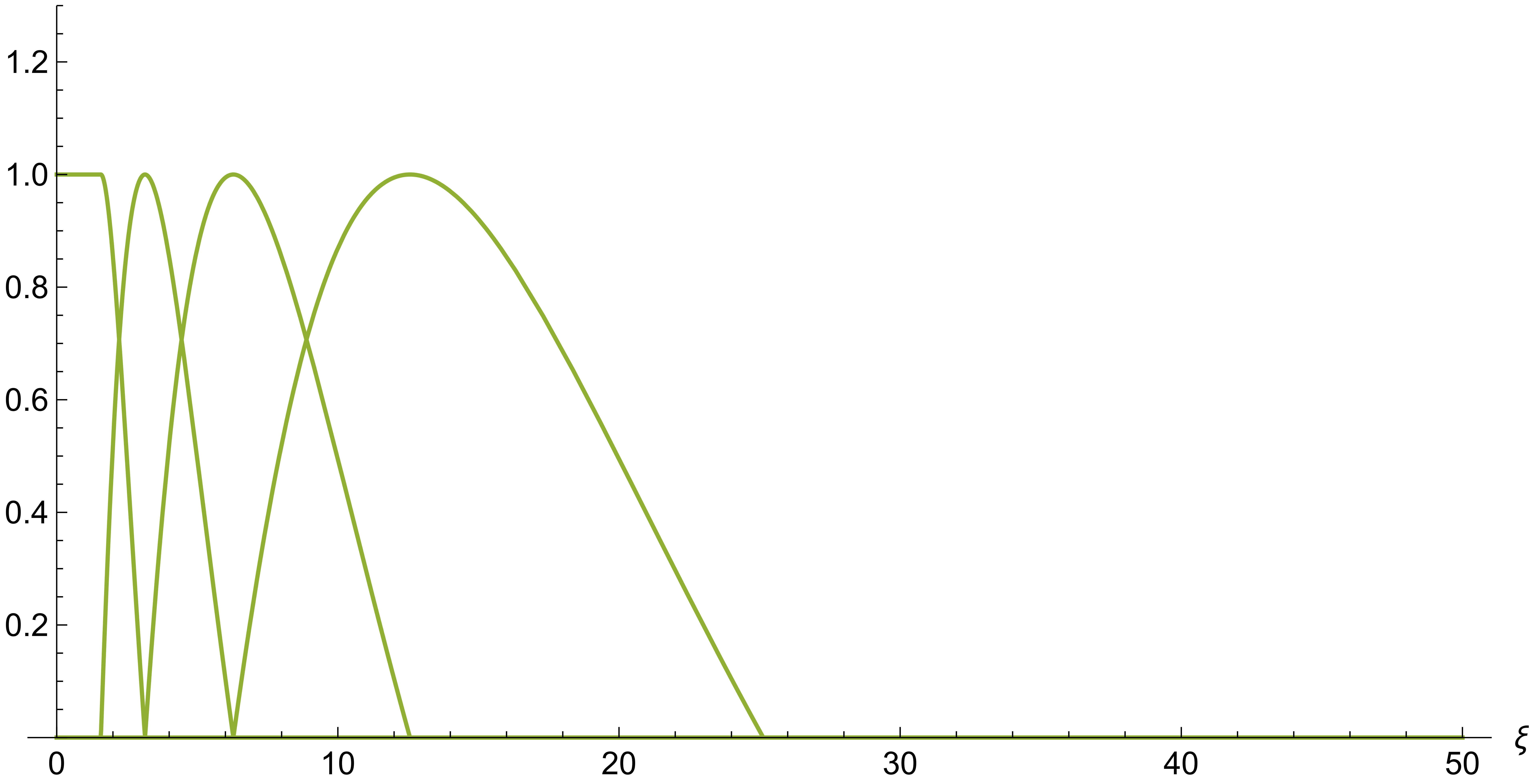

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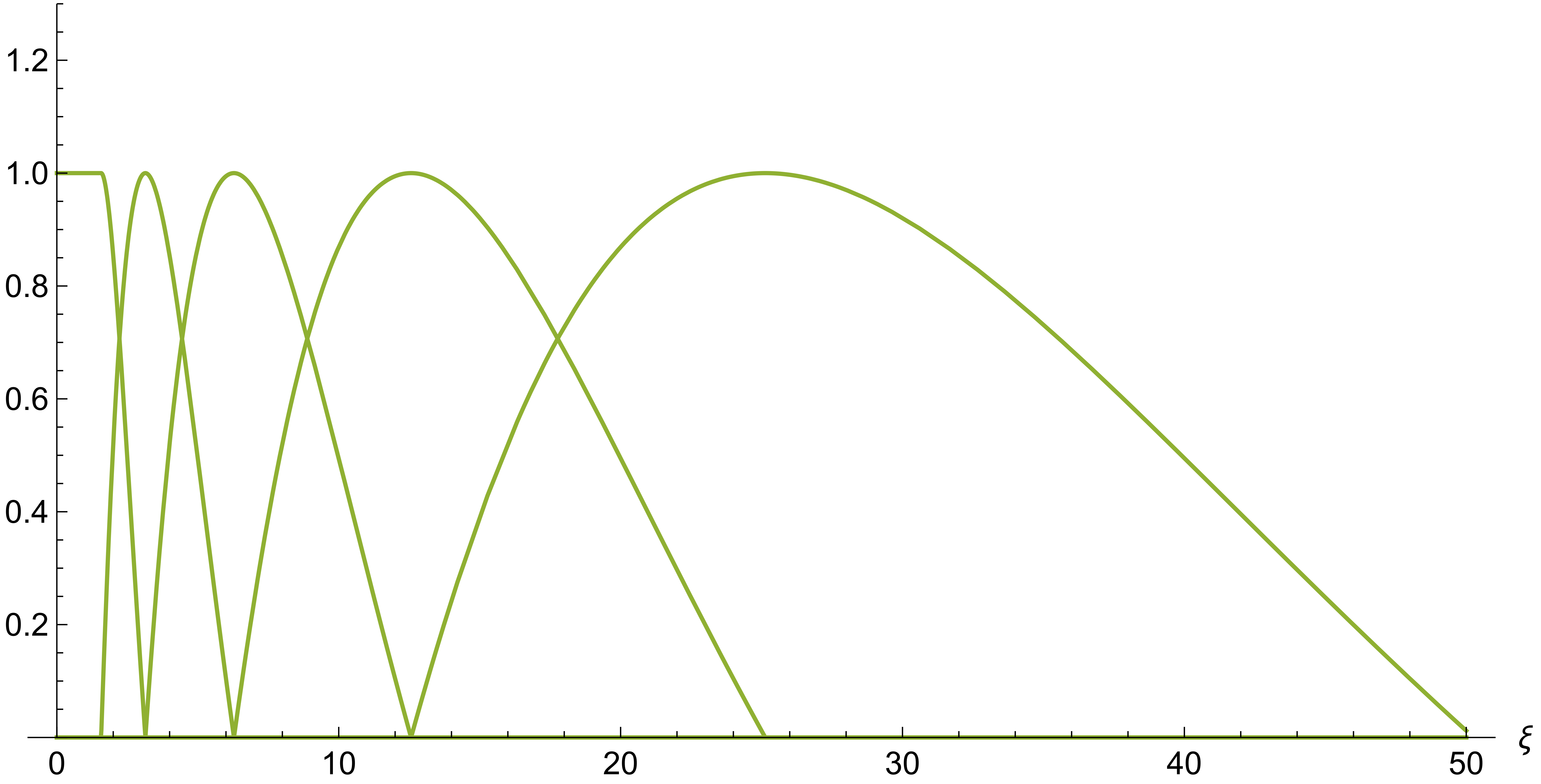

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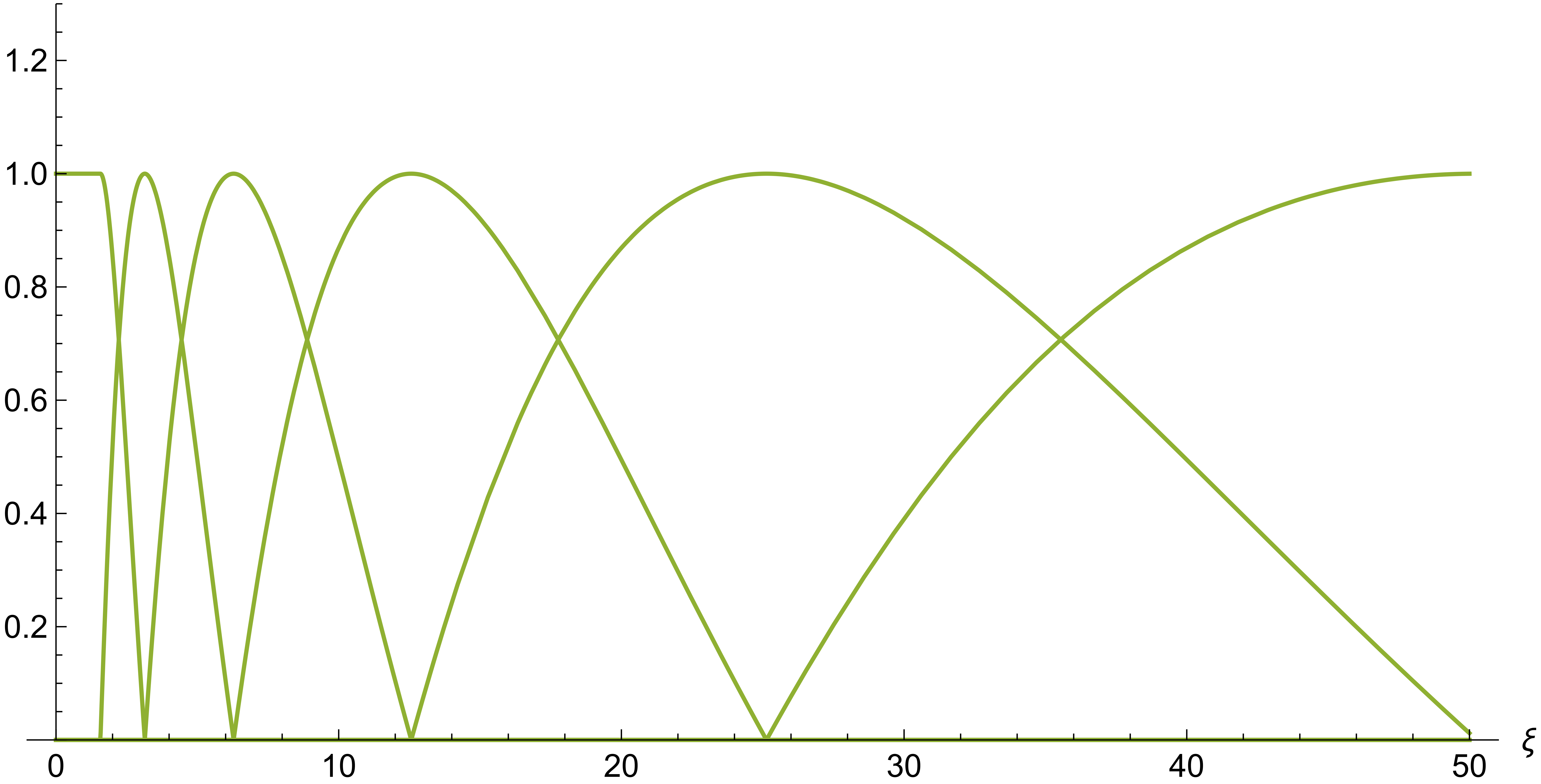

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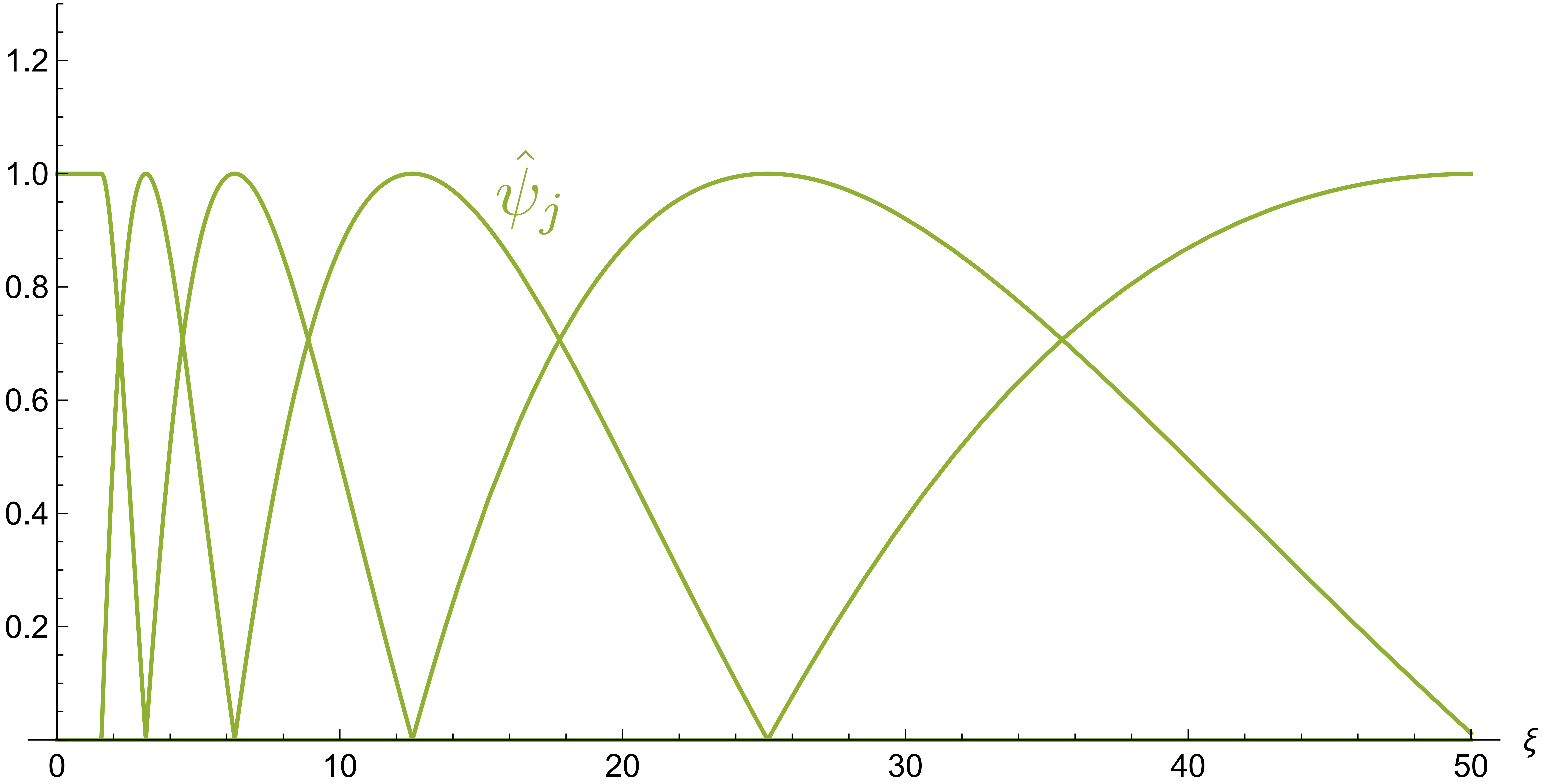

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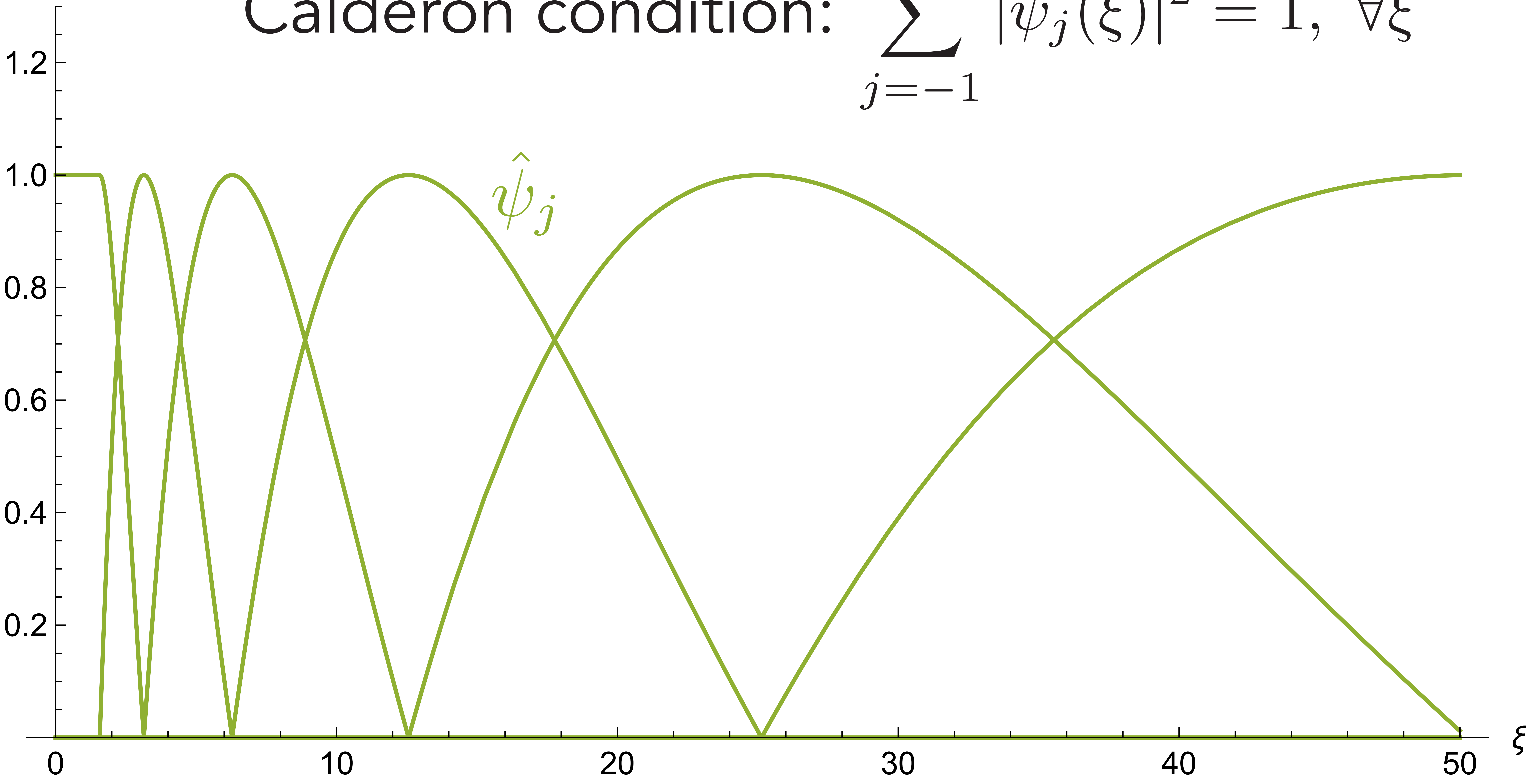

Calderon condition: $\sum_{j=-1}^{\infty} |\hat{\psi}_j(\xi)|^2 = 1, \ \forall \xi$

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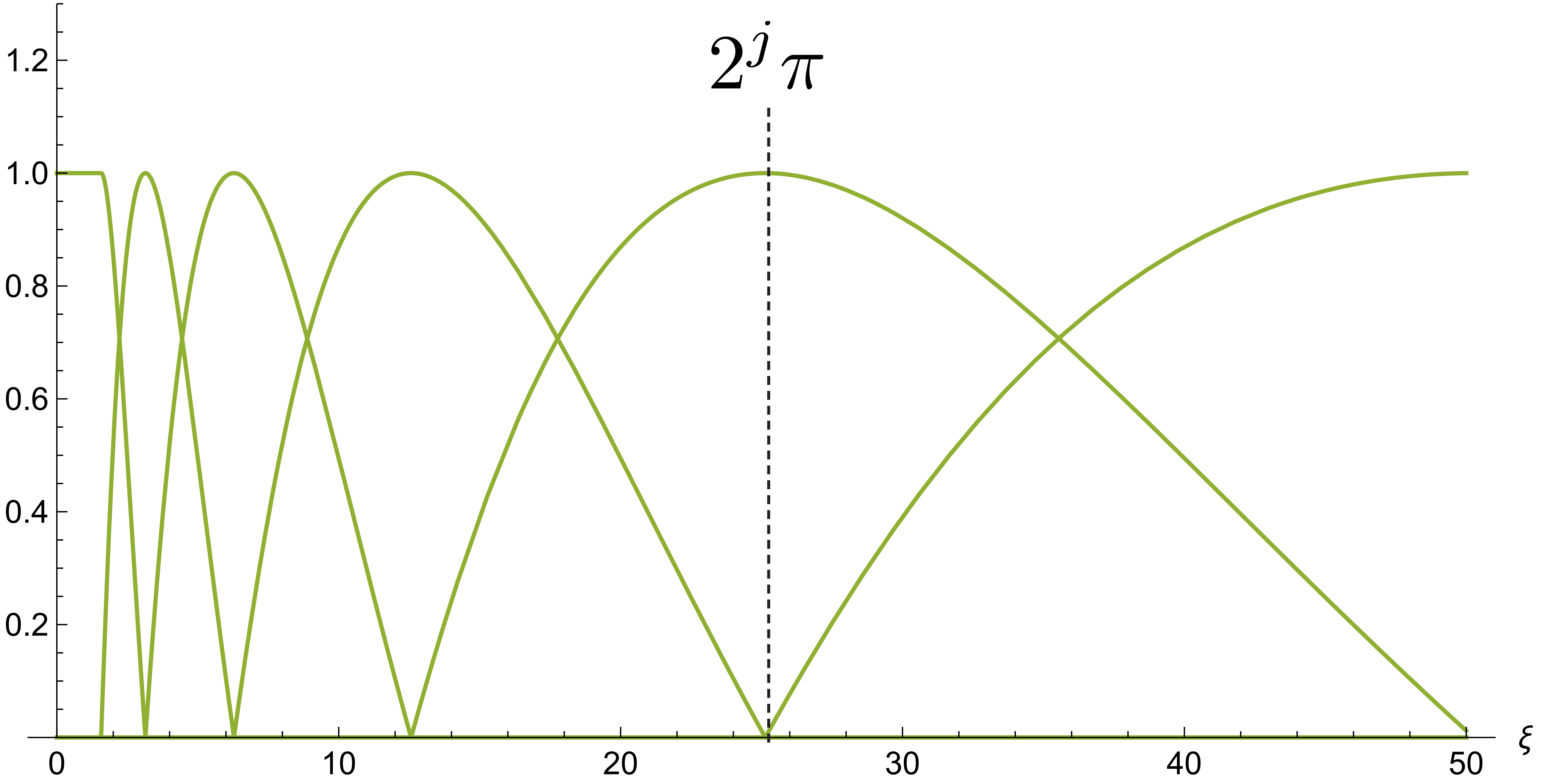

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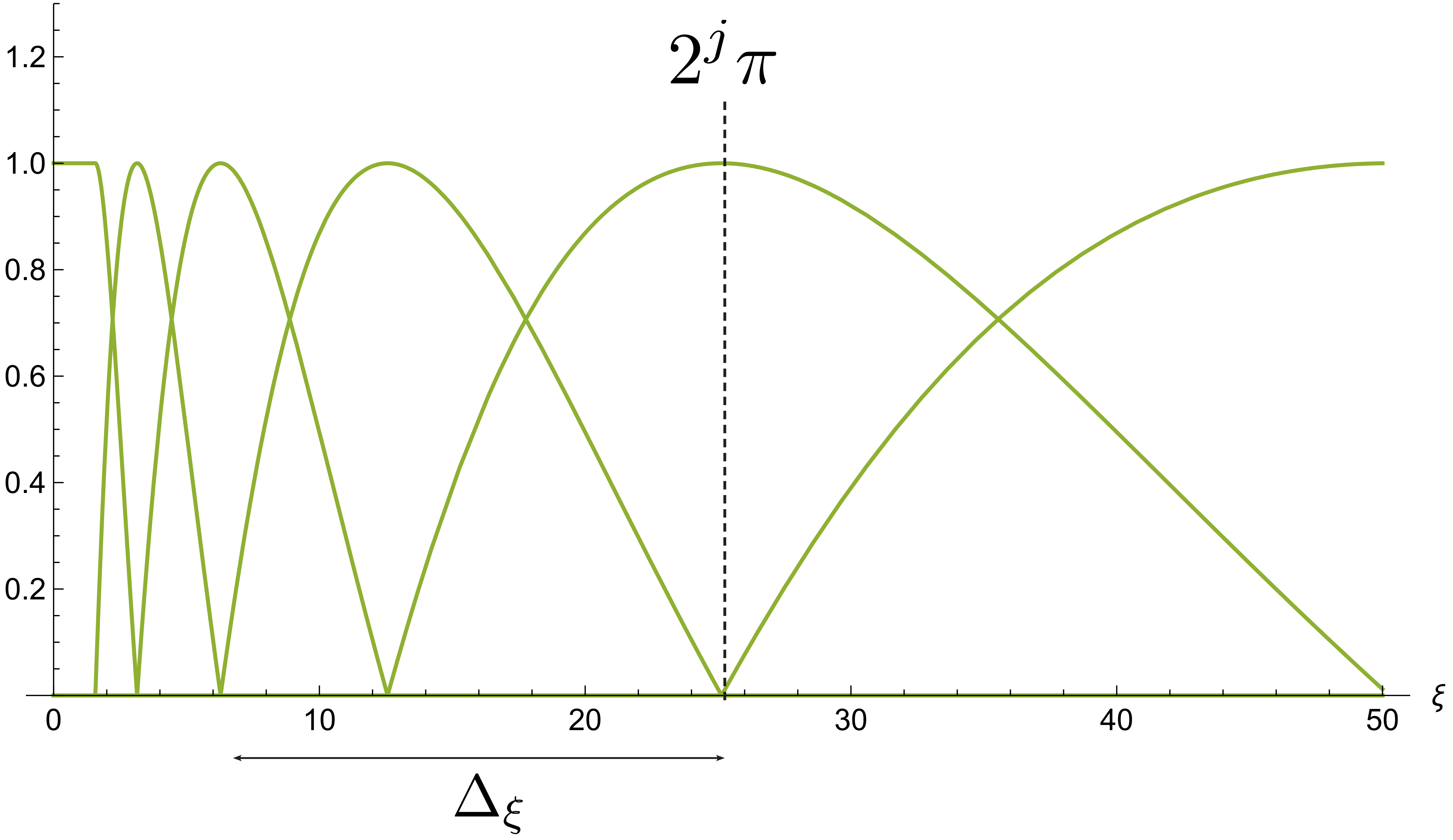

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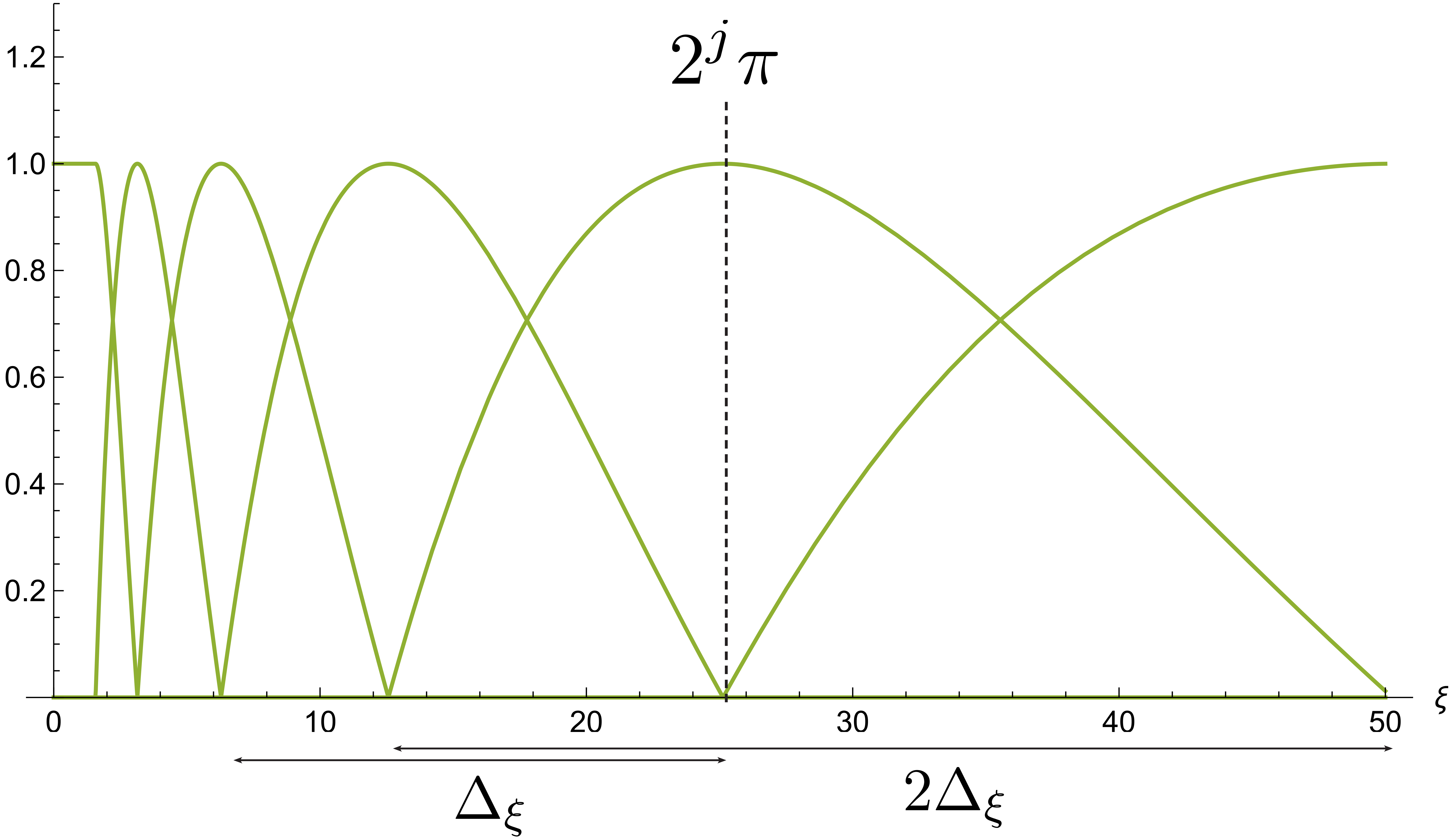

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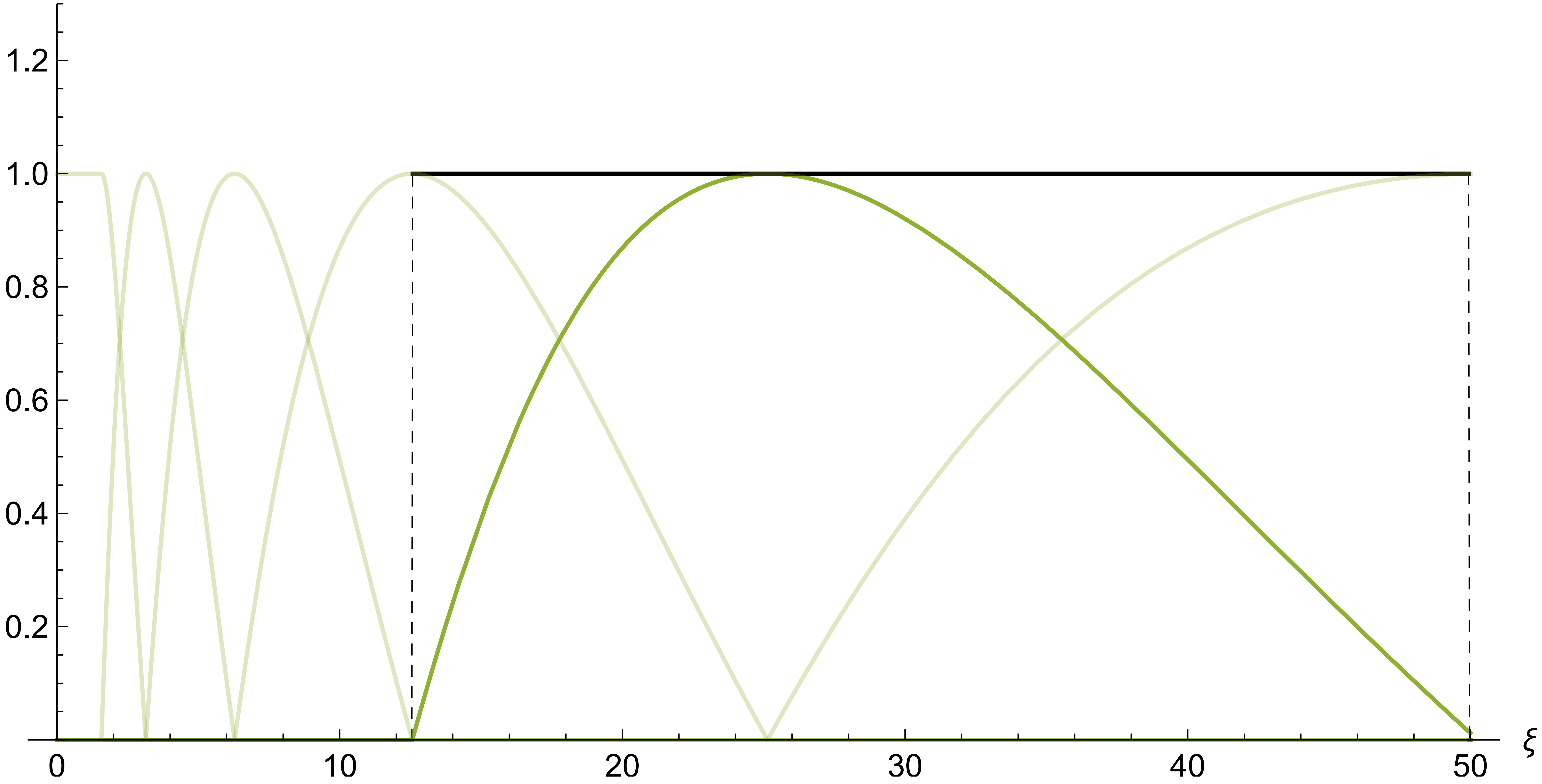

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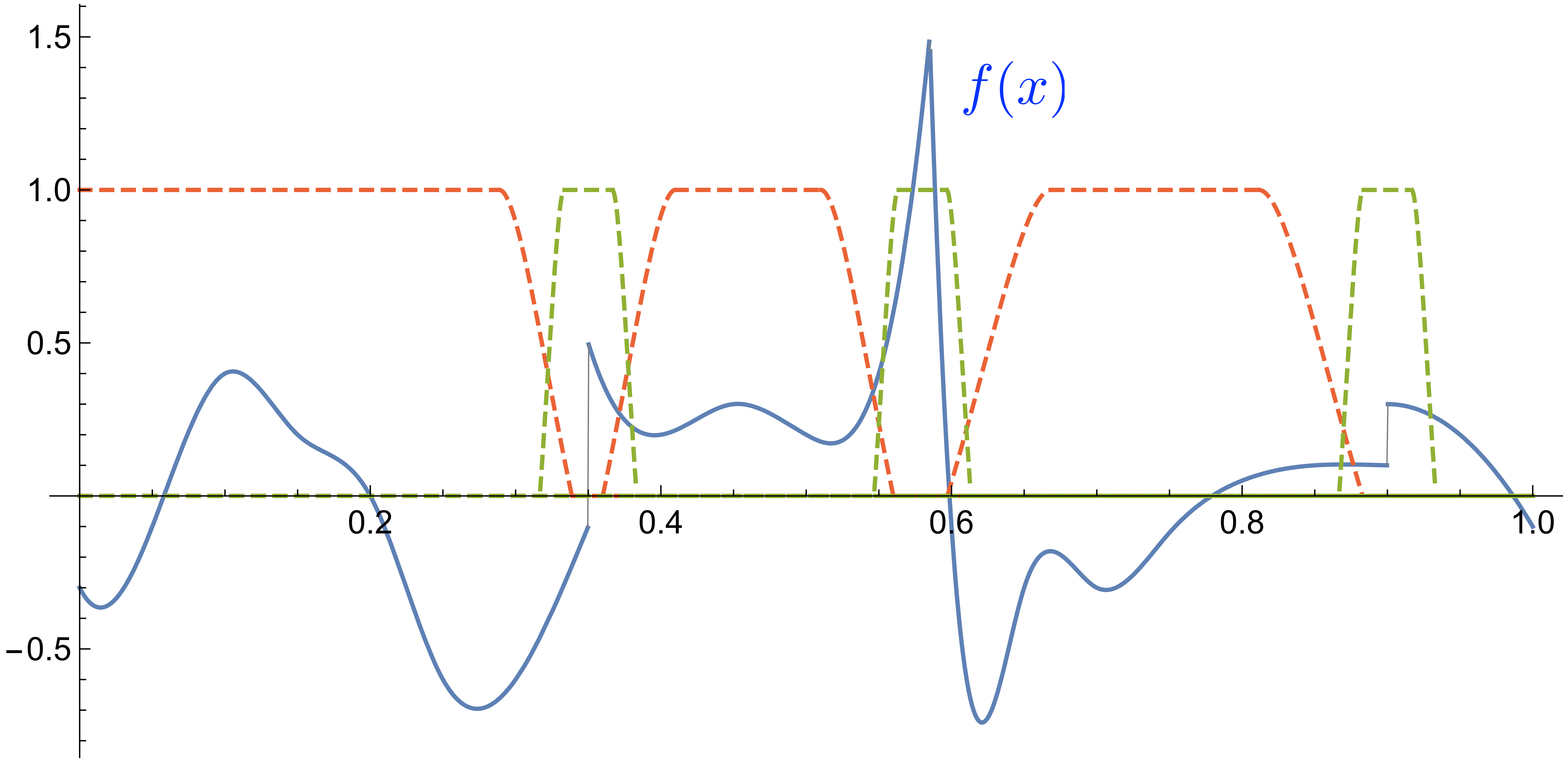

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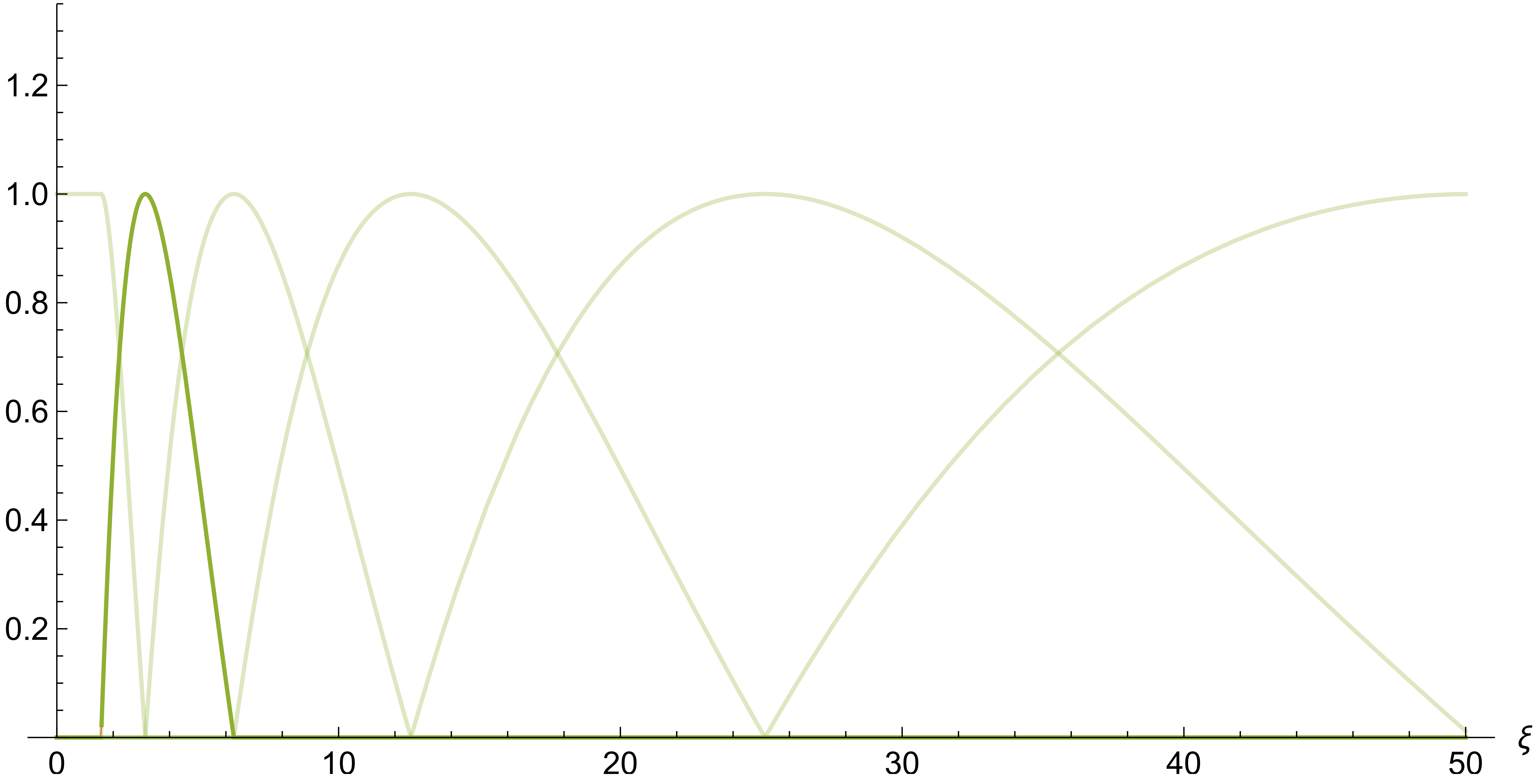

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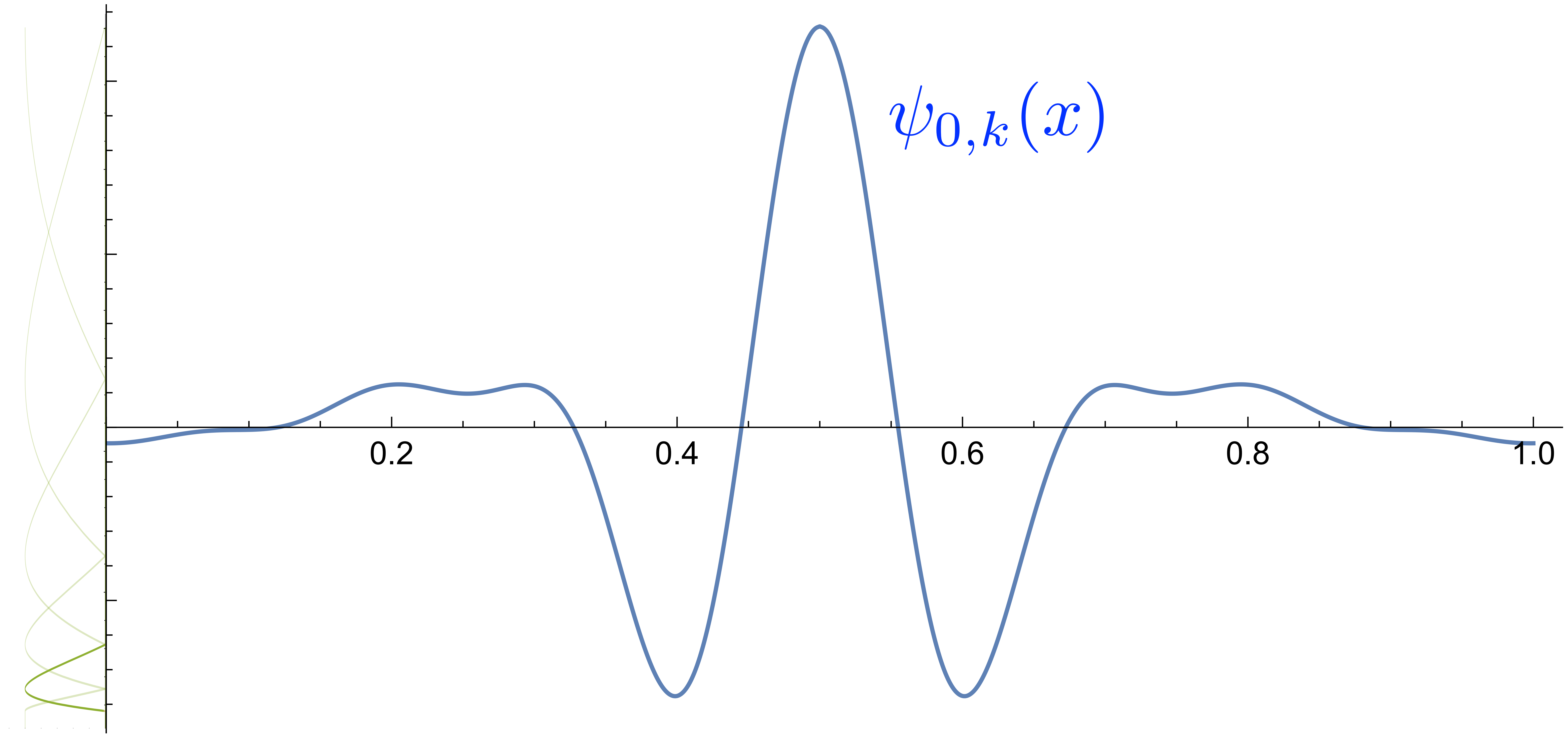

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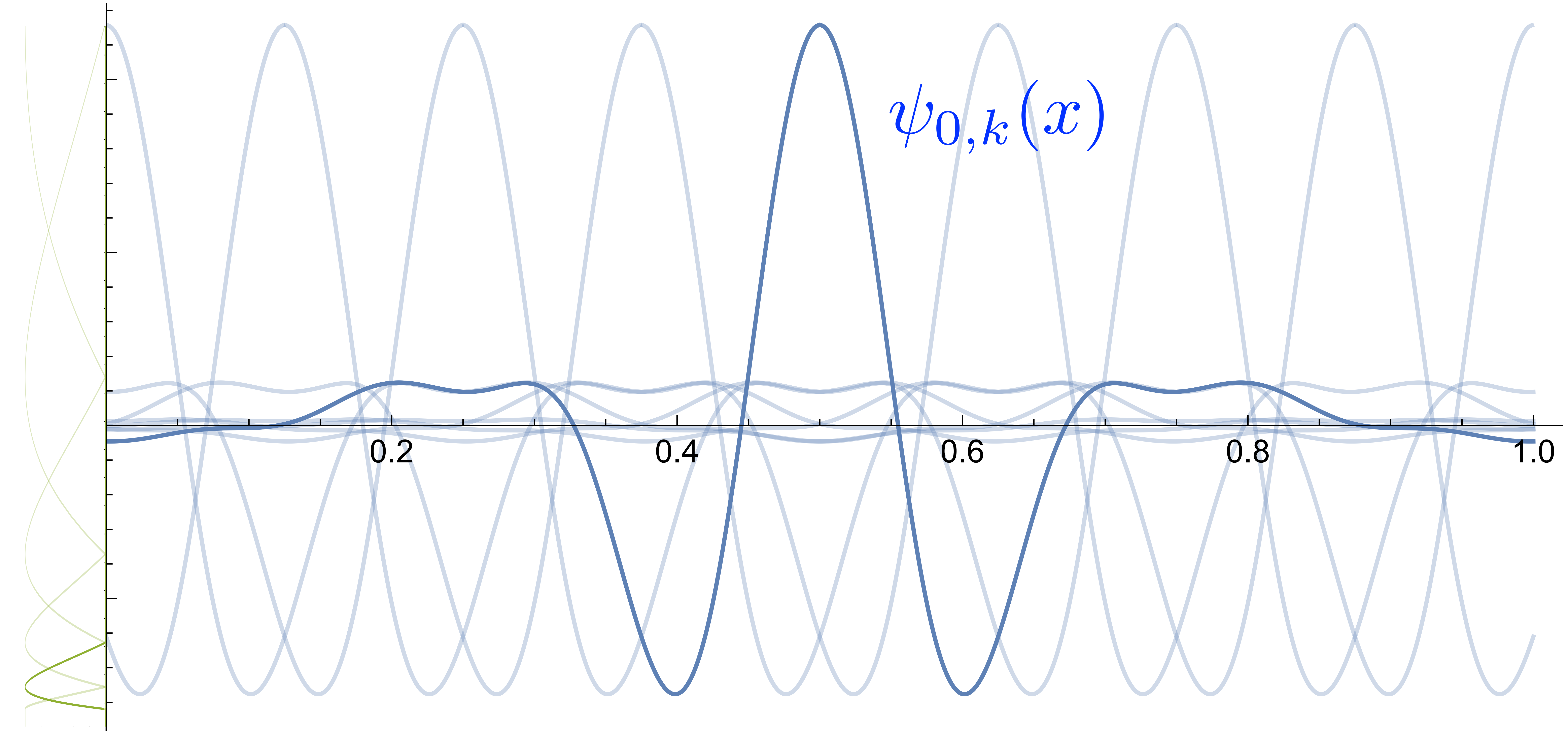

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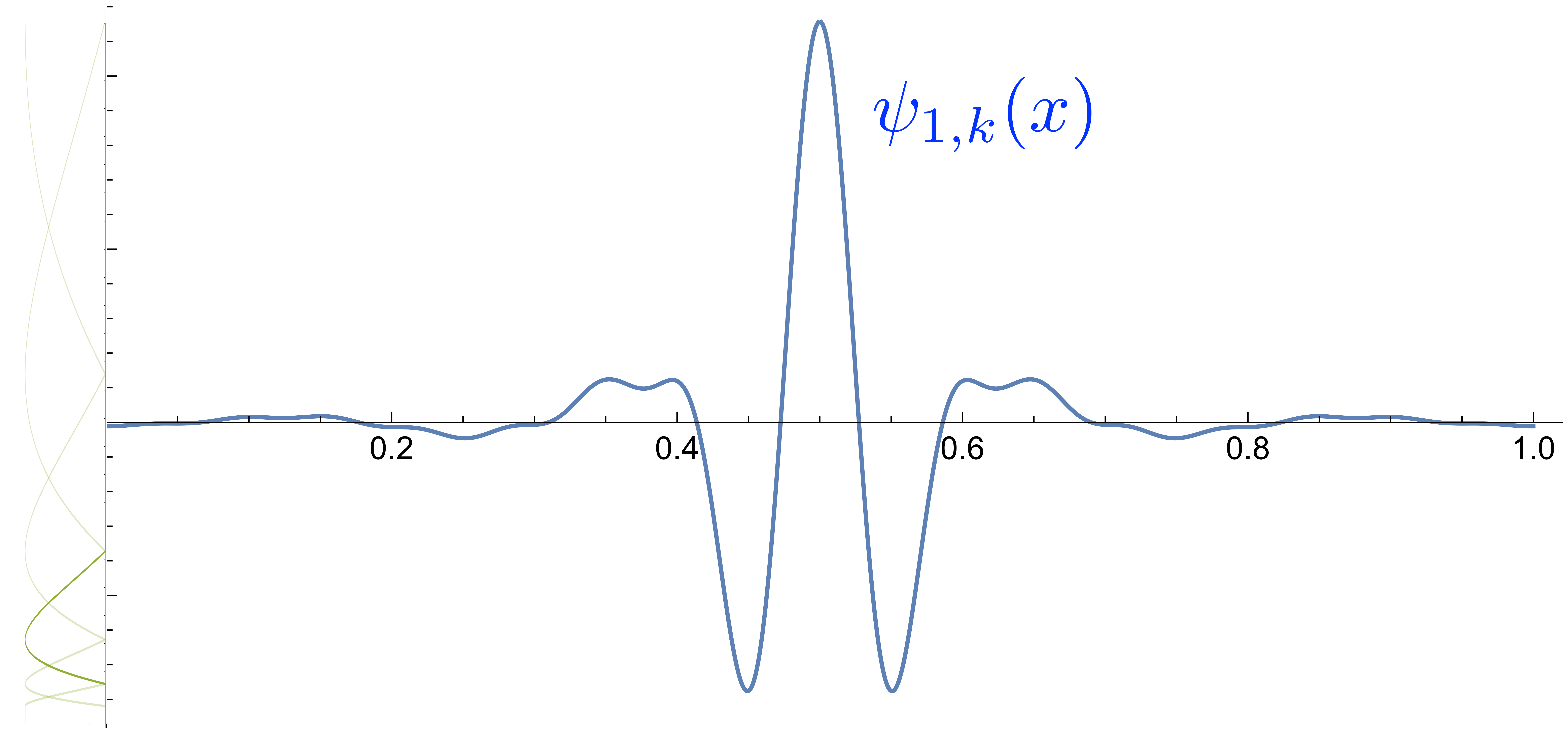

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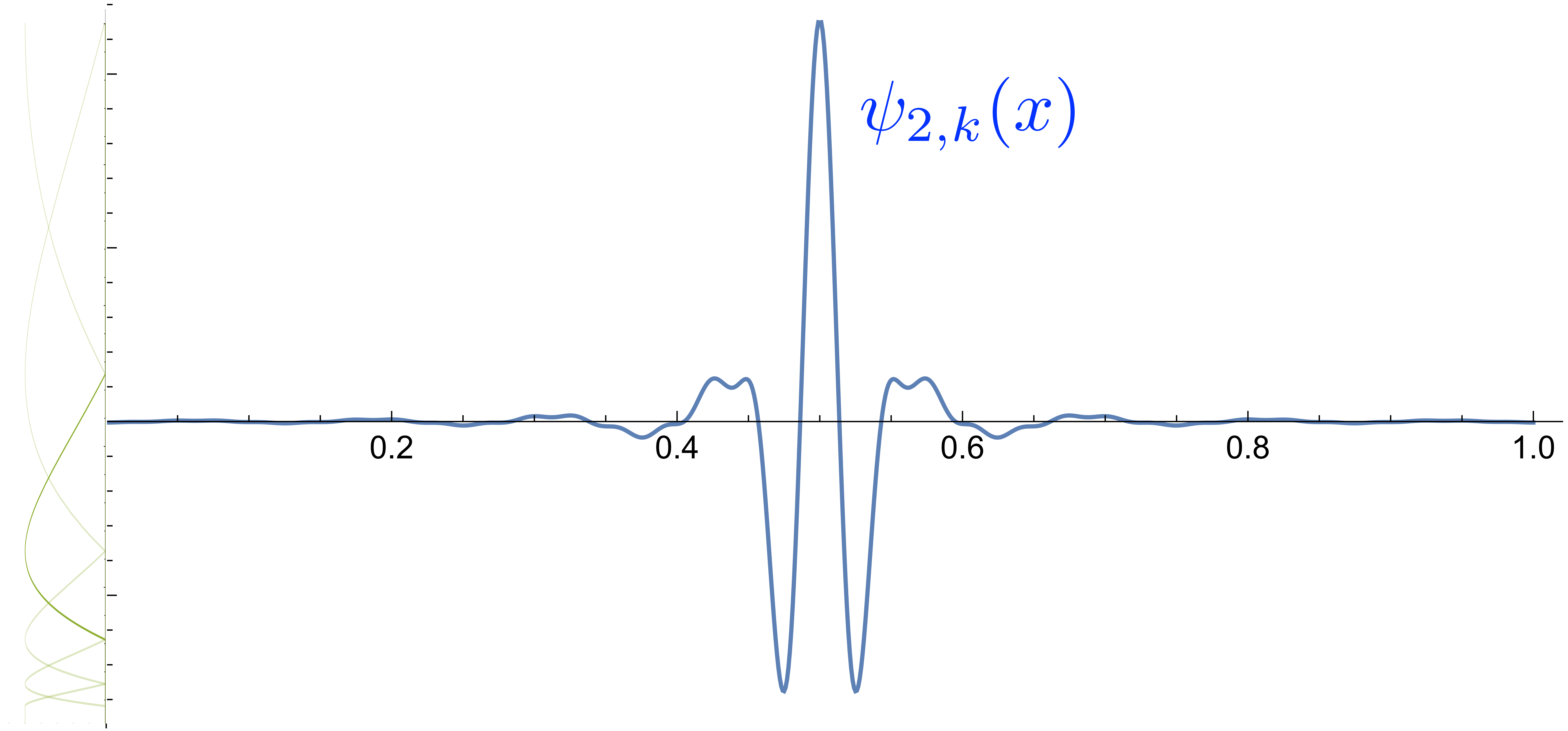

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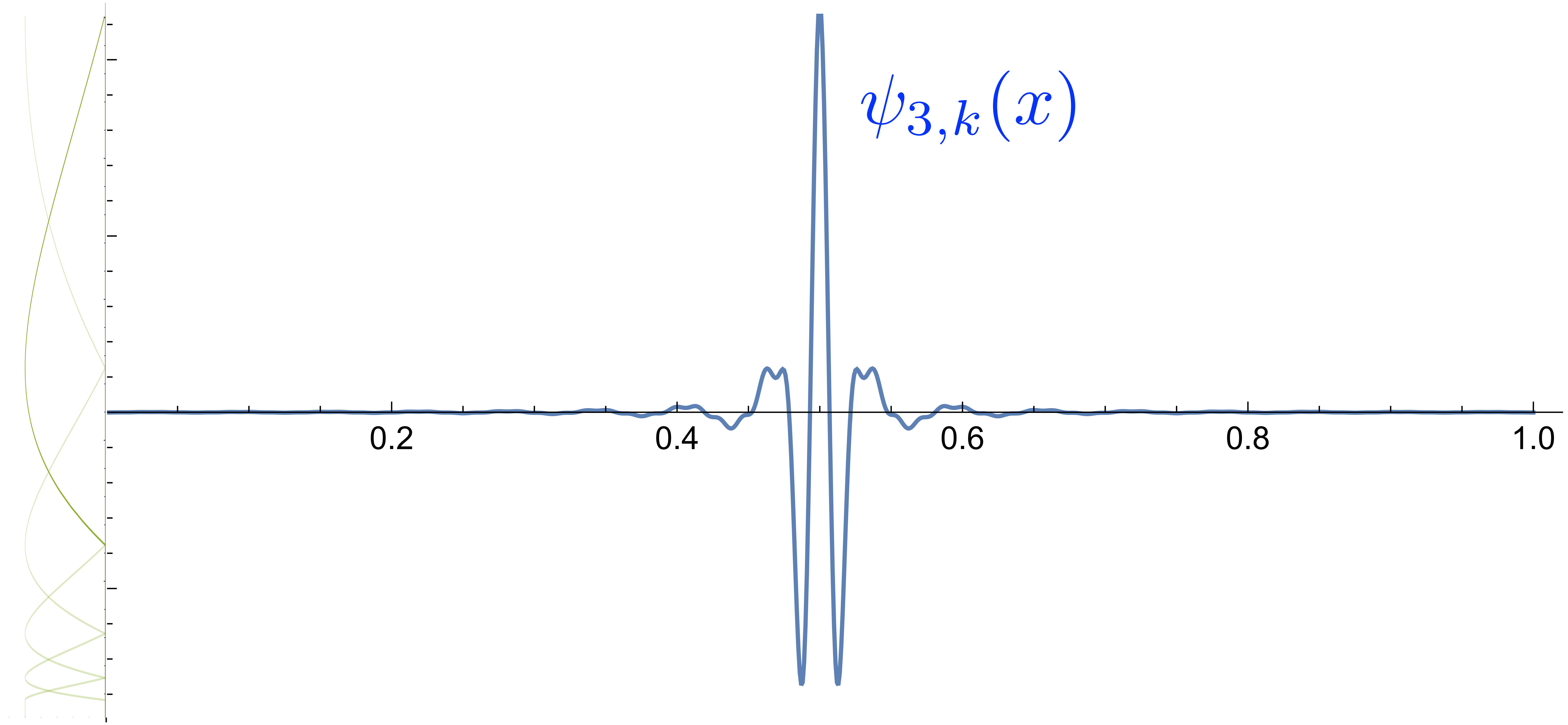

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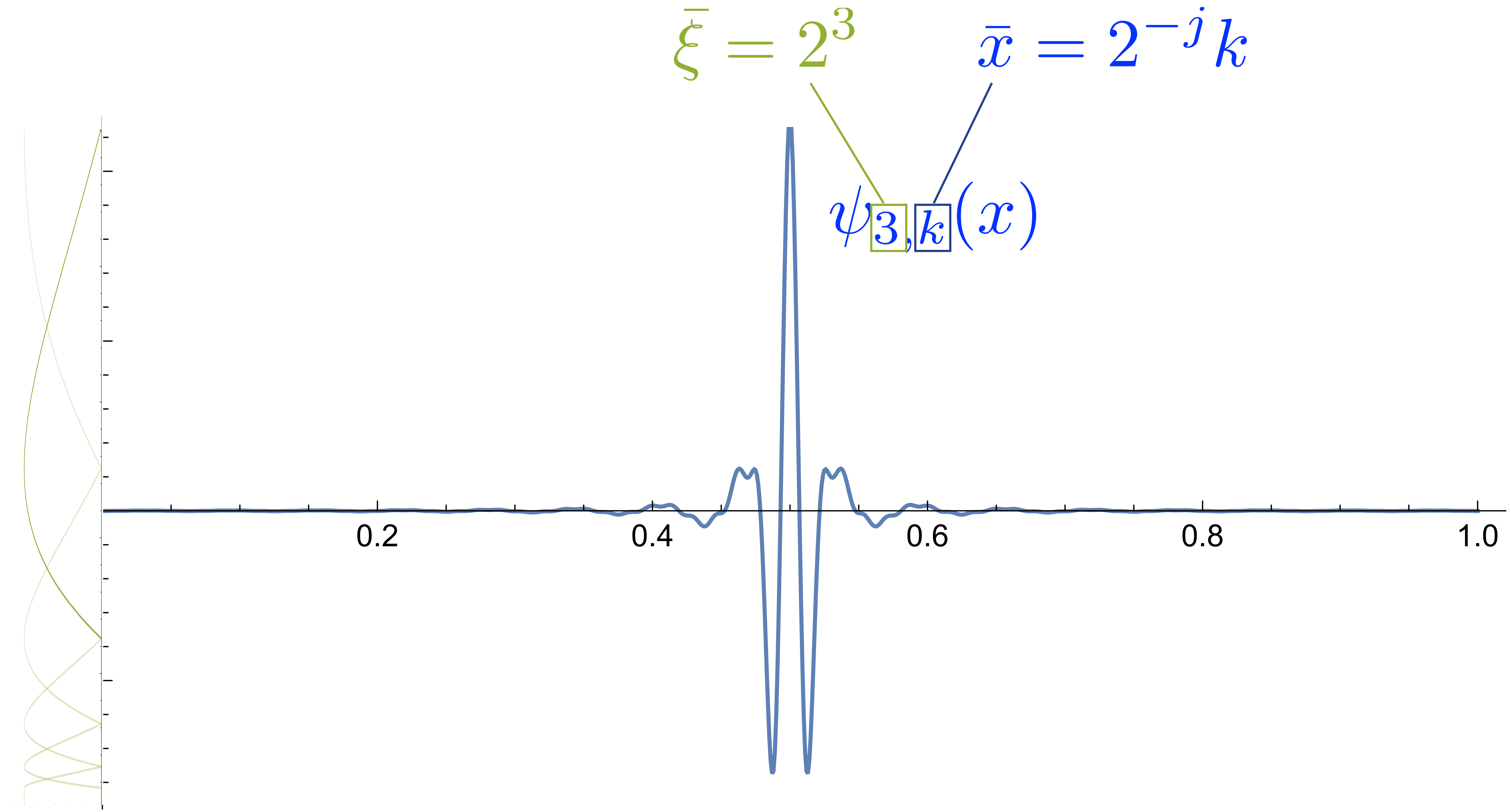

$\bar{\xi} = 2^3$

$\bar{x} = 2^{-j} k$

$\psi_{\boxed{3},\boxed{k}}(x)$

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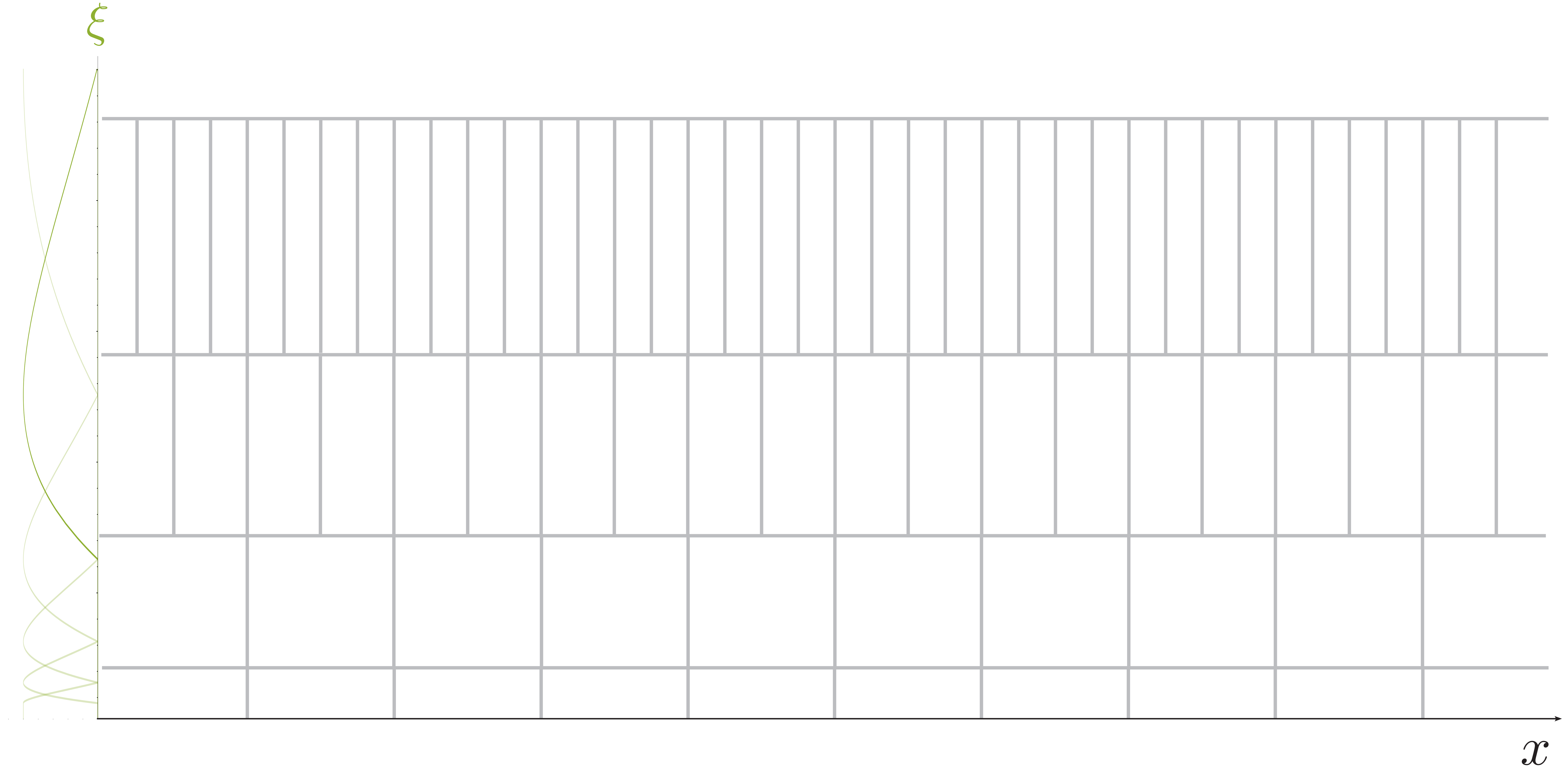

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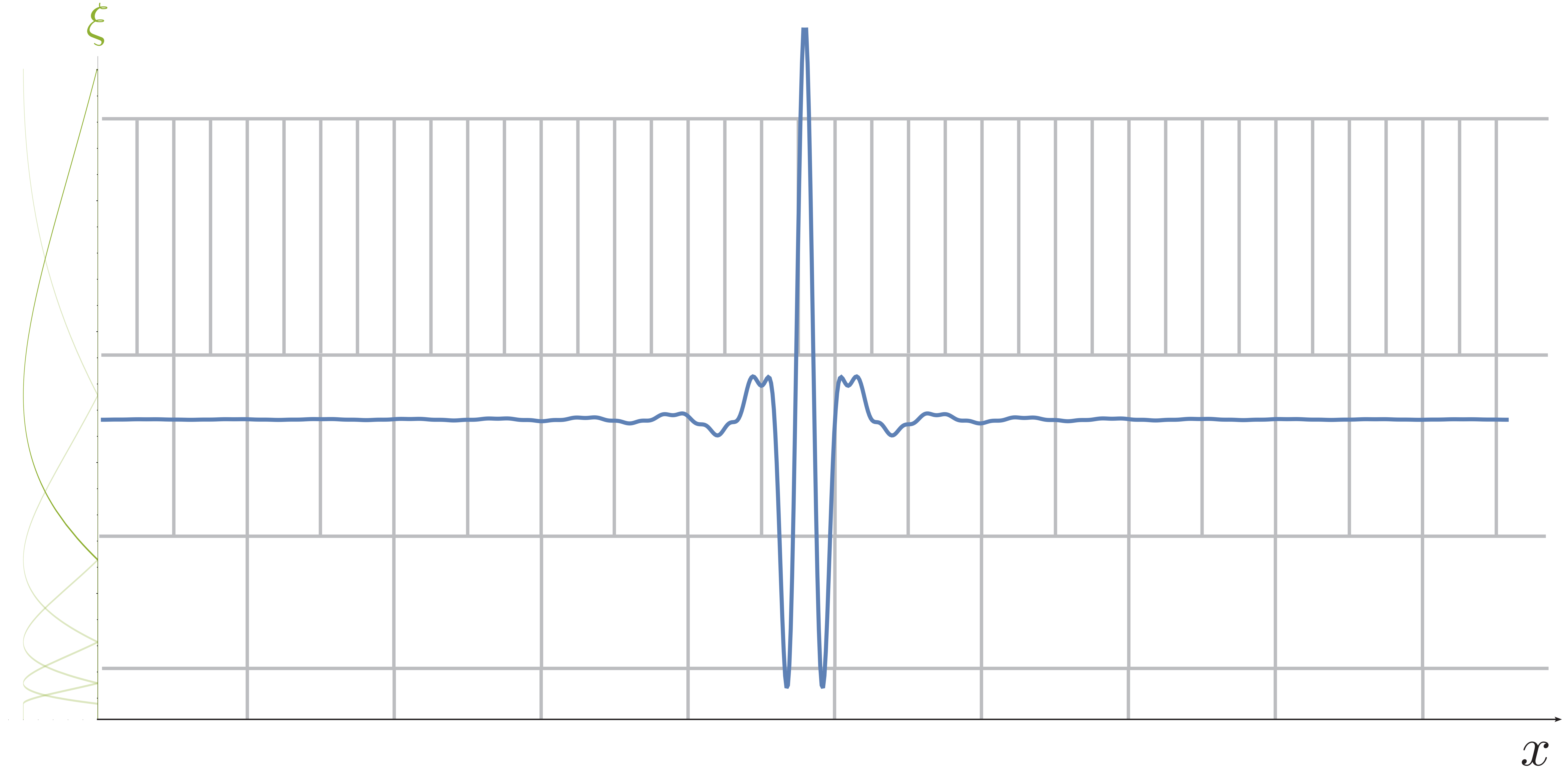

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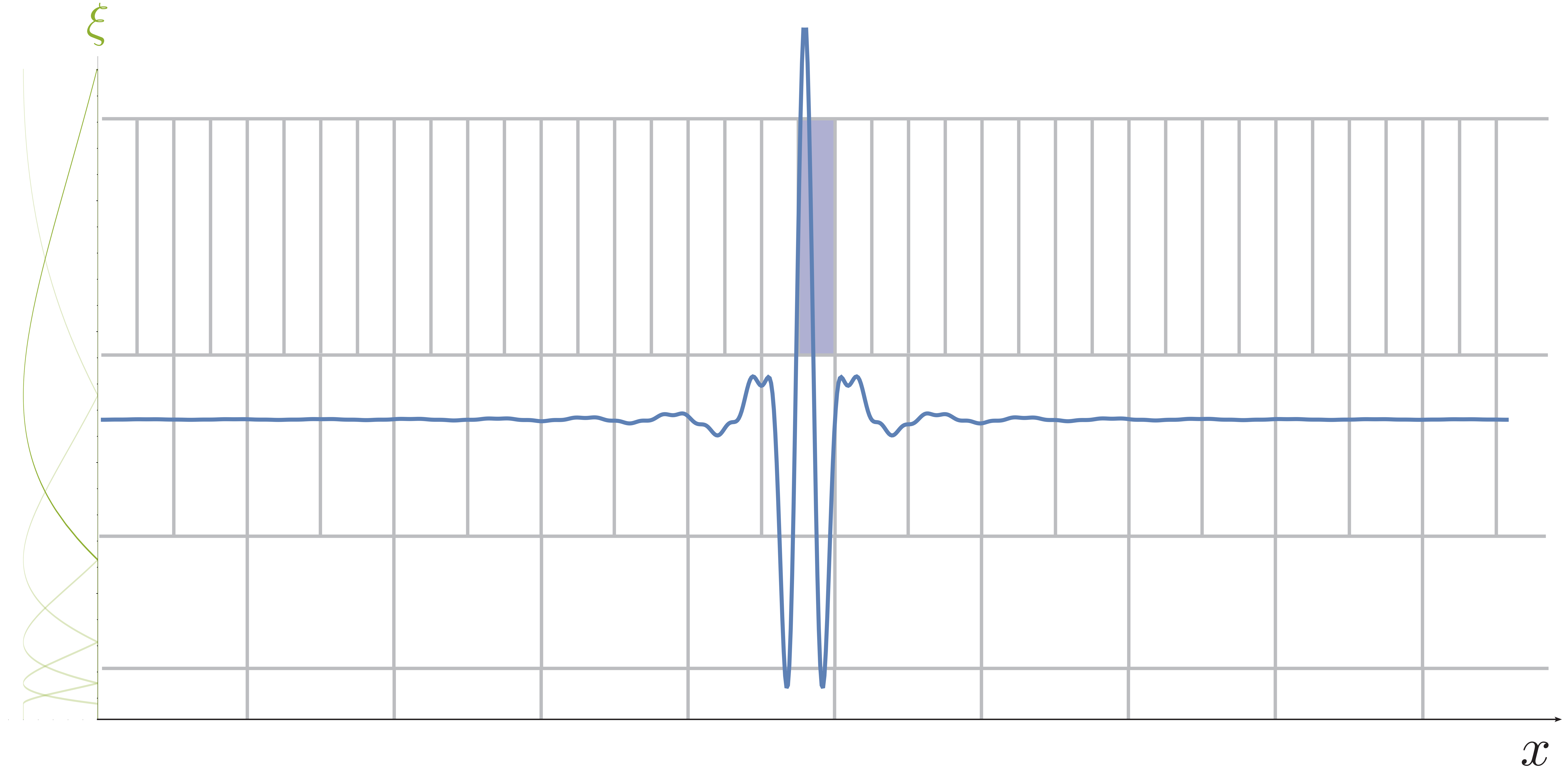

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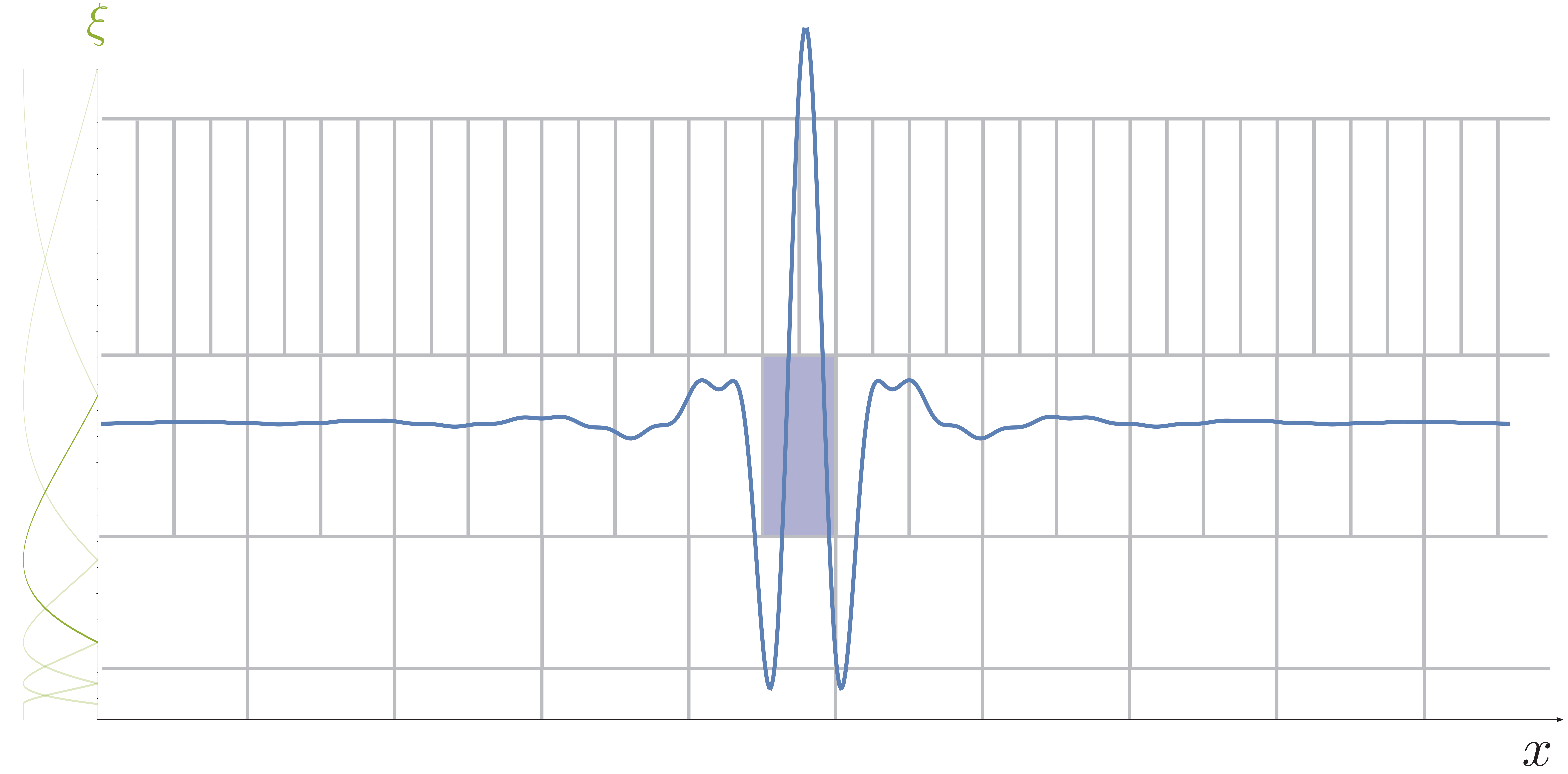

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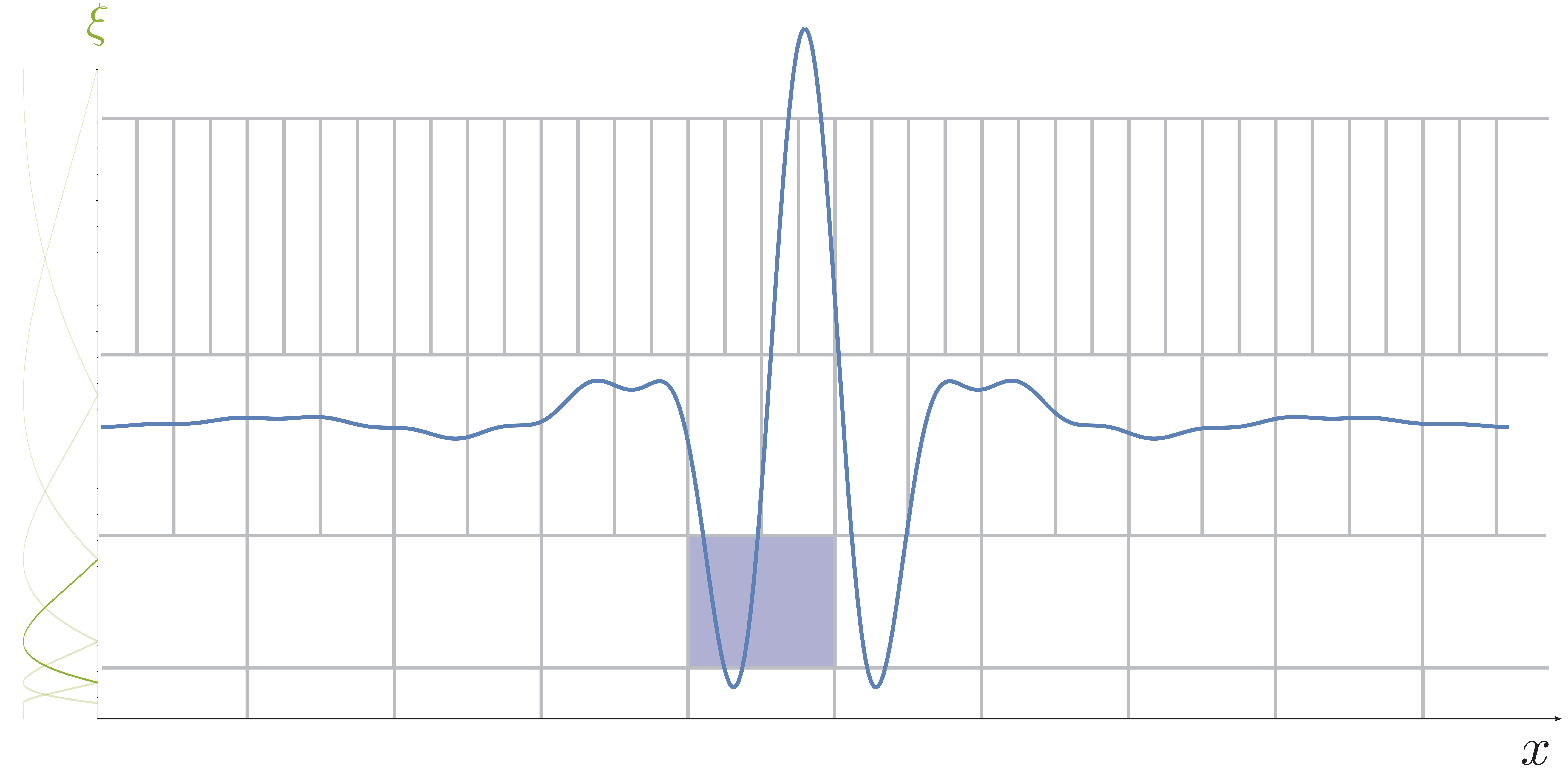

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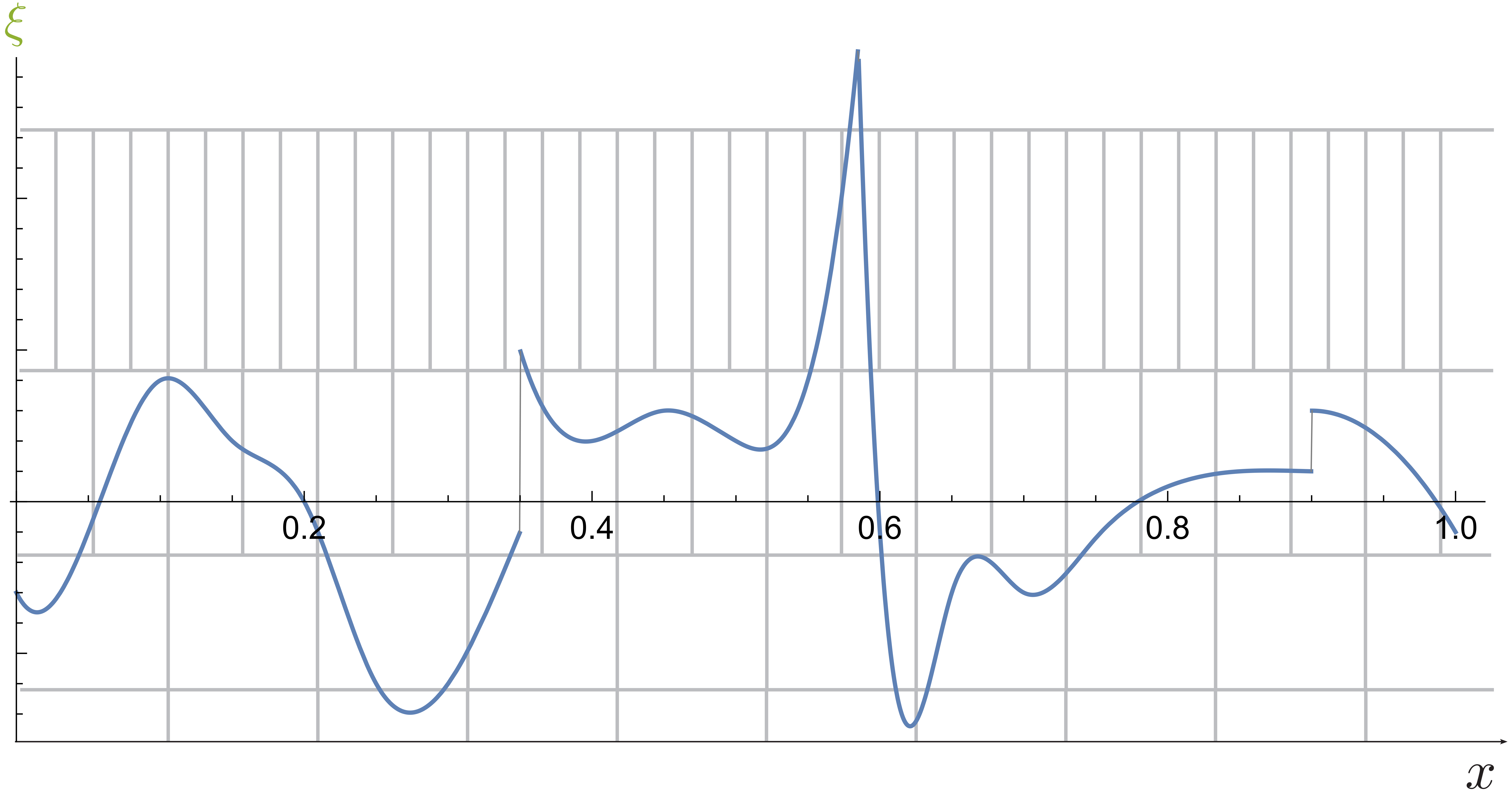

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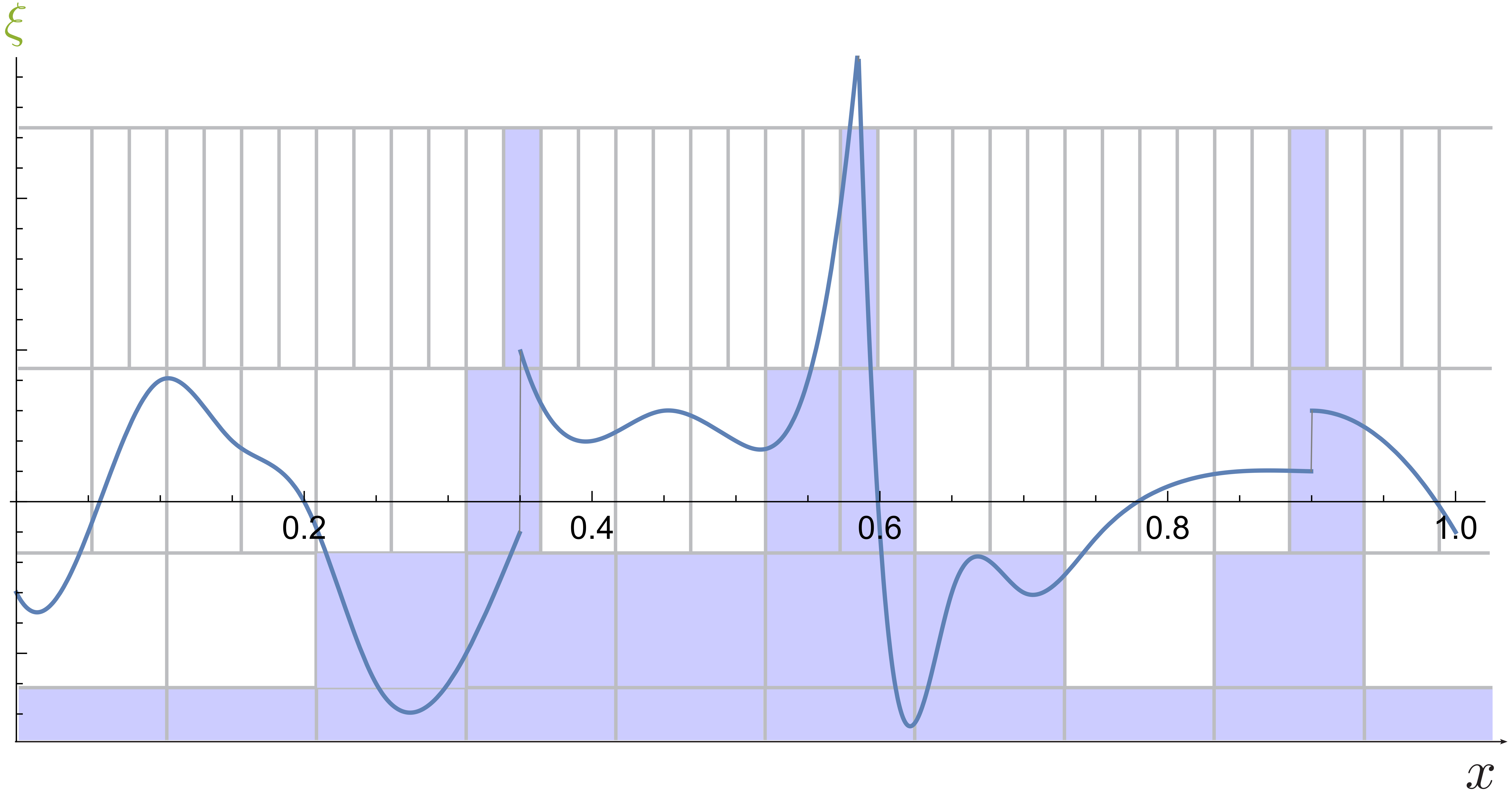

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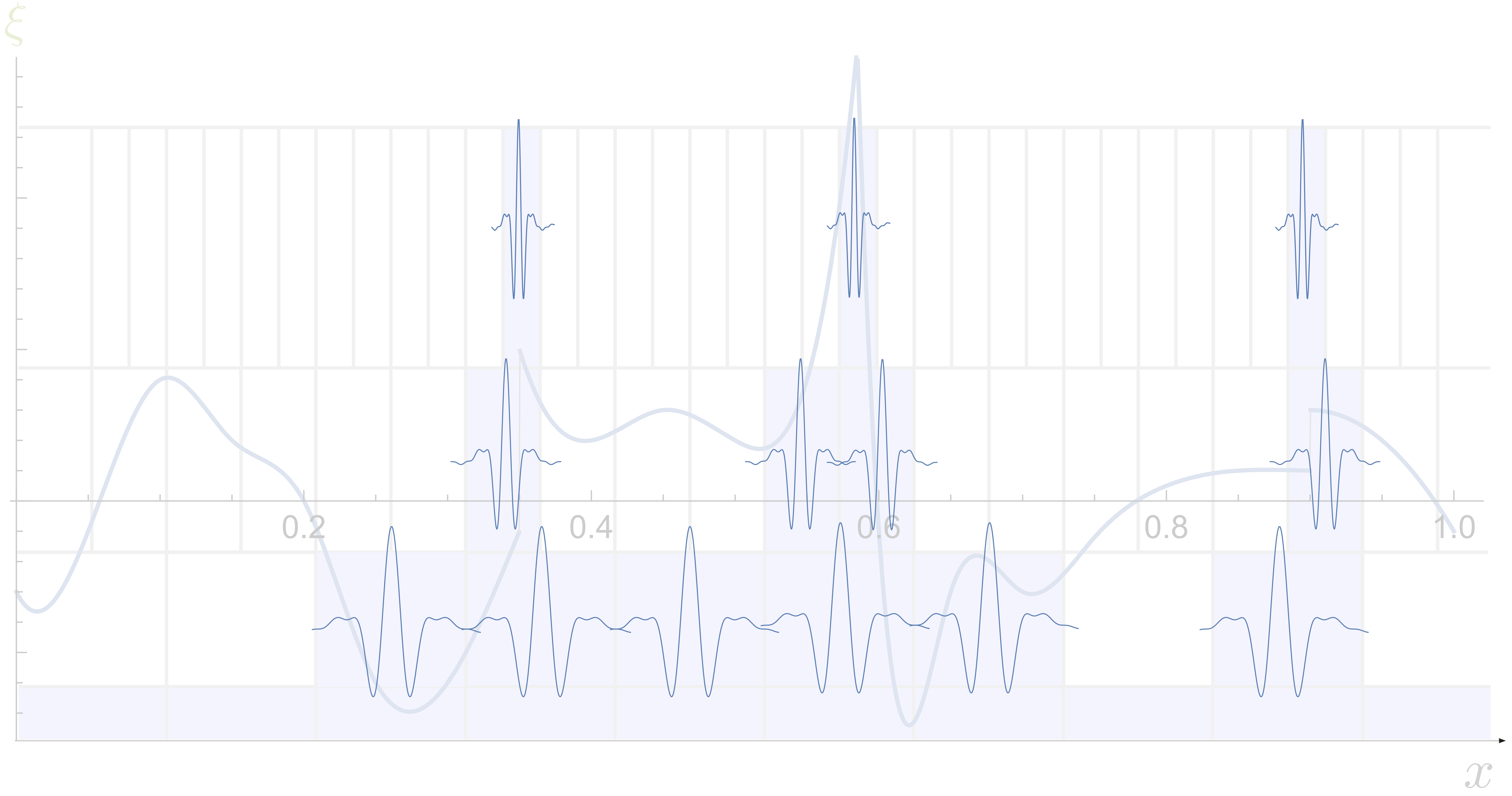

# Approximation of functions

Sparse representation: $f(x) \approx \sum_{j,k \in \mathcal{I}_f} f_{jk} \, \psi_{jk}(x)$

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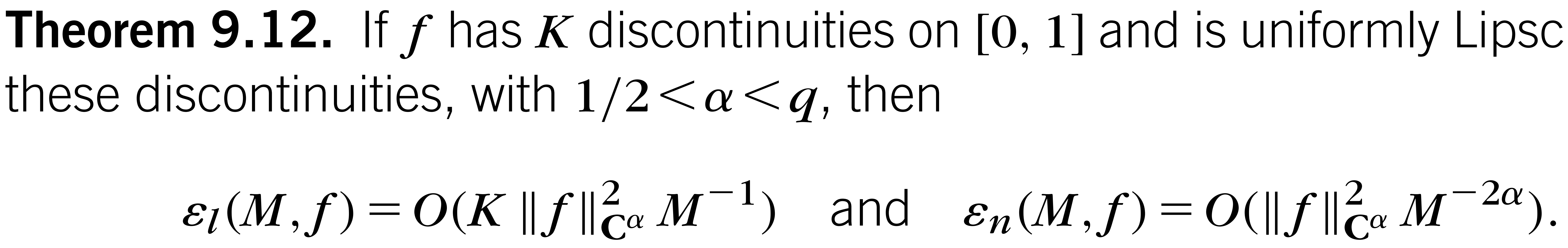
Mallat (2009):



**Theorem 9.12.** If $f$ has $K$ discontinuities on $[0, 1]$ and is uniformly Lipschitz $\alpha$ between these discontinuities, with $1/2 < \alpha < q$, then

$$\varepsilon_l(M,f) = O(K \, \|f\|^2_{\mathbf{C}^\alpha} \, M^{-1}) \quad \text{and} \quad \varepsilon_n(M,f) = O(\|f\|^2_{\mathbf{C}^\alpha} \, M^{-2\alpha}).$$

# Approximation of functions on $\mathbb{R}^n$

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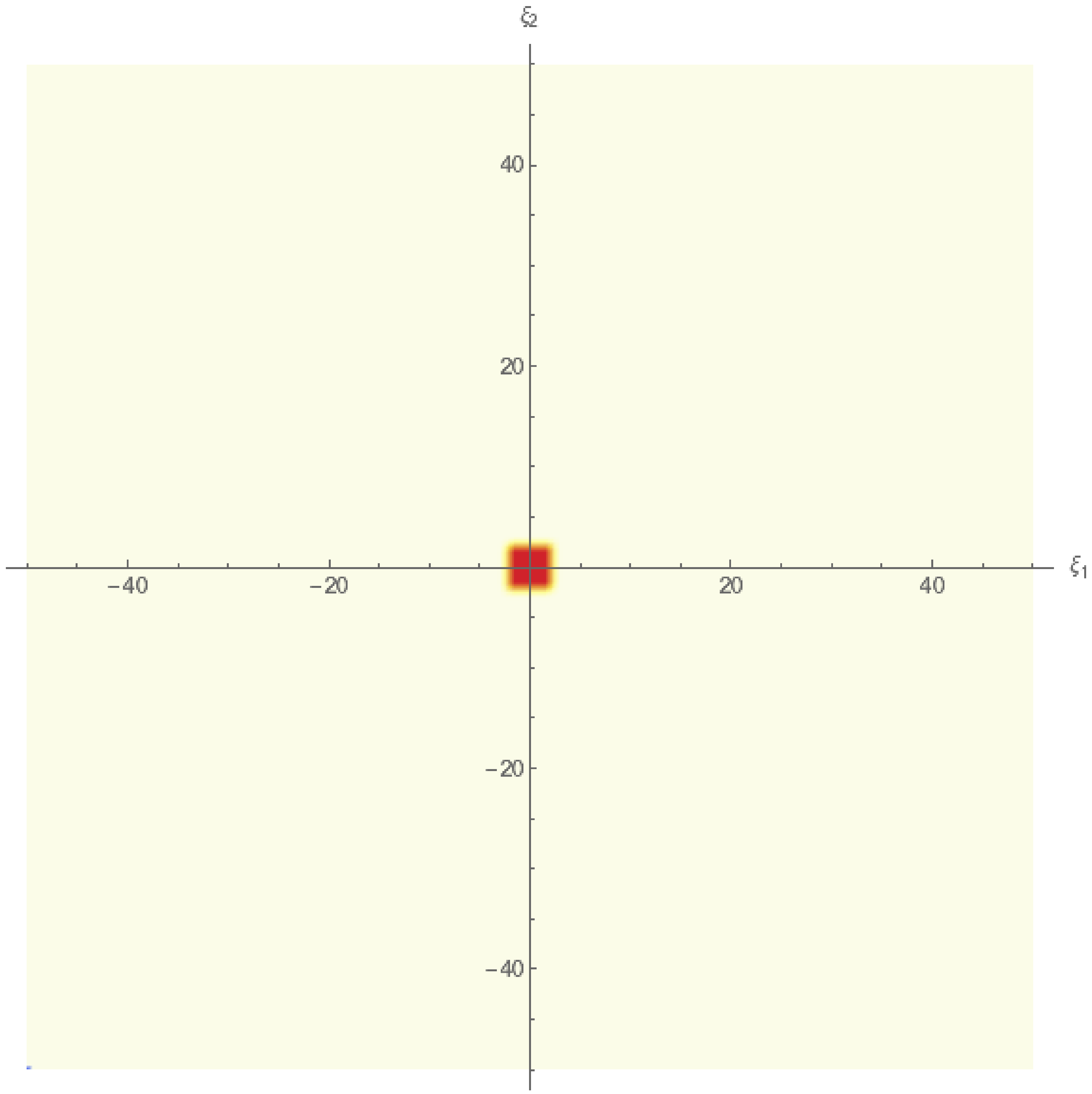
- Use that $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$

# Approximation of functions on $\mathbb{R}^n$



$$\hat{\psi}_j(\xi_1, \xi_2) = \hat{\psi}_j(\xi_1)\, \hat{\psi}_j(\xi_2)$$

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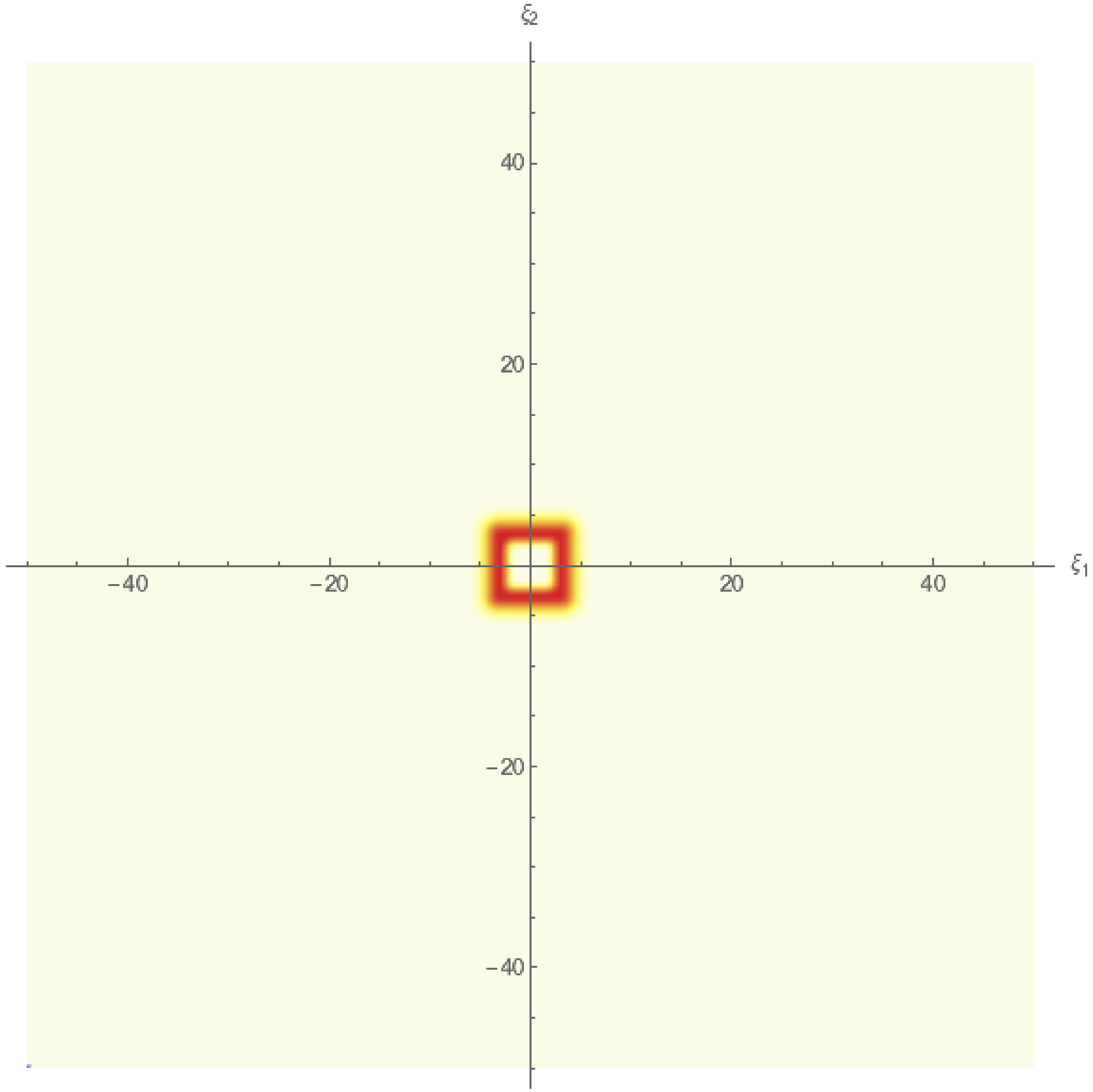

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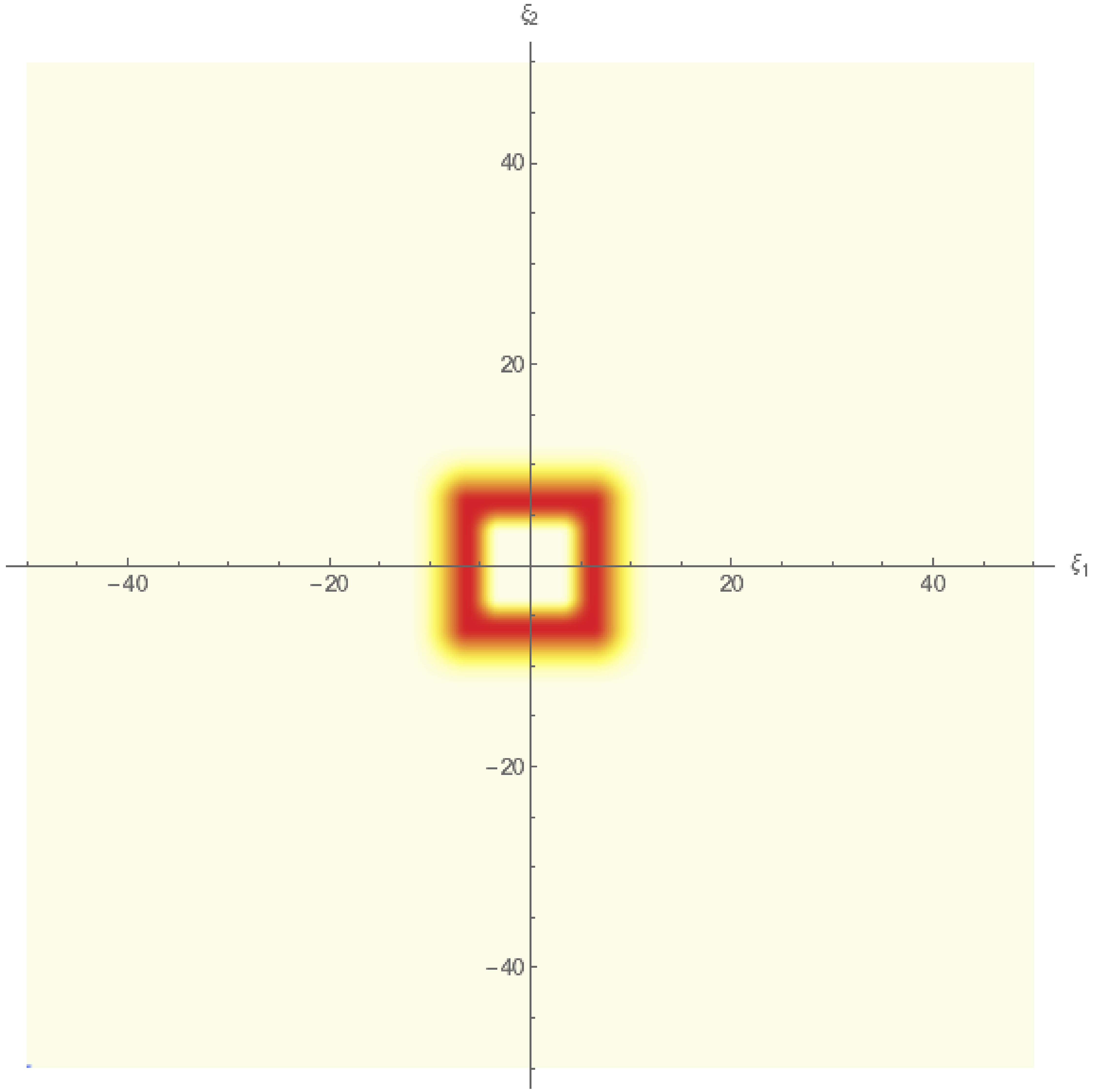

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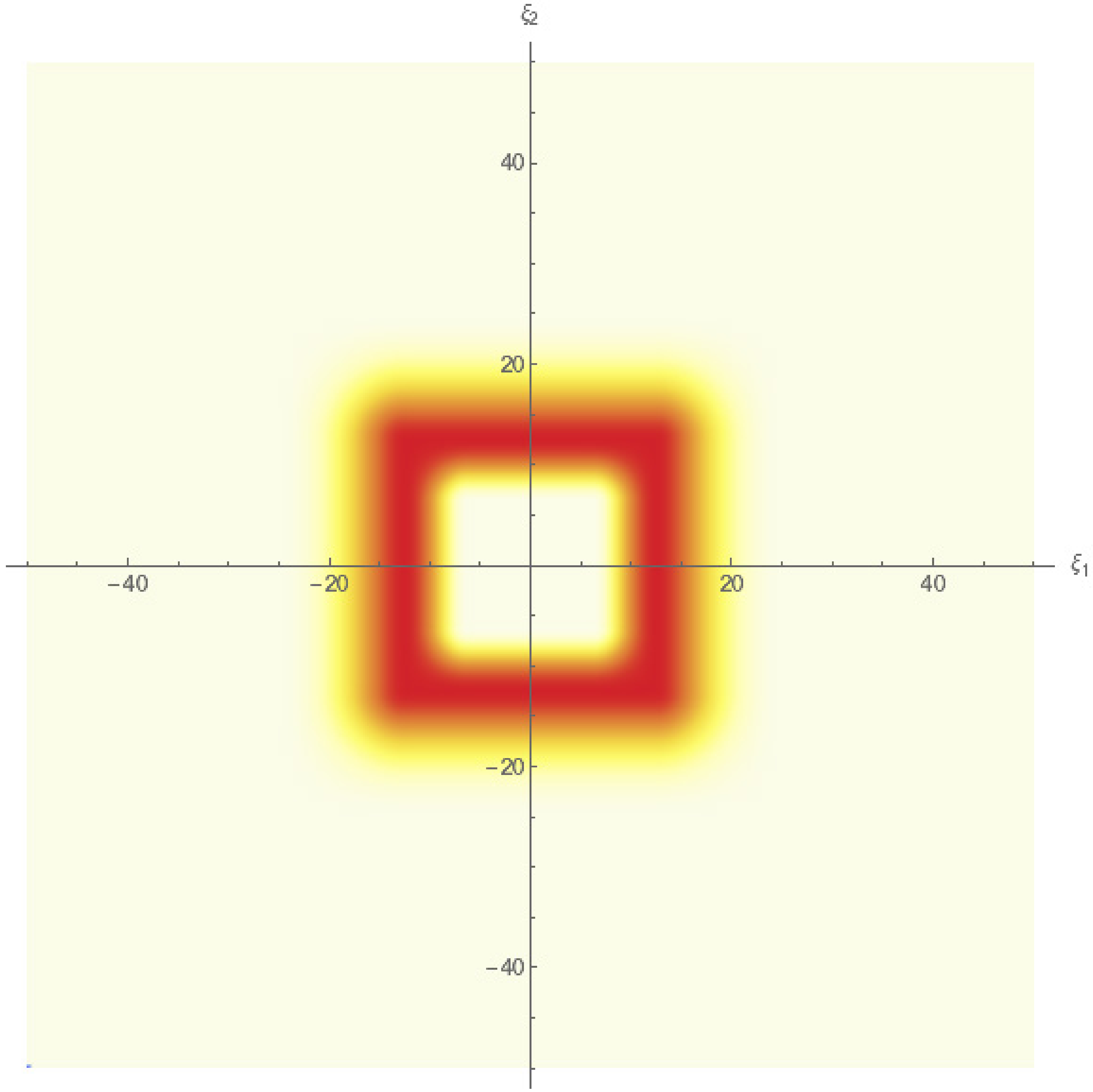

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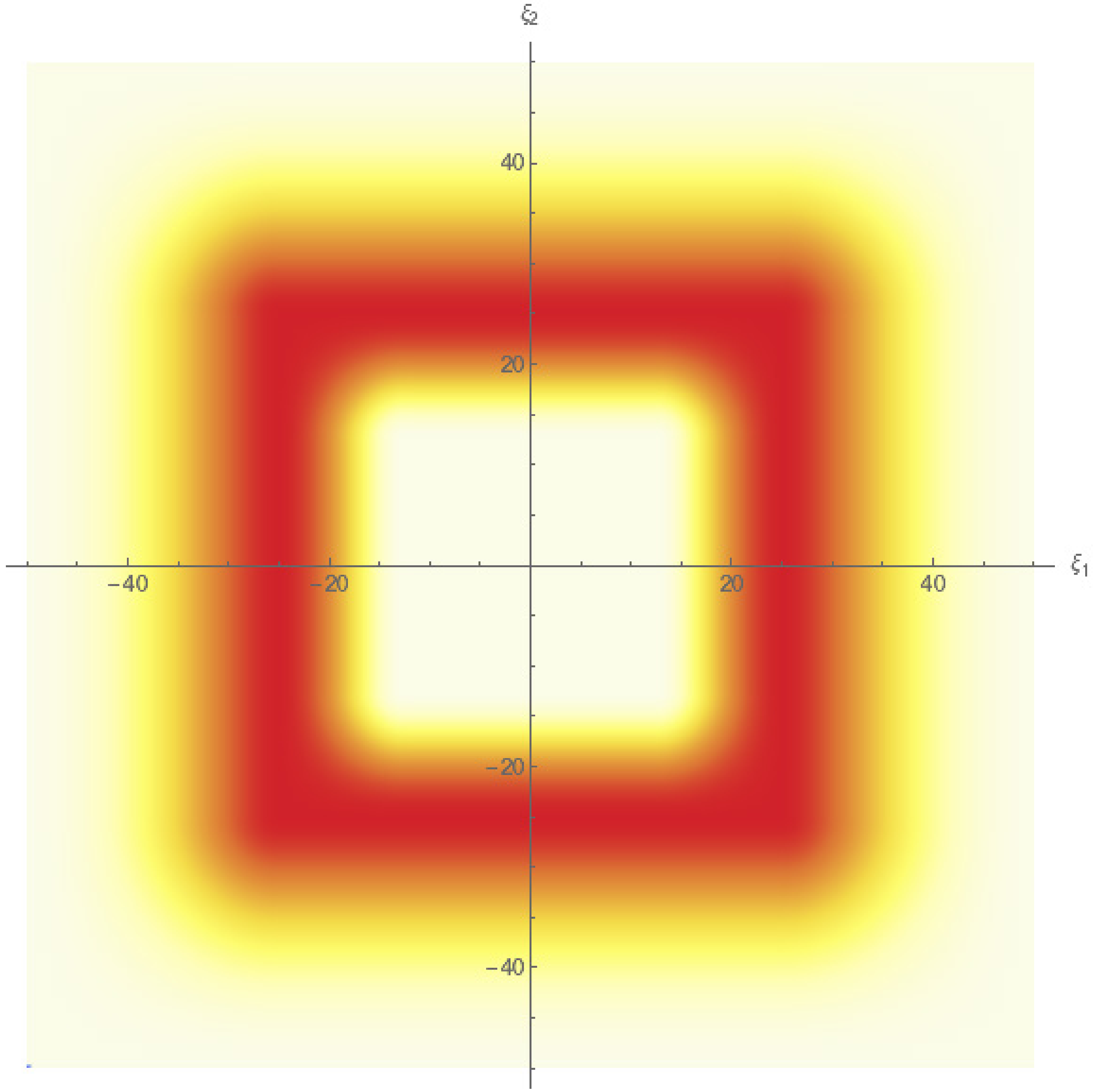

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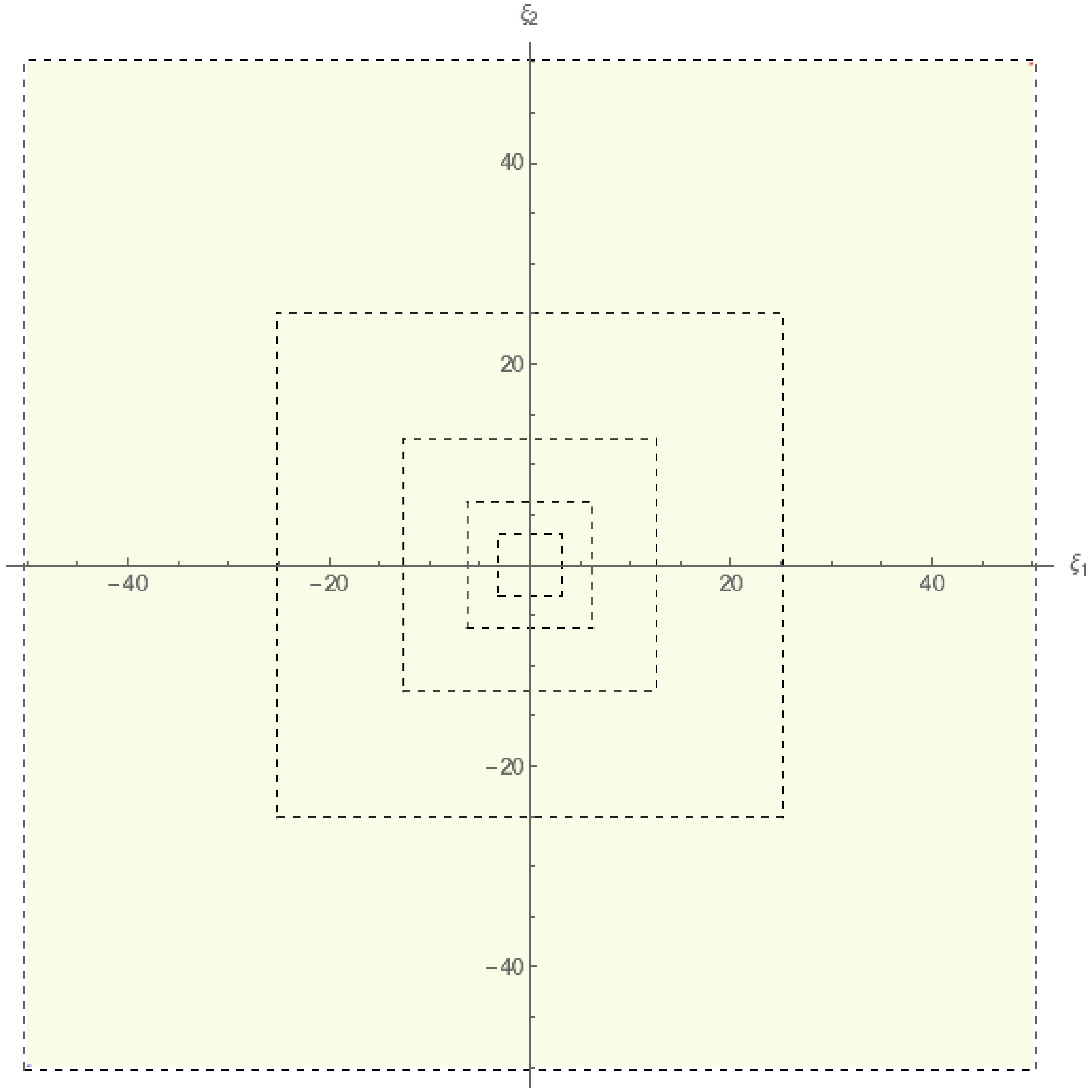

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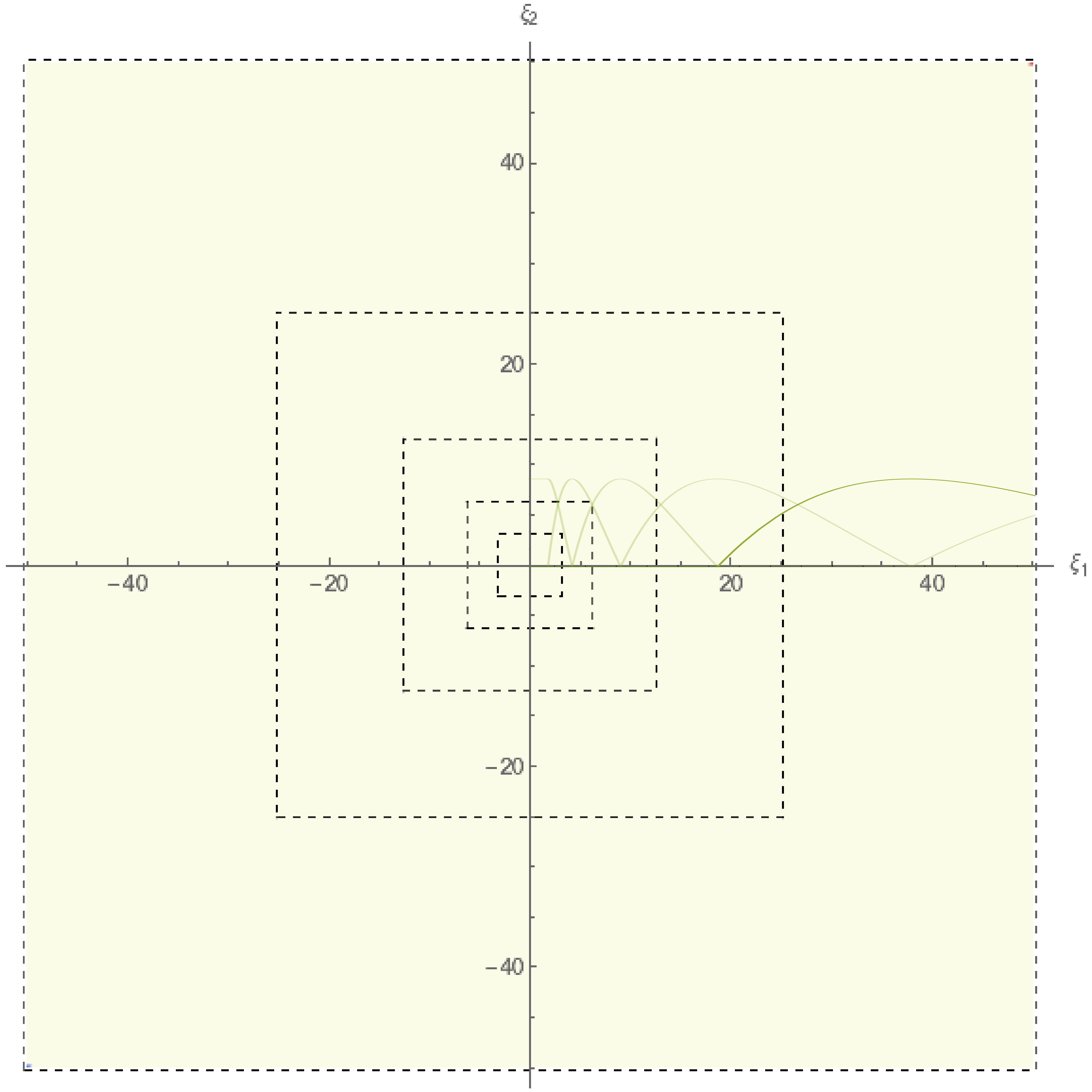
# Approximation of functions on $\mathbb{R}^n$



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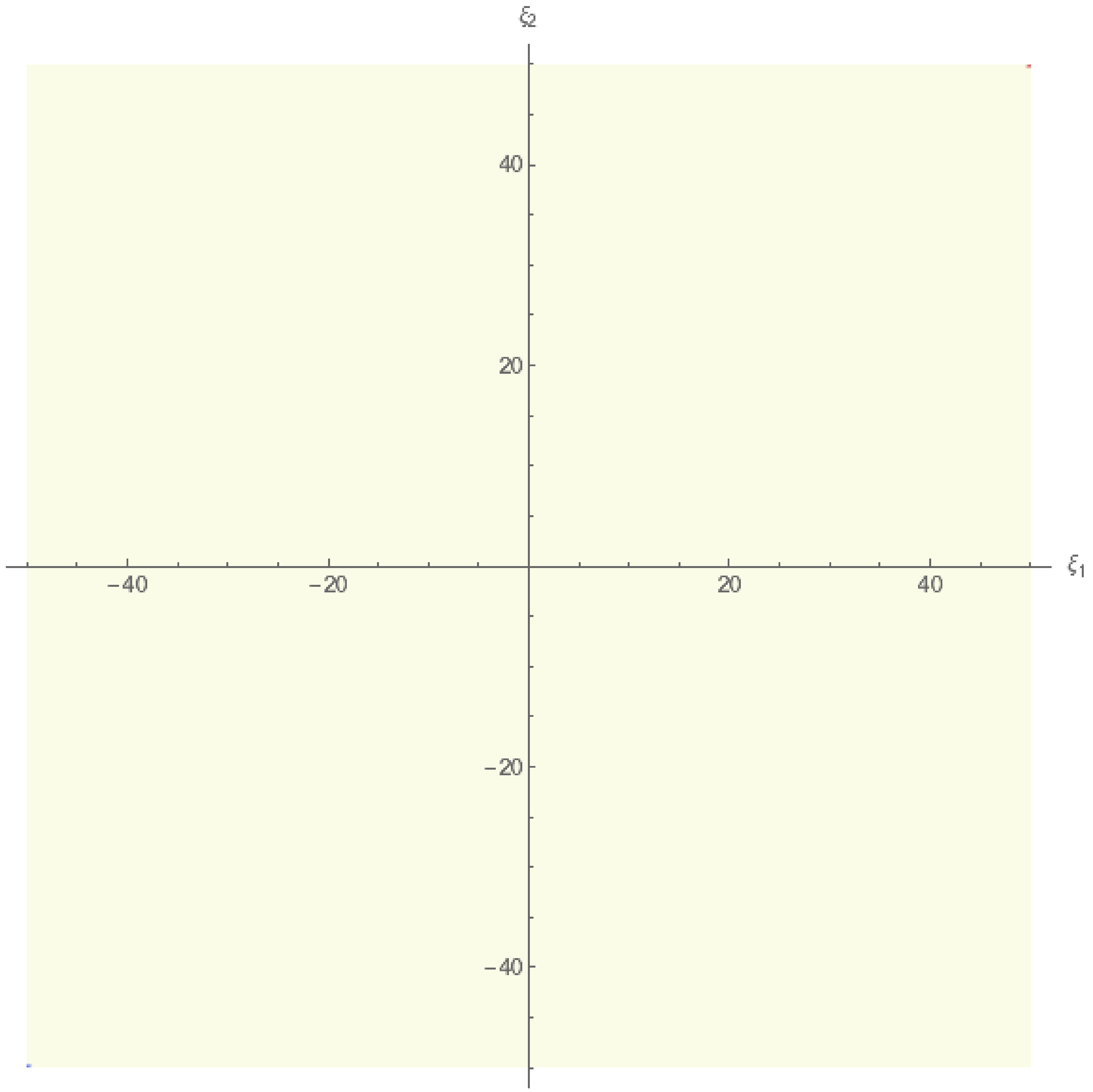
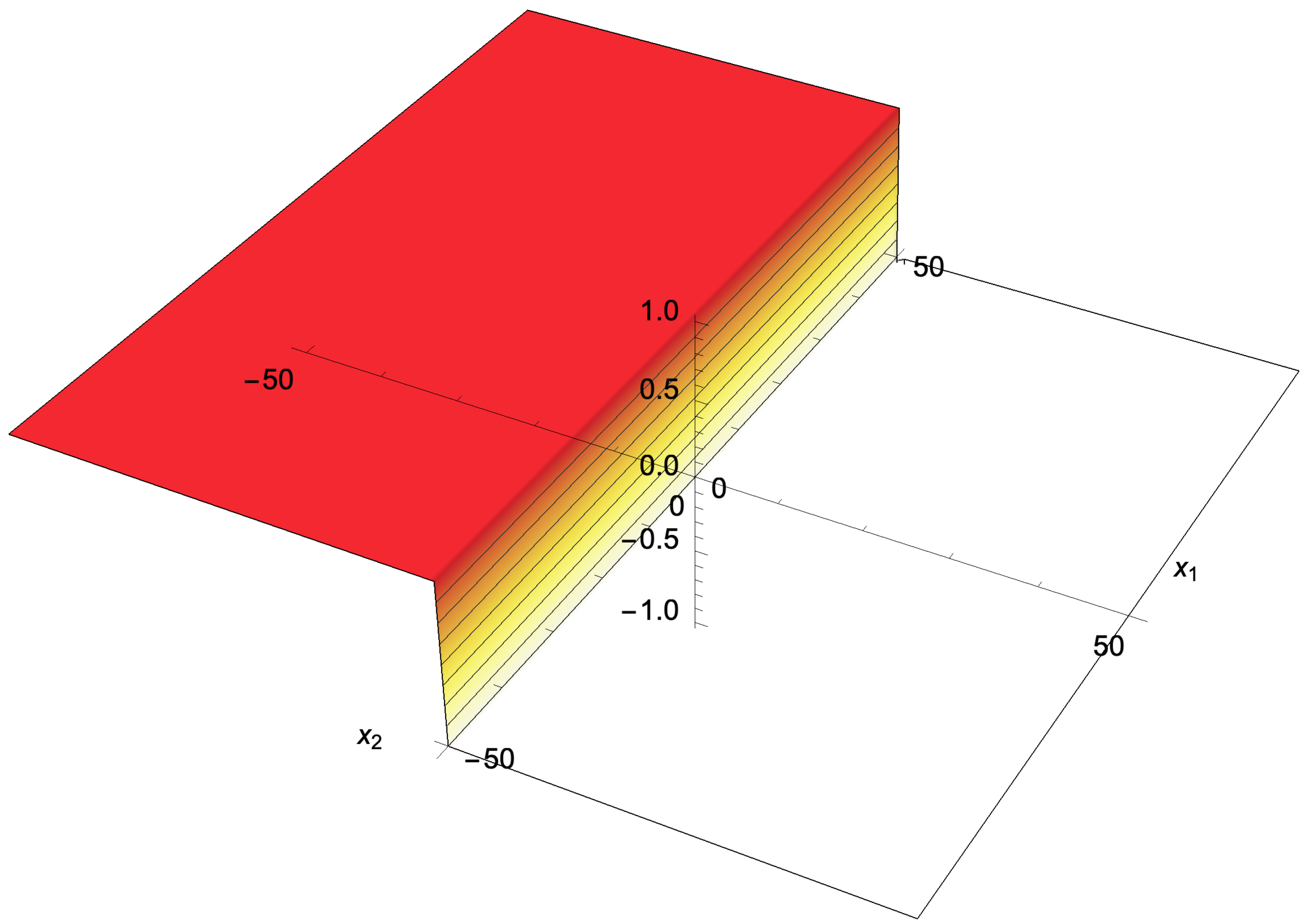
# Approximation of functions on $\mathbb{R}^n$





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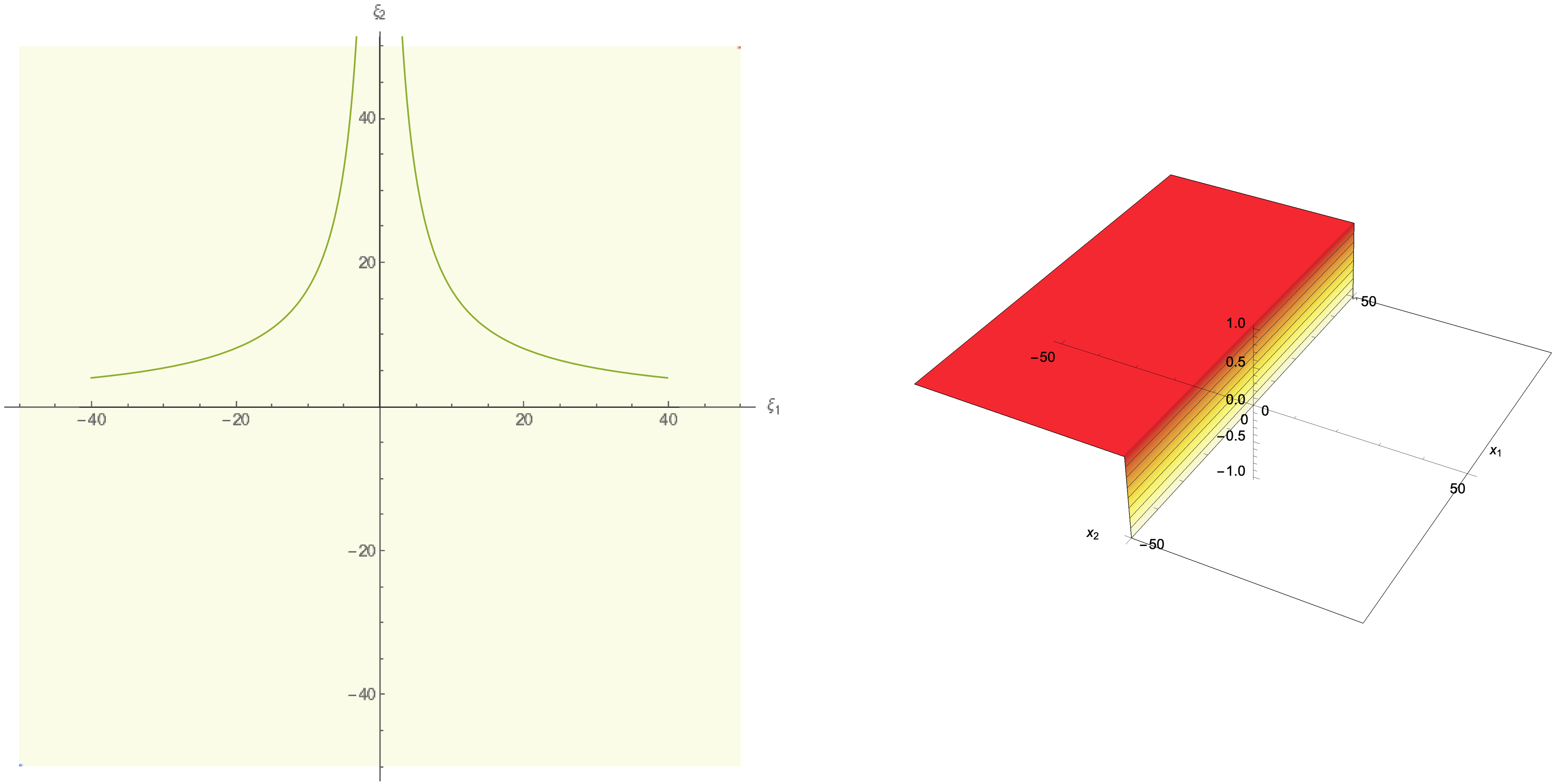

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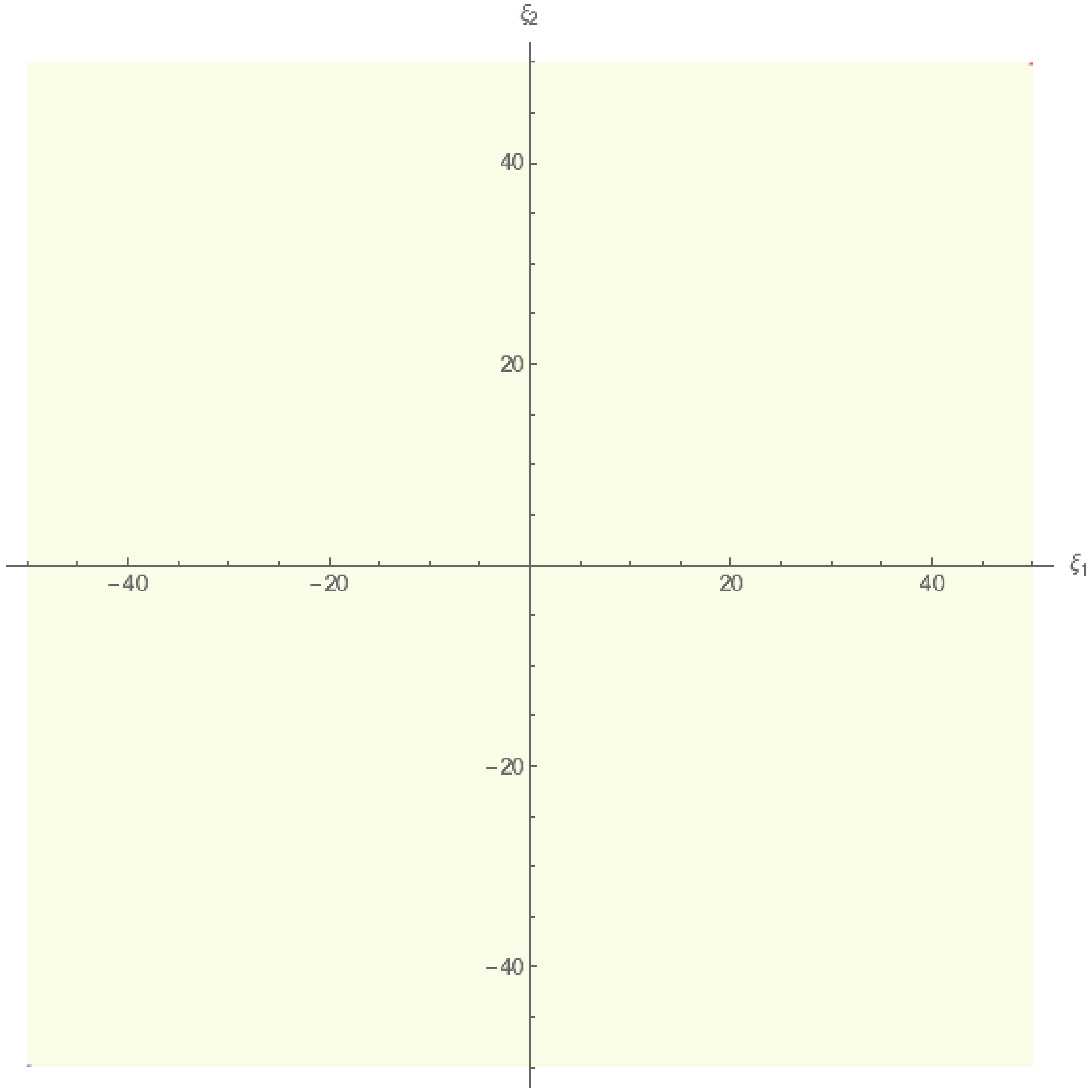

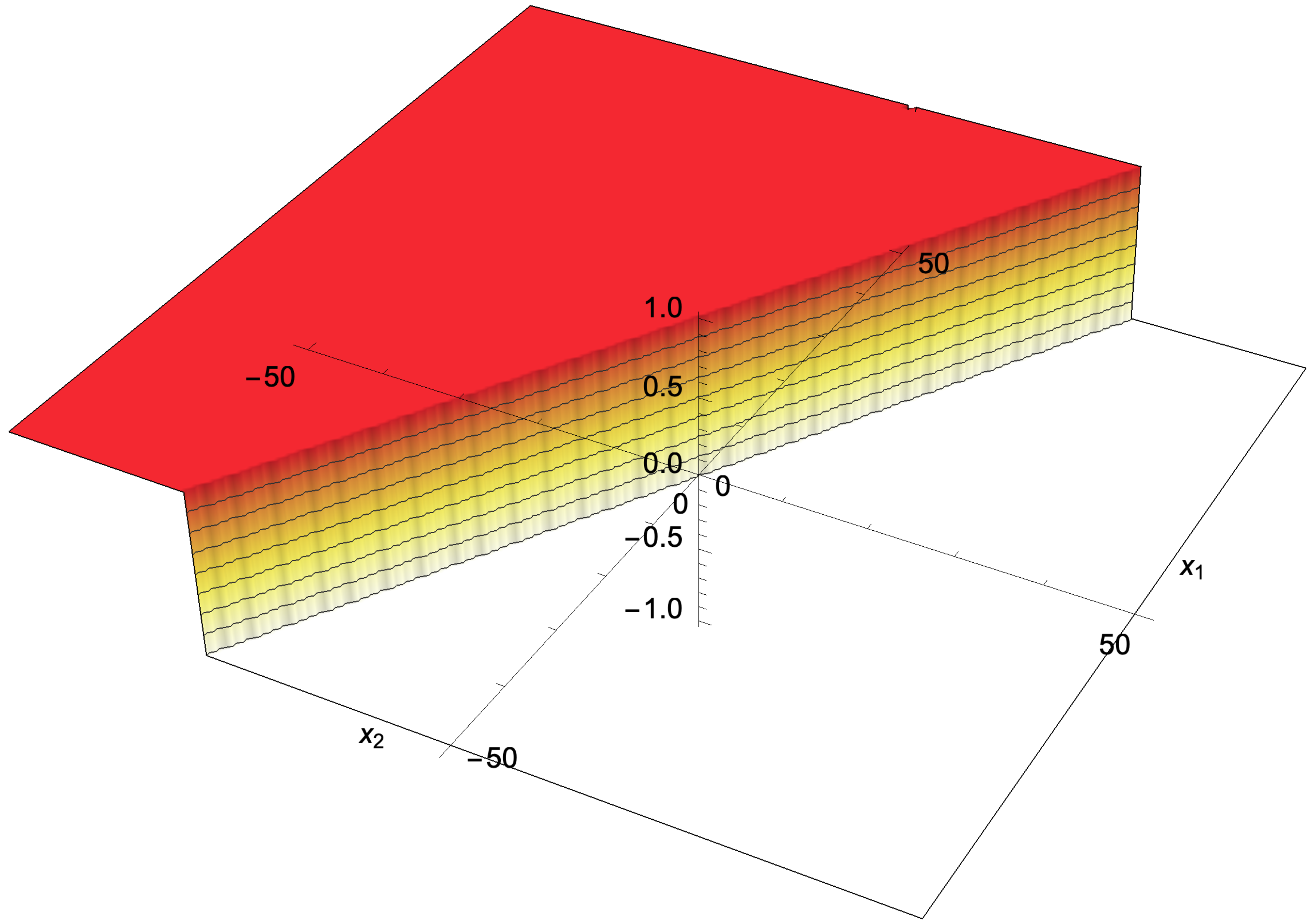

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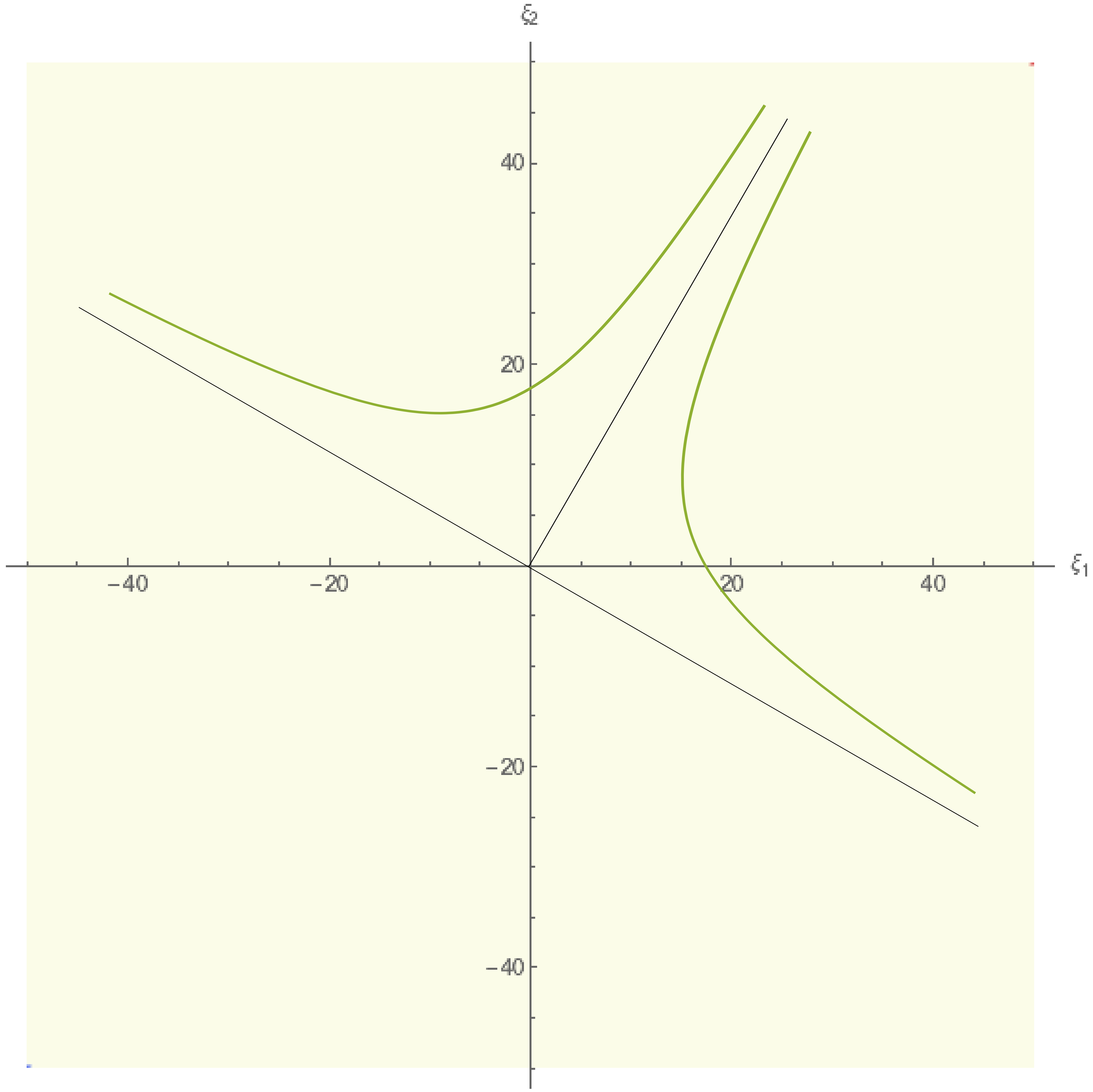
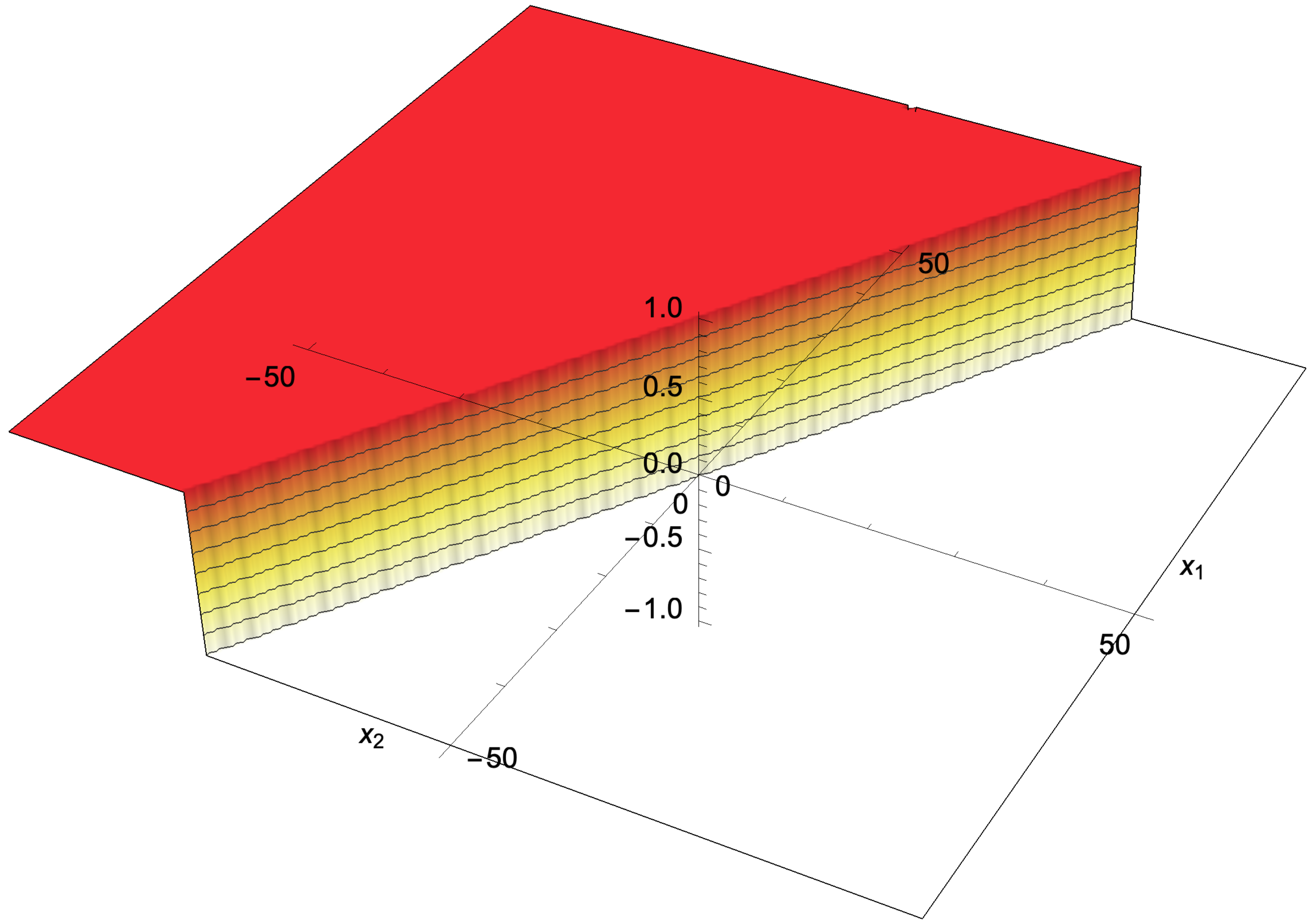
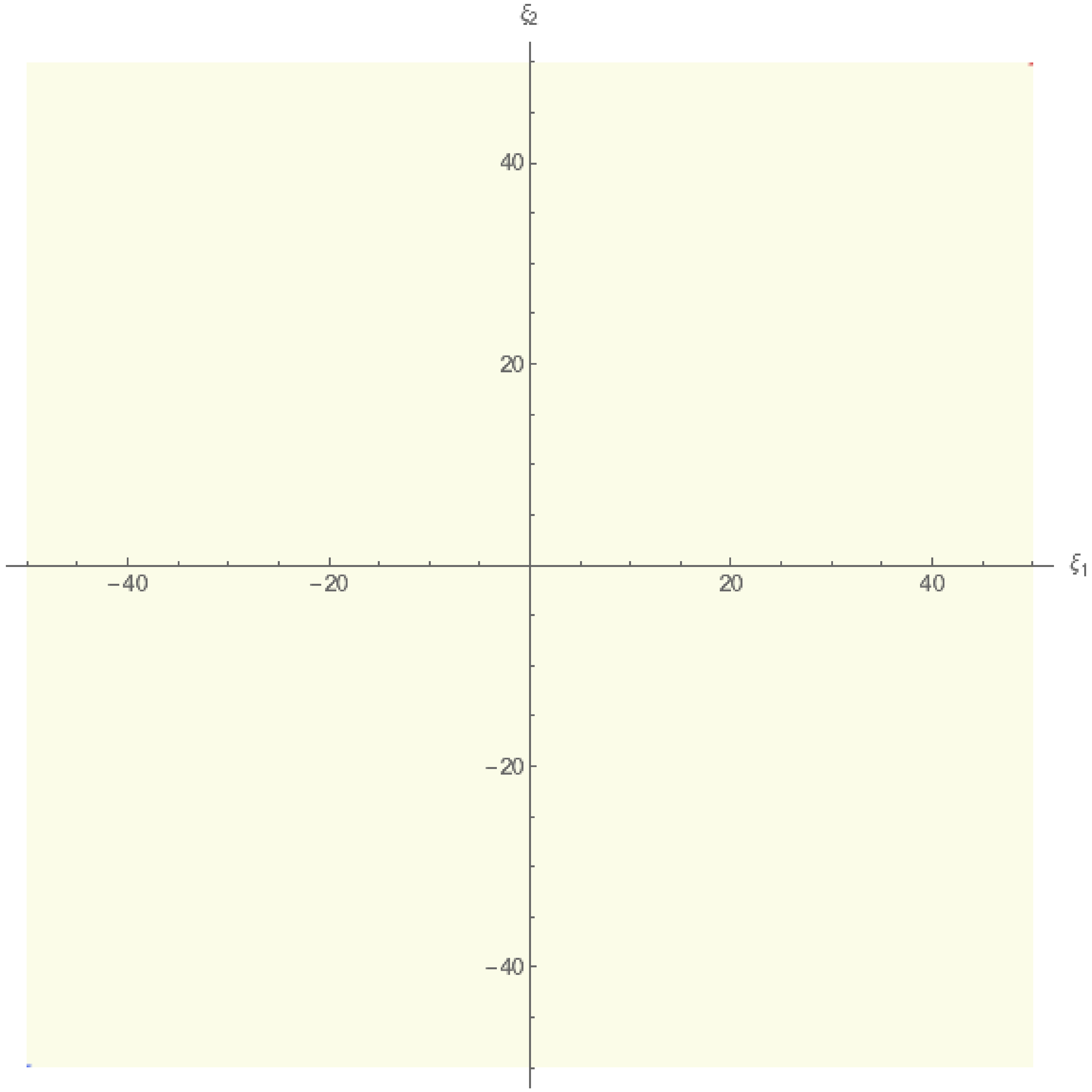
# Approximation of functions on $\mathbb{R}^n$





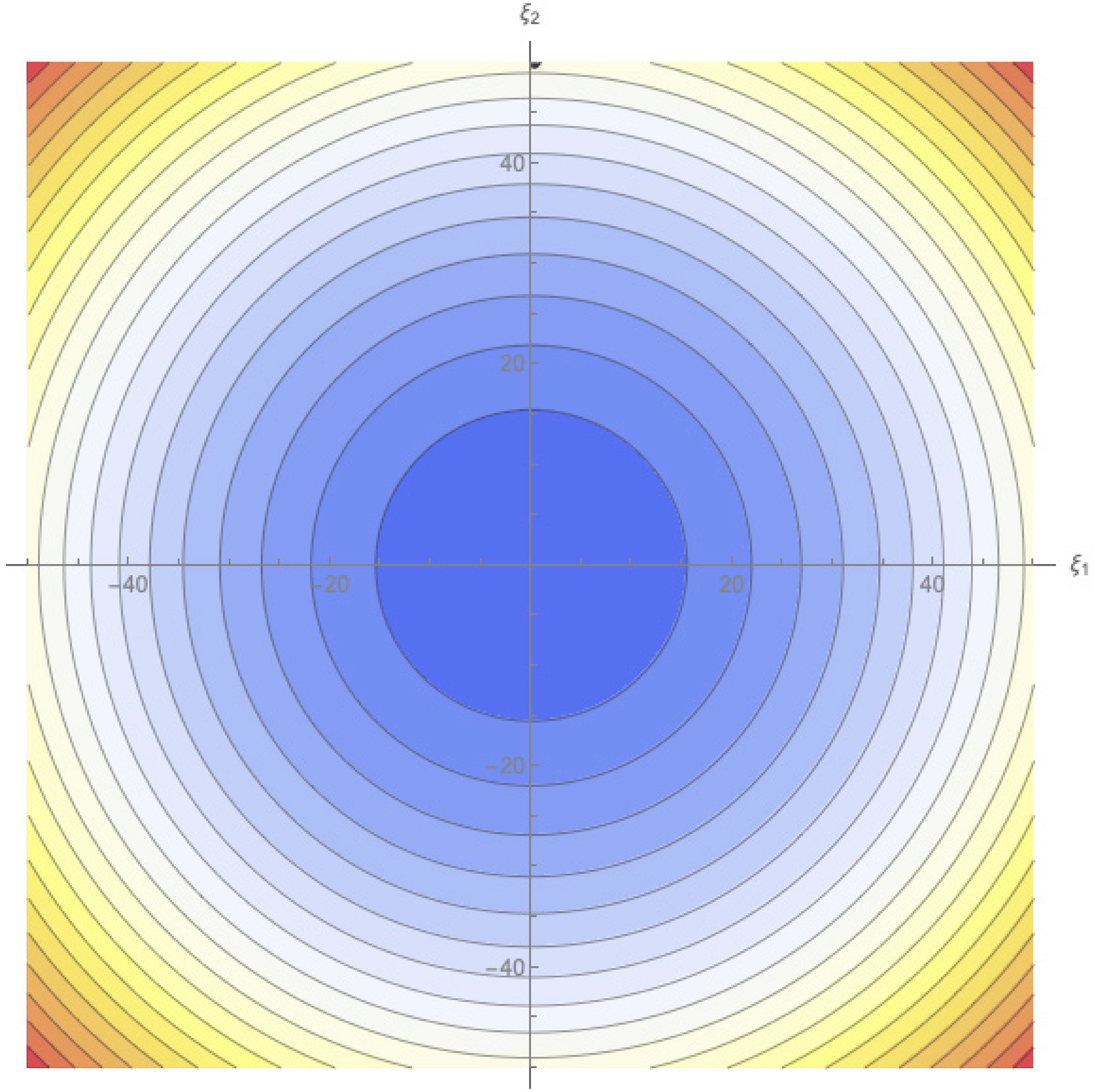
# Approximation of functions on $\mathbb{R}^n$



Laplace operator:

$$\hat{\Delta} = -|\xi|^2$$

# Approximation of functions on $\mathbb{R}^n$



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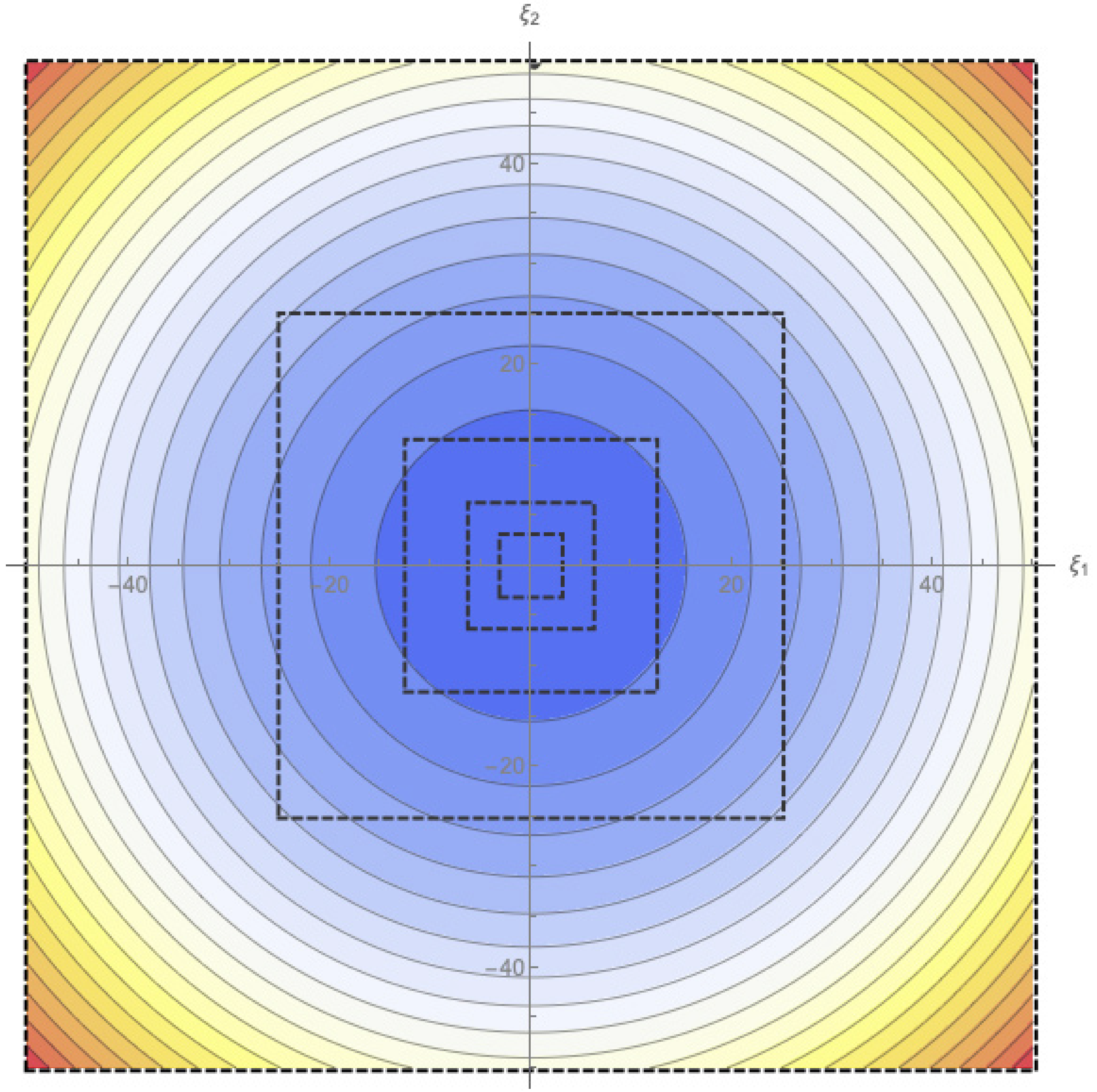
# Approximation of functions on $\mathbb{R}^n$



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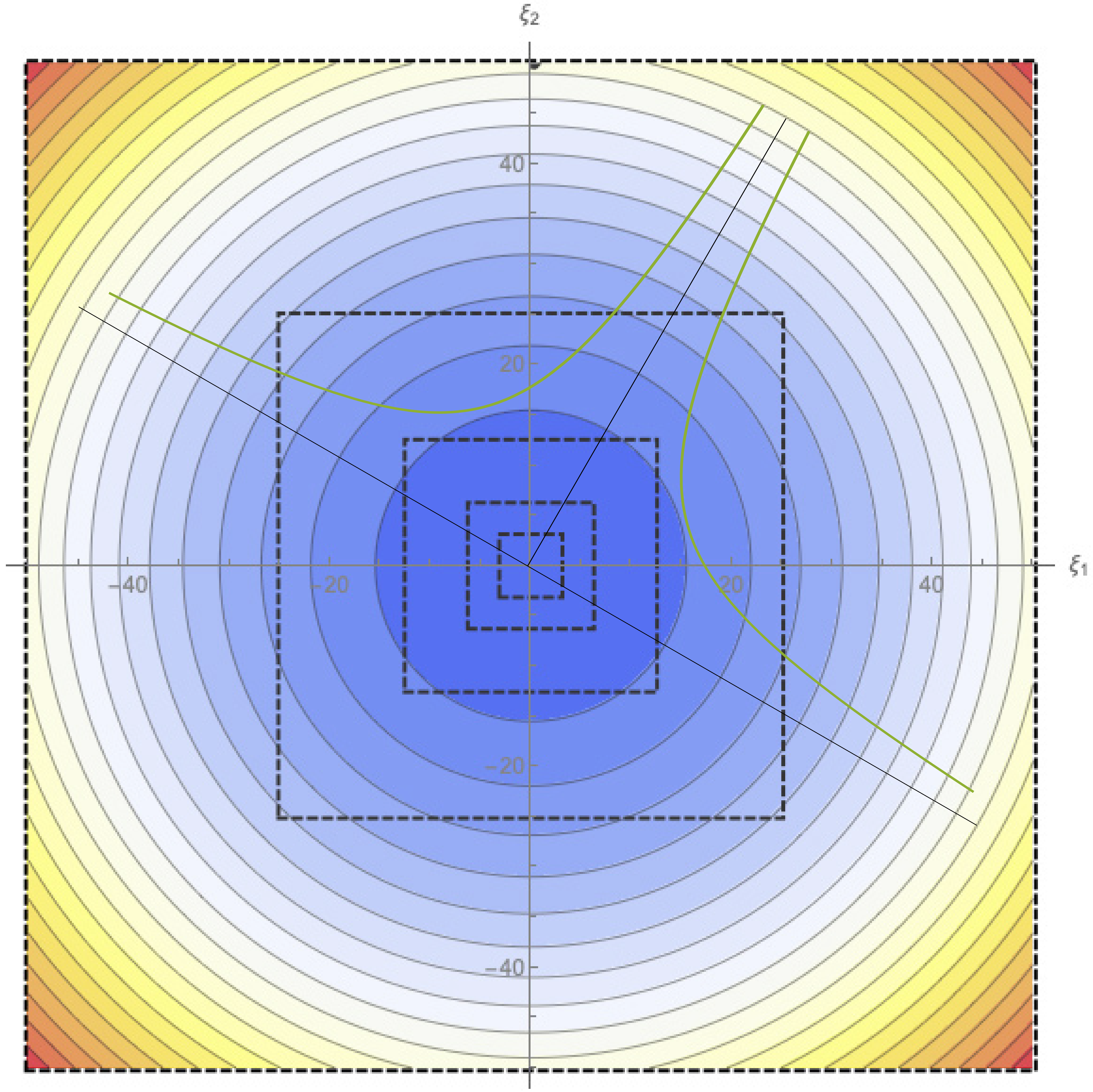
# Approximation of functions on $\mathbb{R}^n$



Laplace operator:

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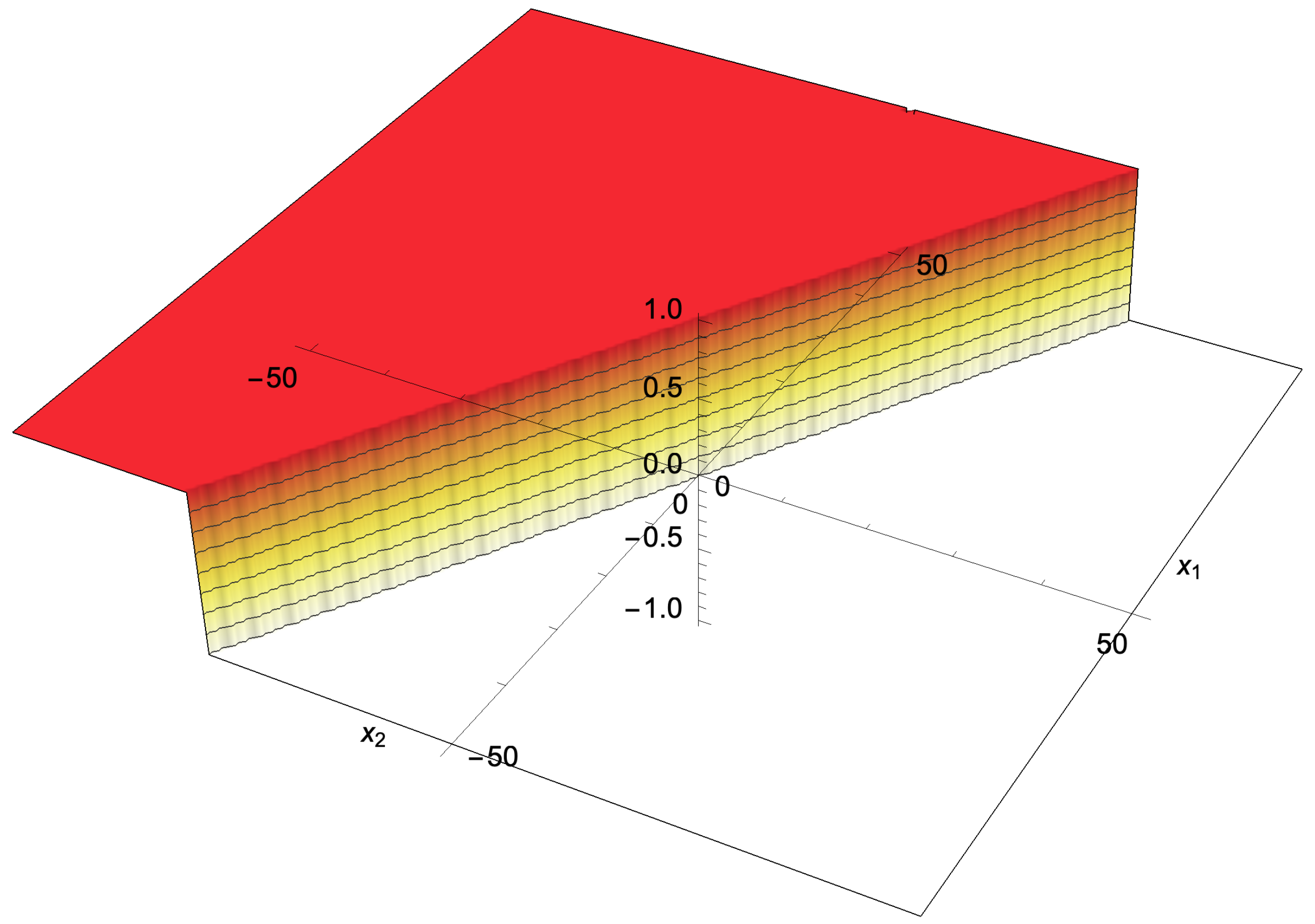
Discontinuity:



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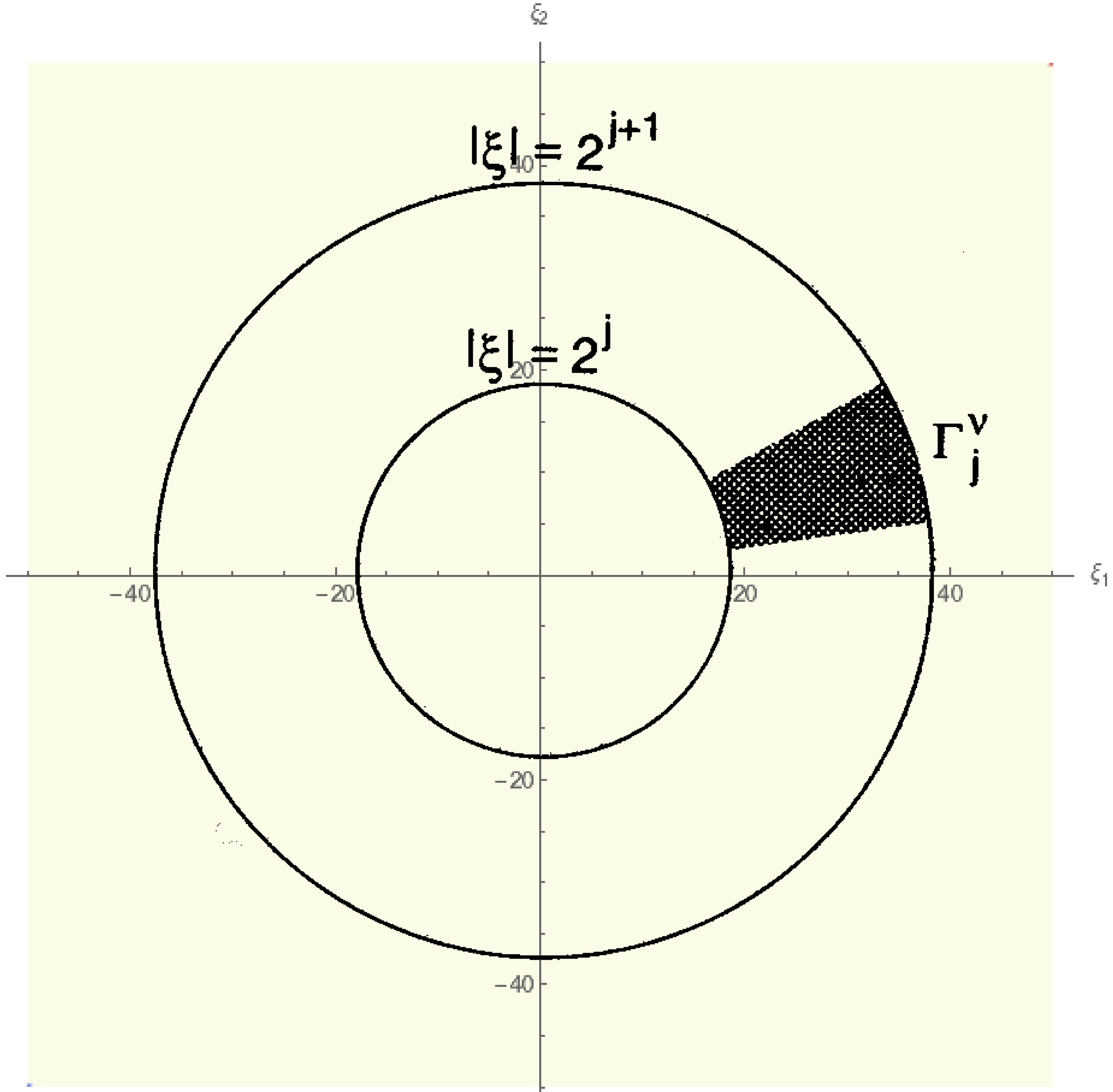

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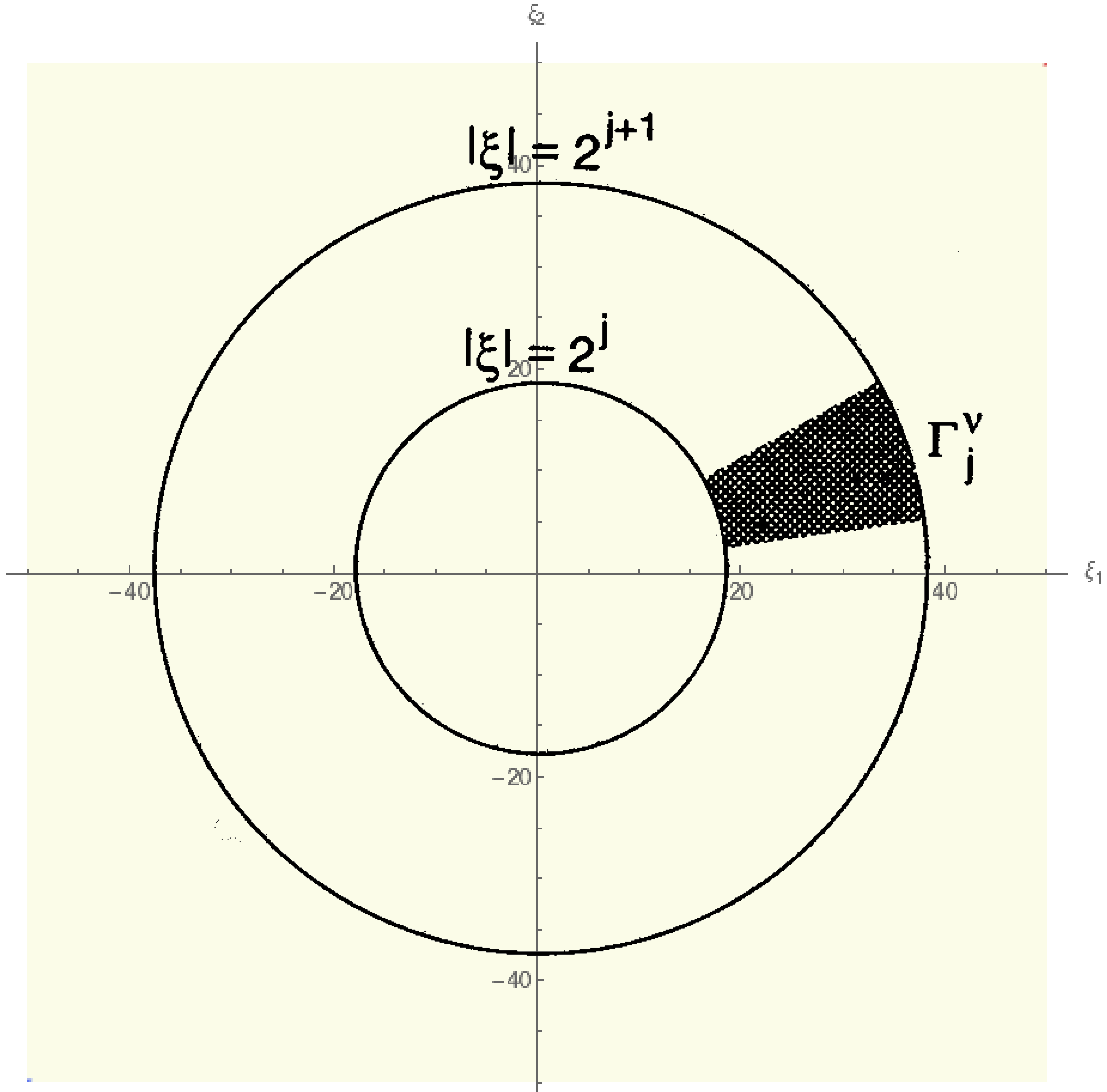

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$$\hat{\psi}_j(\xi_1, \xi_2) = \hat{\psi}_j(\xi_1)\,\hat{\psi}_j(\xi_2)$$

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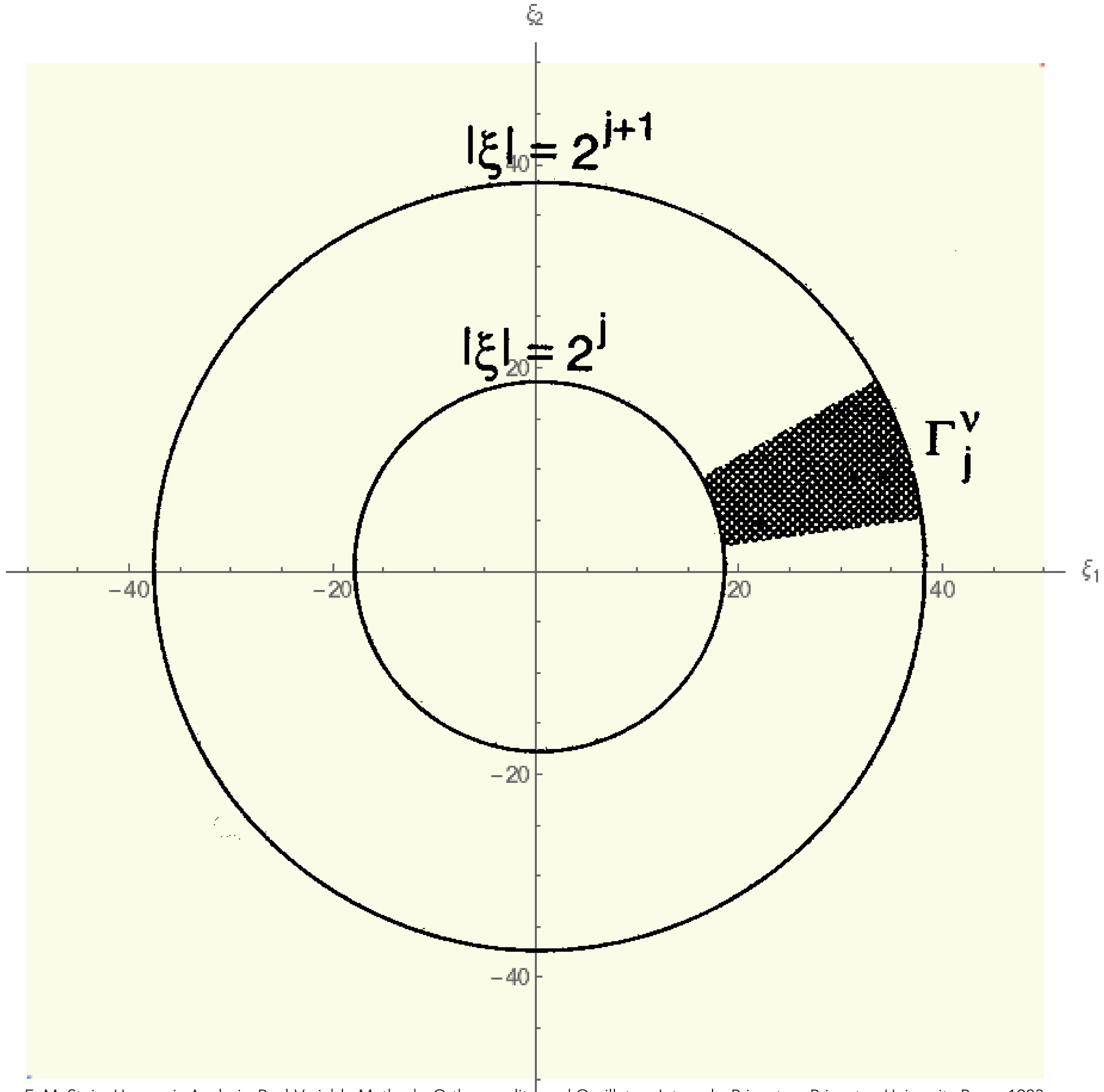
# Approximation of functions on $\mathbb{R}^n$



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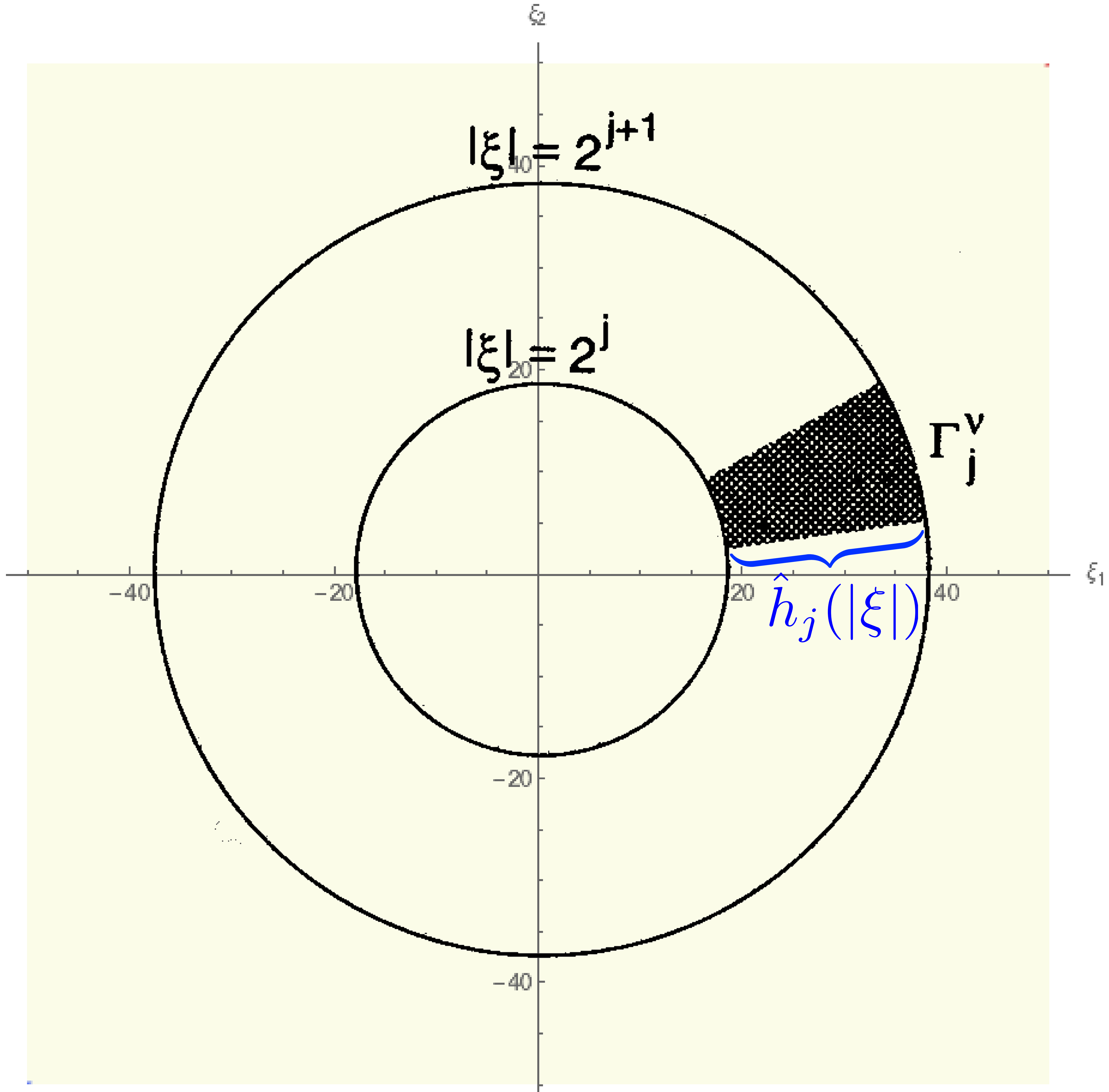
~~$\hat{\psi}_j(\xi_1,\xi_2) = \hat{\psi}_j(\xi_1)\,\hat{\psi}_j(\xi_2)$~~

$$\hat{\psi}_j(\xi_1,\xi_2) = \hat{h}_j(|\xi|)$$

# Approximation of functions on $\mathbb{R}^n$



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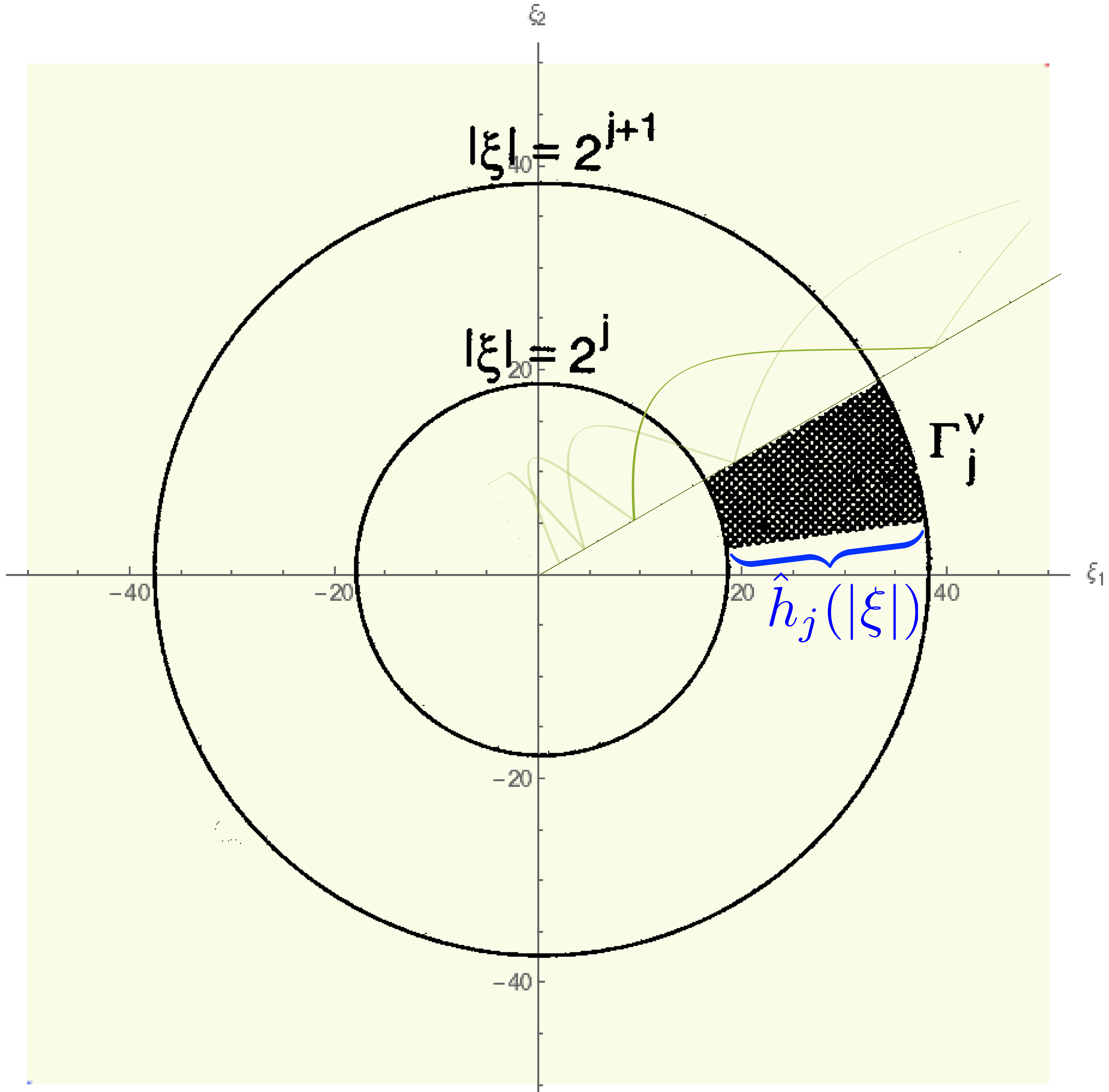
# Approximation of functions on $\mathbb{R}^n$



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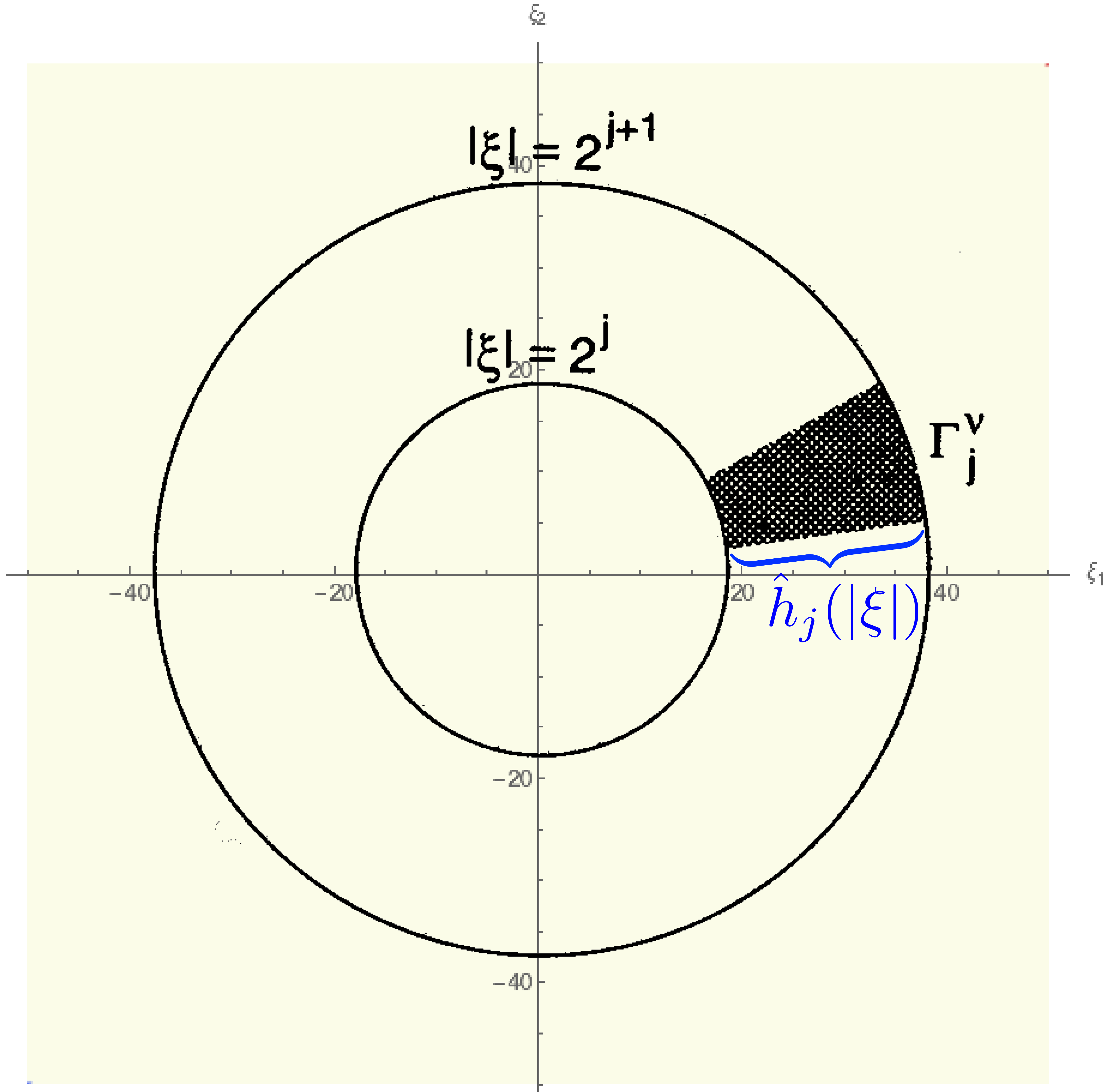

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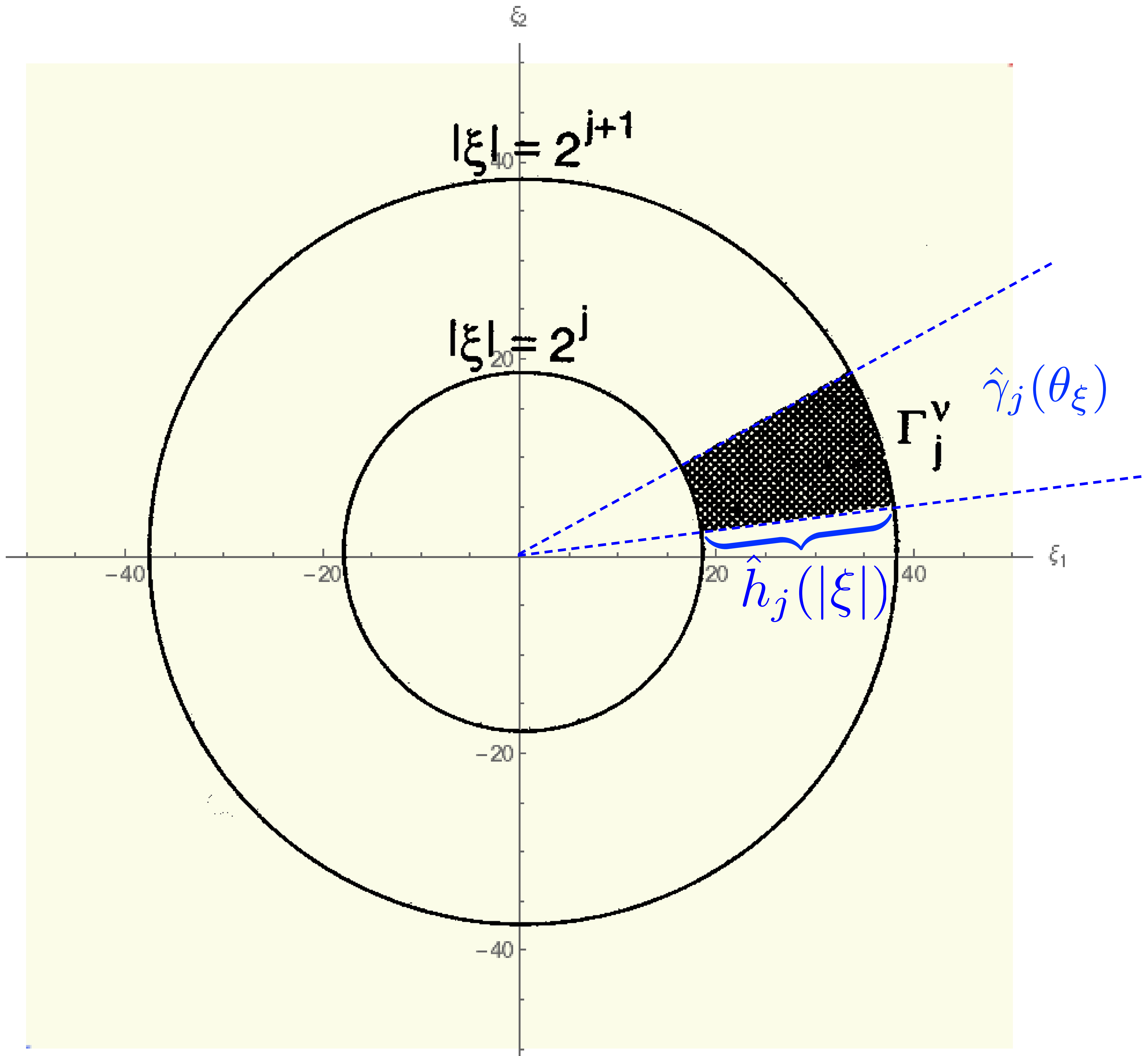
# Approximation of functions on $\mathbb{R}^n$



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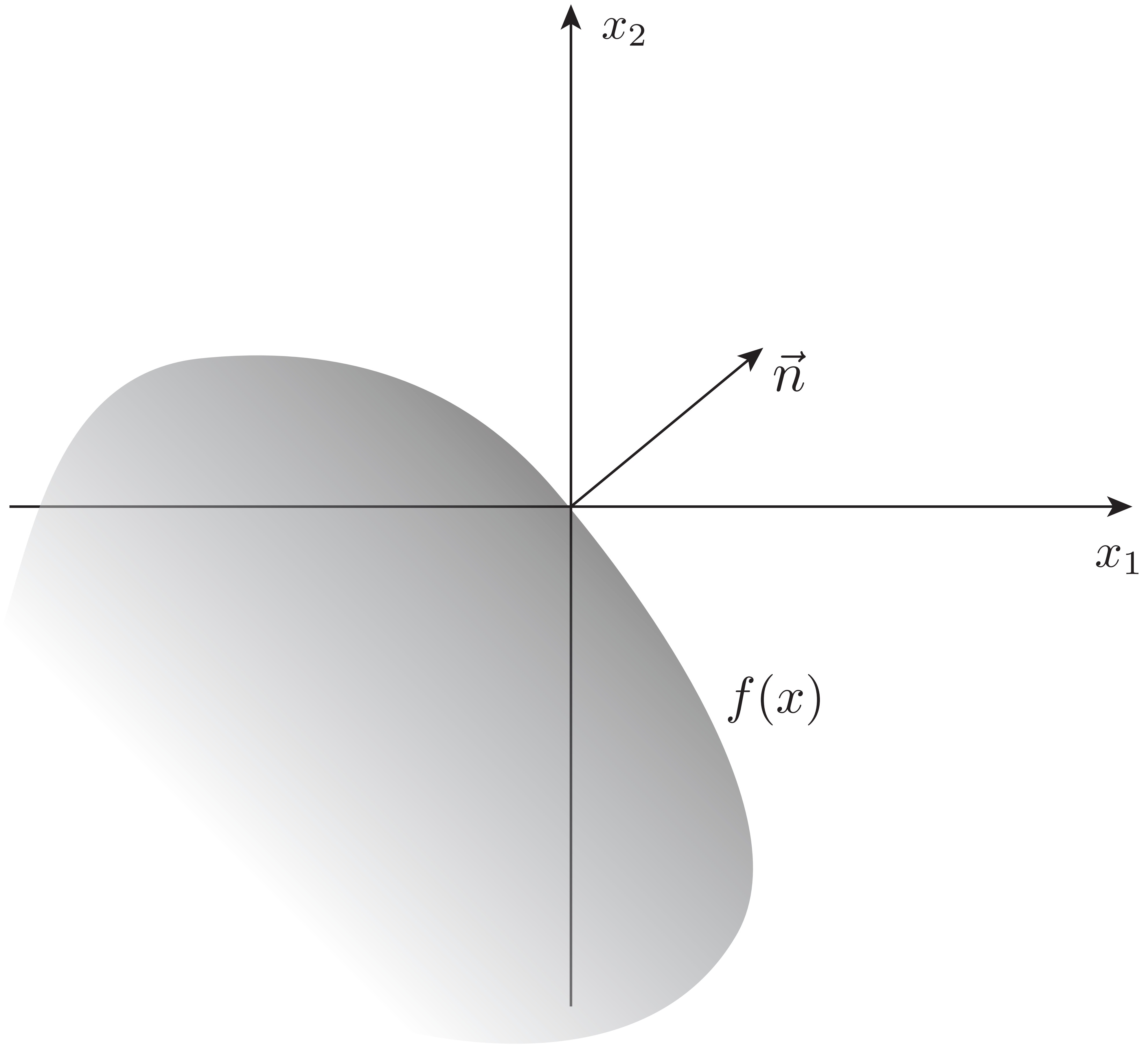
# Approximation of functions on $\mathbb{R}^n$



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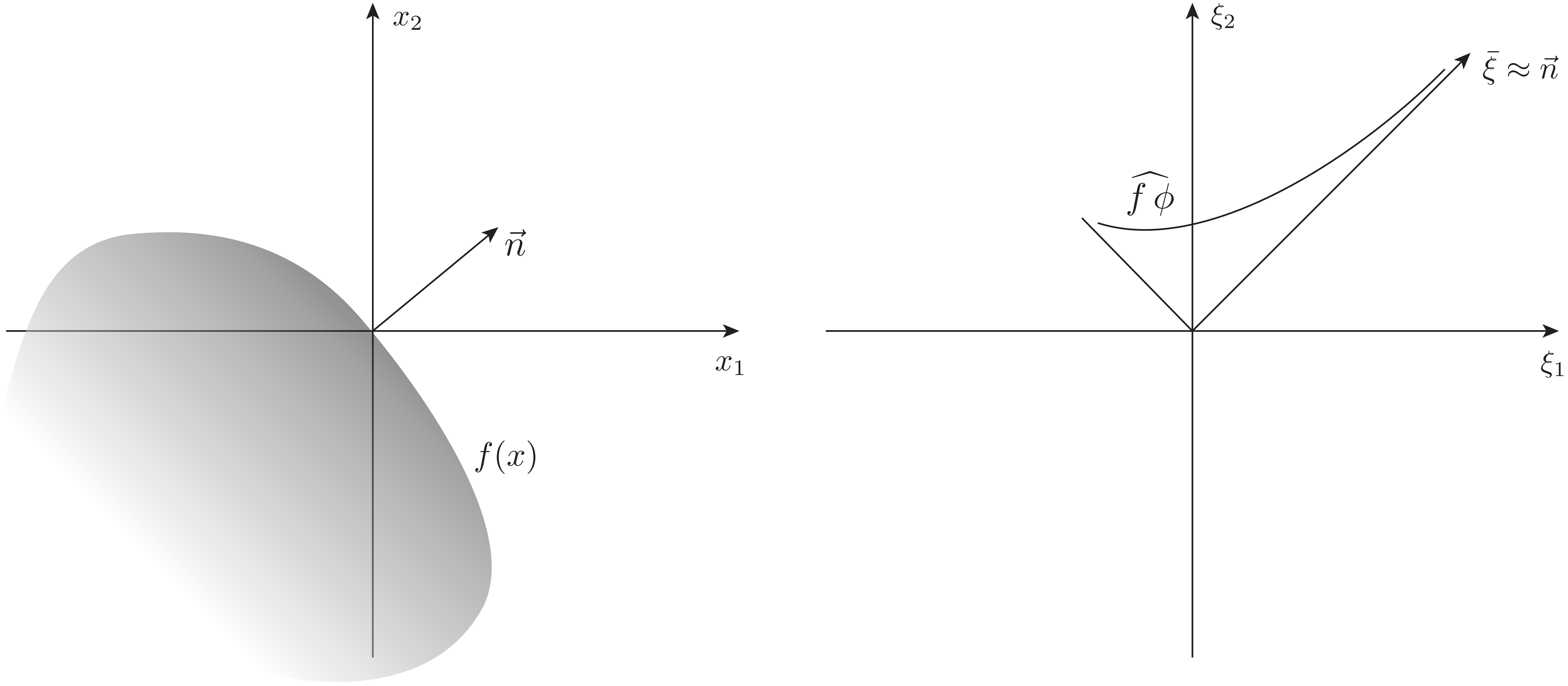

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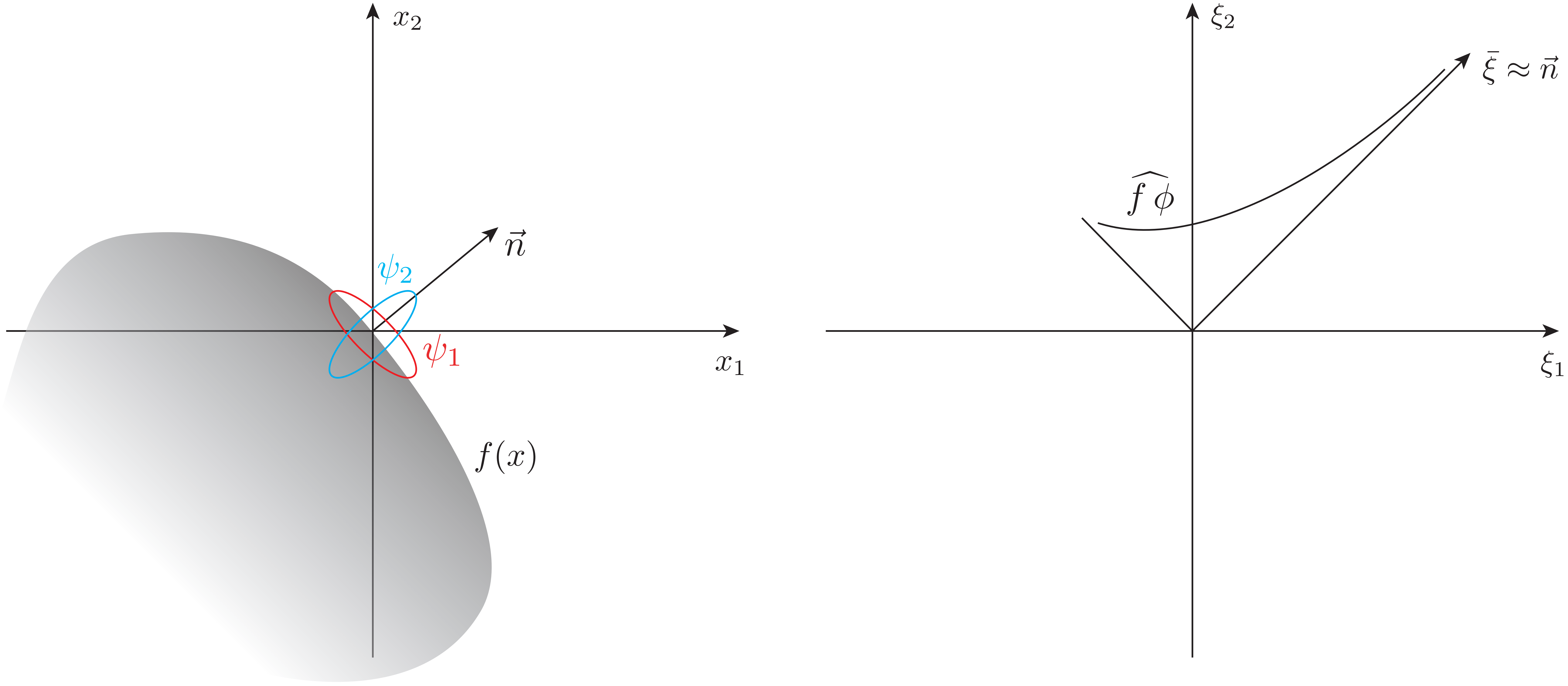

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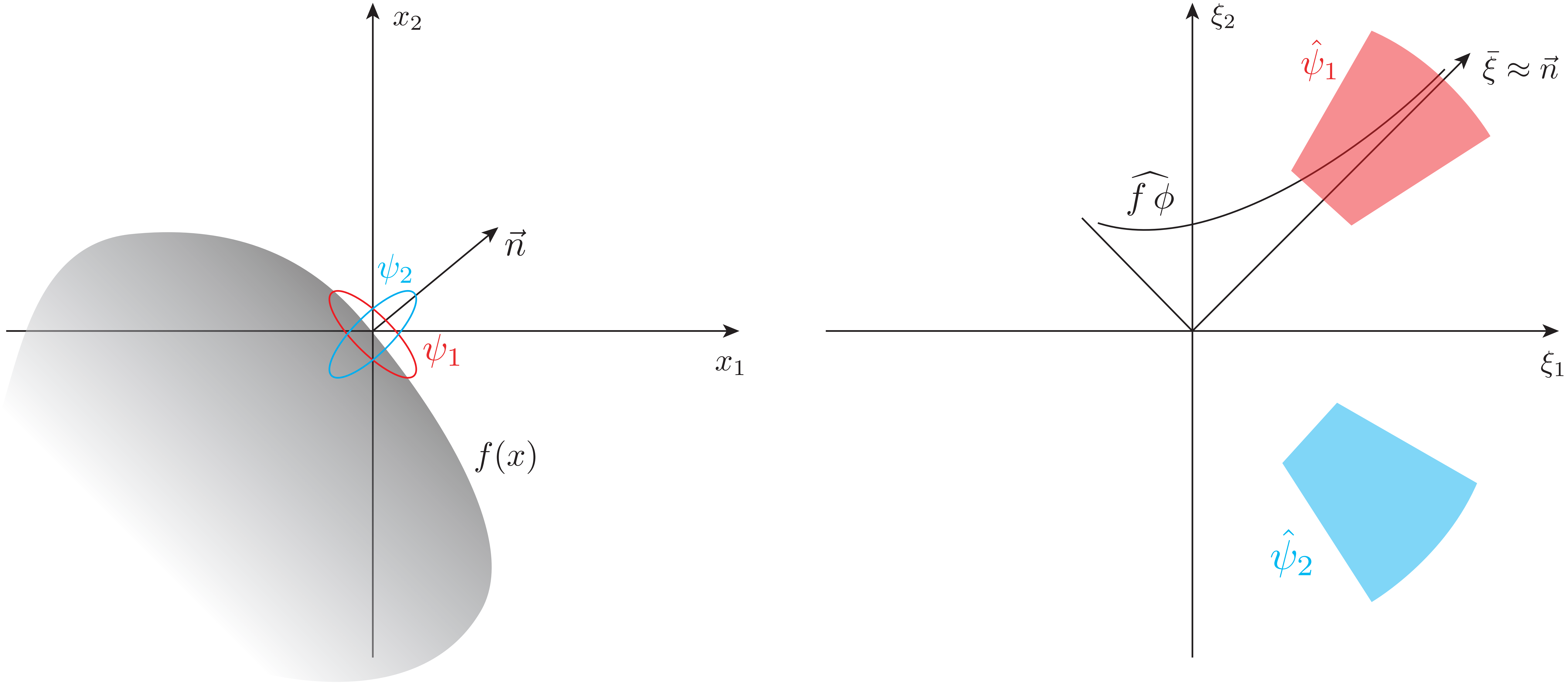
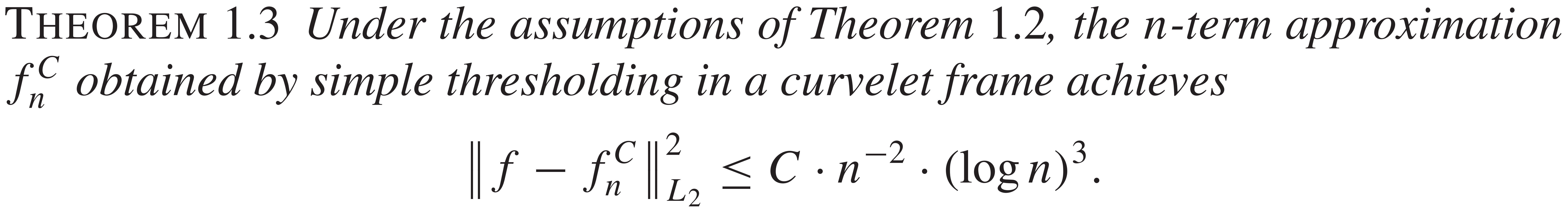


# Approximation of functions on $\mathbb{R}^n$

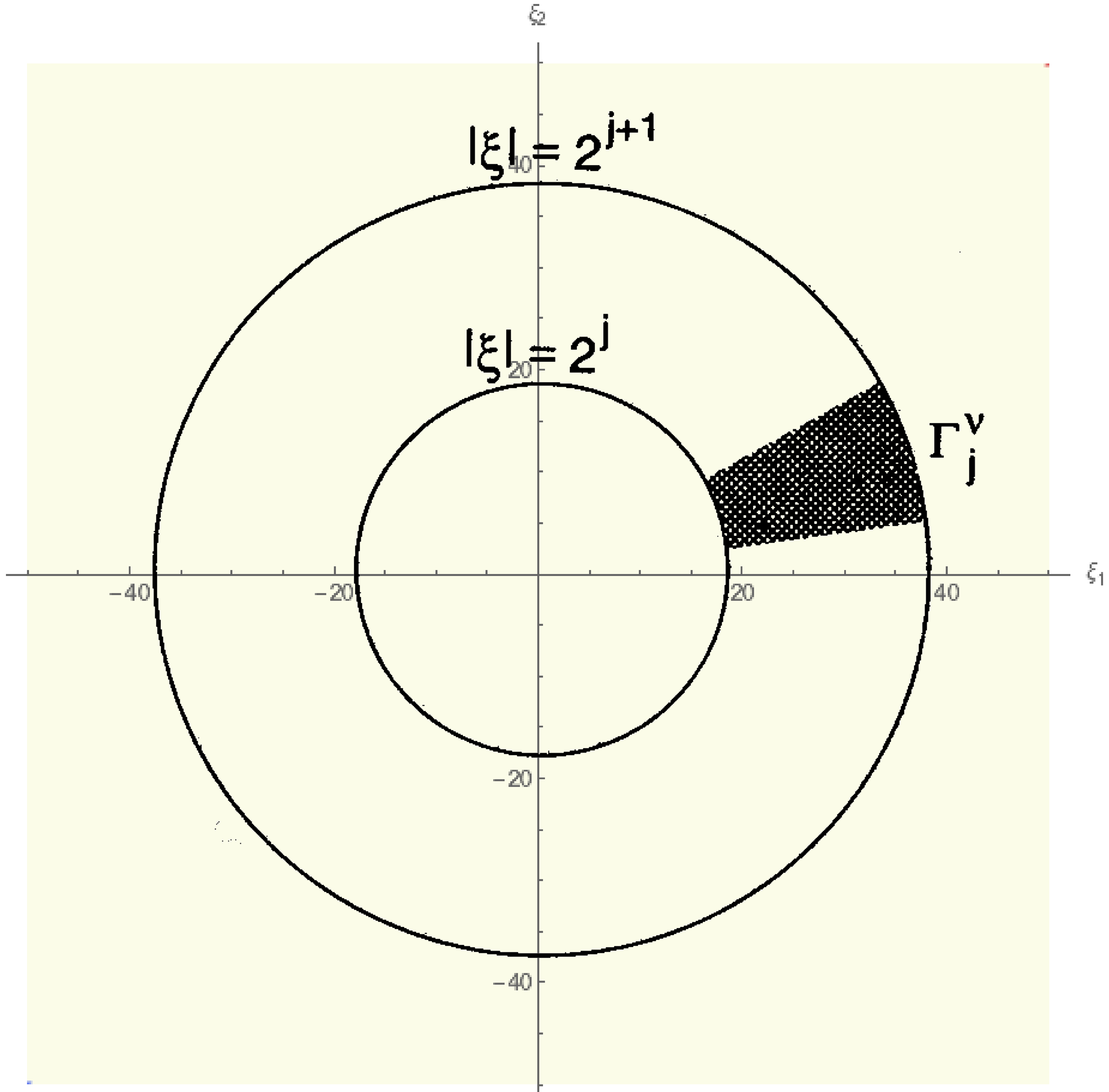
Candès & Donoho (2004):



THEOREM 1.3 *Under the assumptions of Theorem 1.2, the $n$-term approximation $f_n^C$ obtained by simple thresholding in a curvelet frame achieves*

$$\| f - f_n^C \|_{L_2}^2 \leq C \cdot n^{-2} \cdot (\log n)^3.$$

# Approximation of functions on $\mathbb{R}^n$



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- Fefferman [1973]

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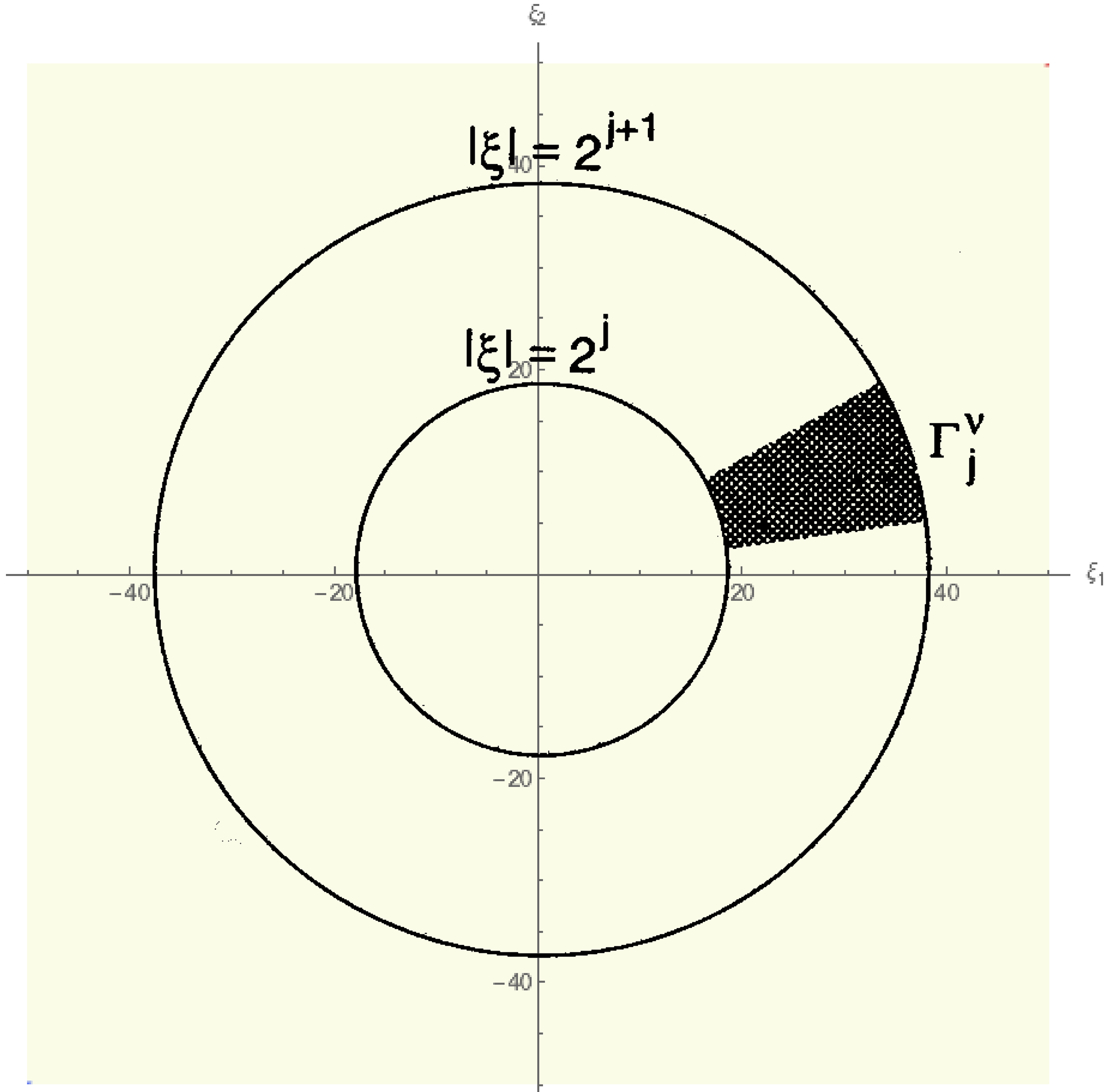

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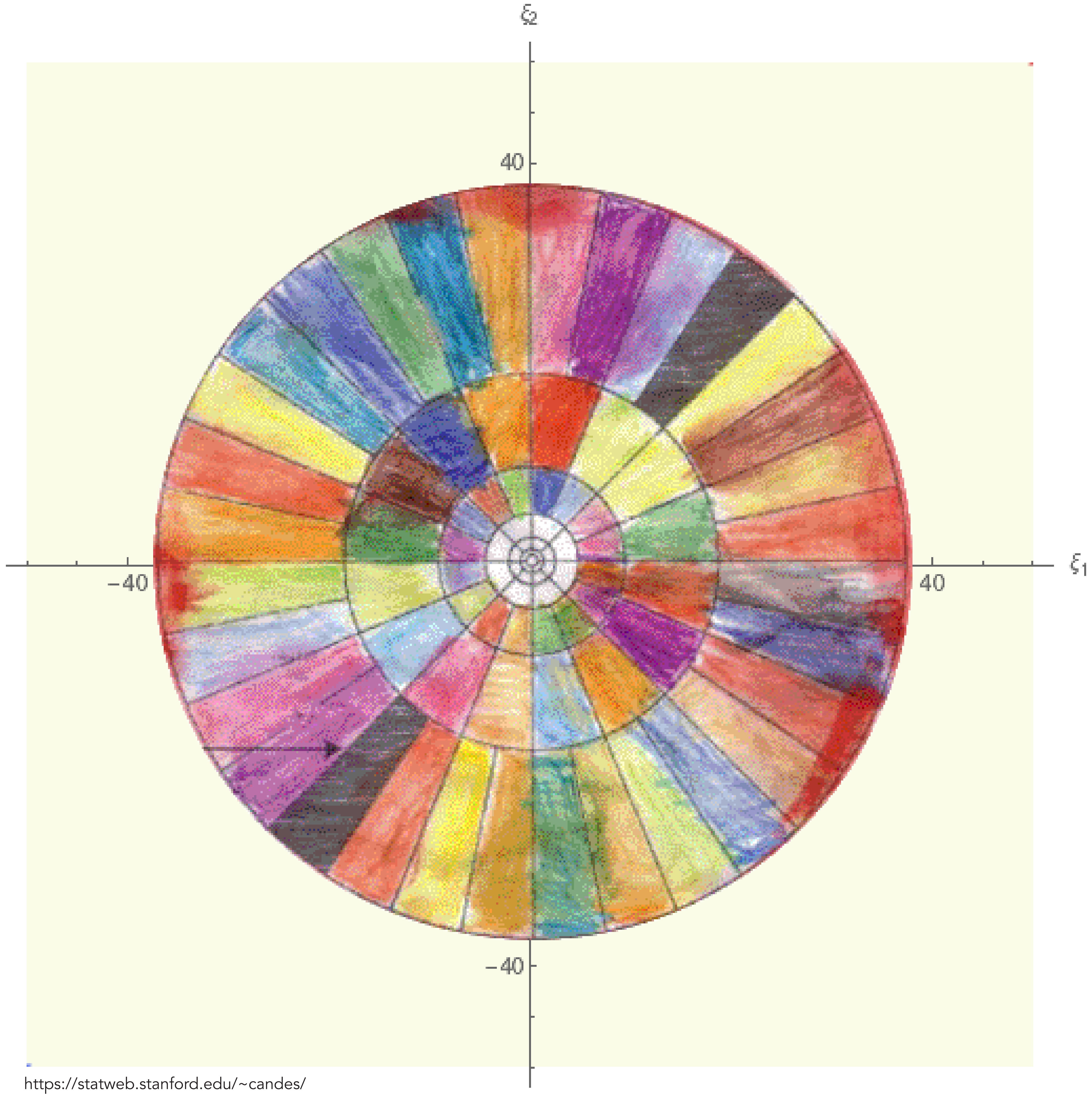

https://statweb.stanford.edu/~candes/

- Fefferman [1973]
- Perona [1991], Simoncelli & Freeman [1995]
- Candès [1999,2005]

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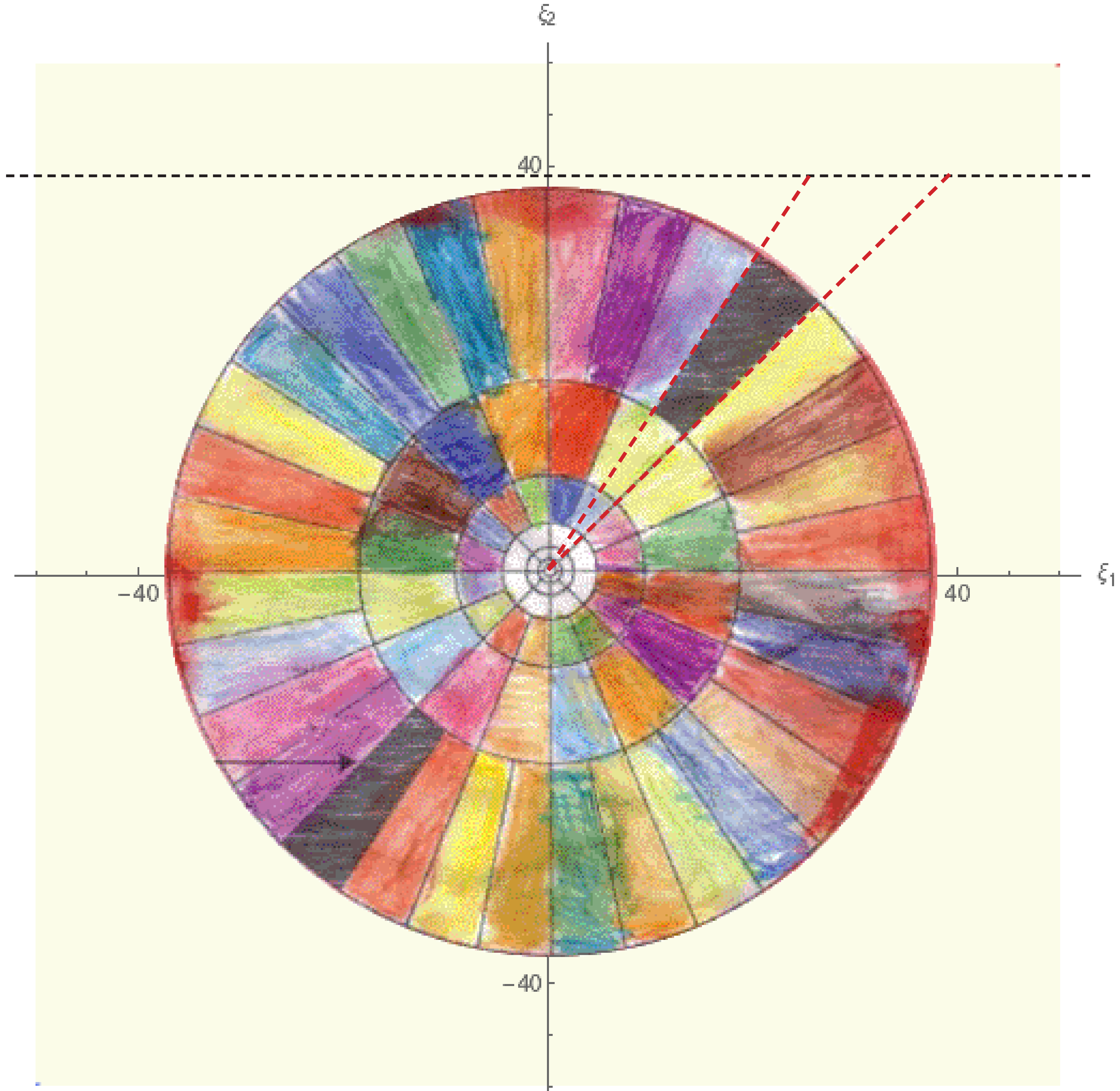

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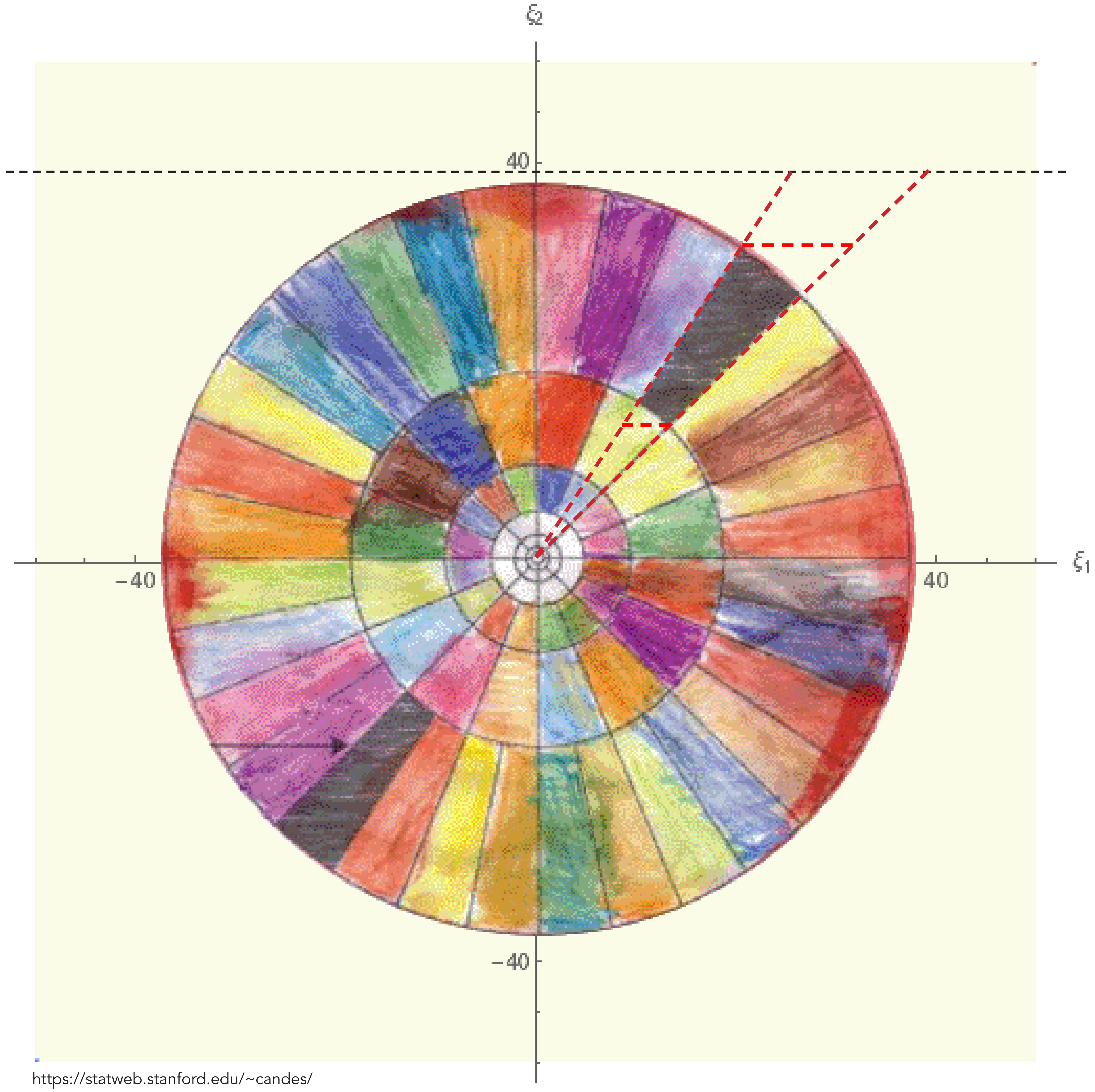

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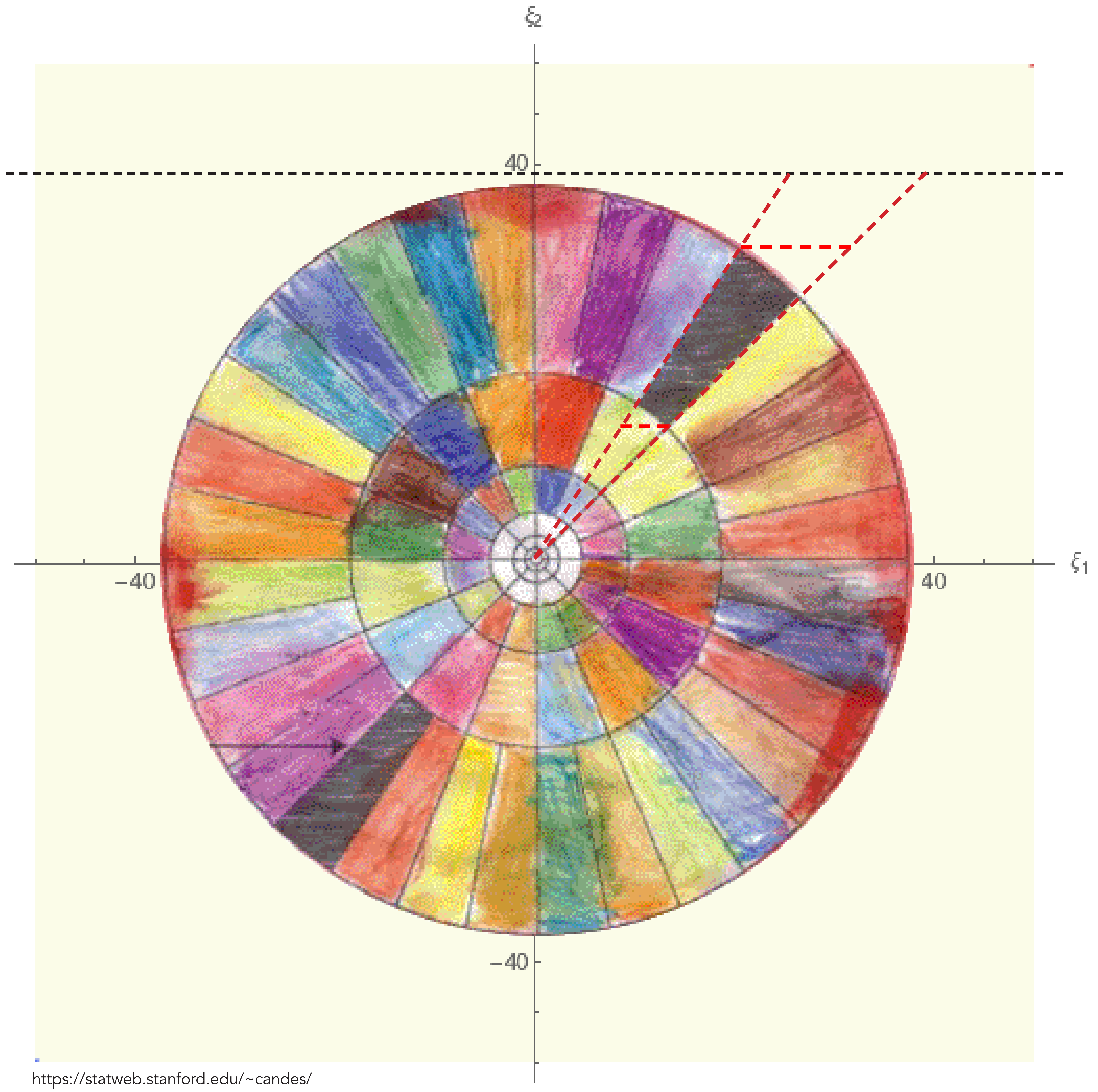
# Approximation of functions on $\mathbb{R}^n$



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- Fefferman [1973]
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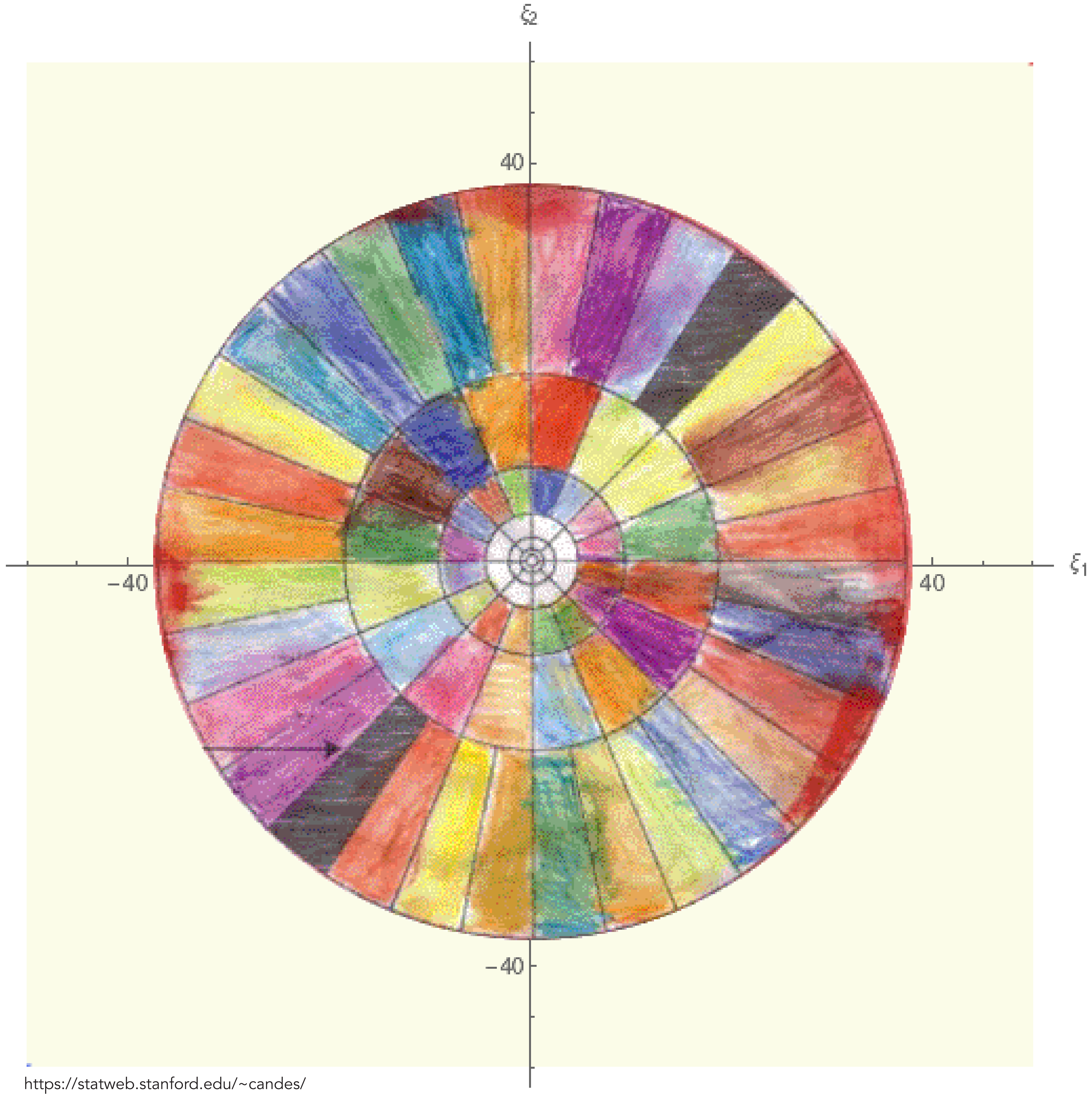
# Approximation of functions on $\mathbb{R}^n$



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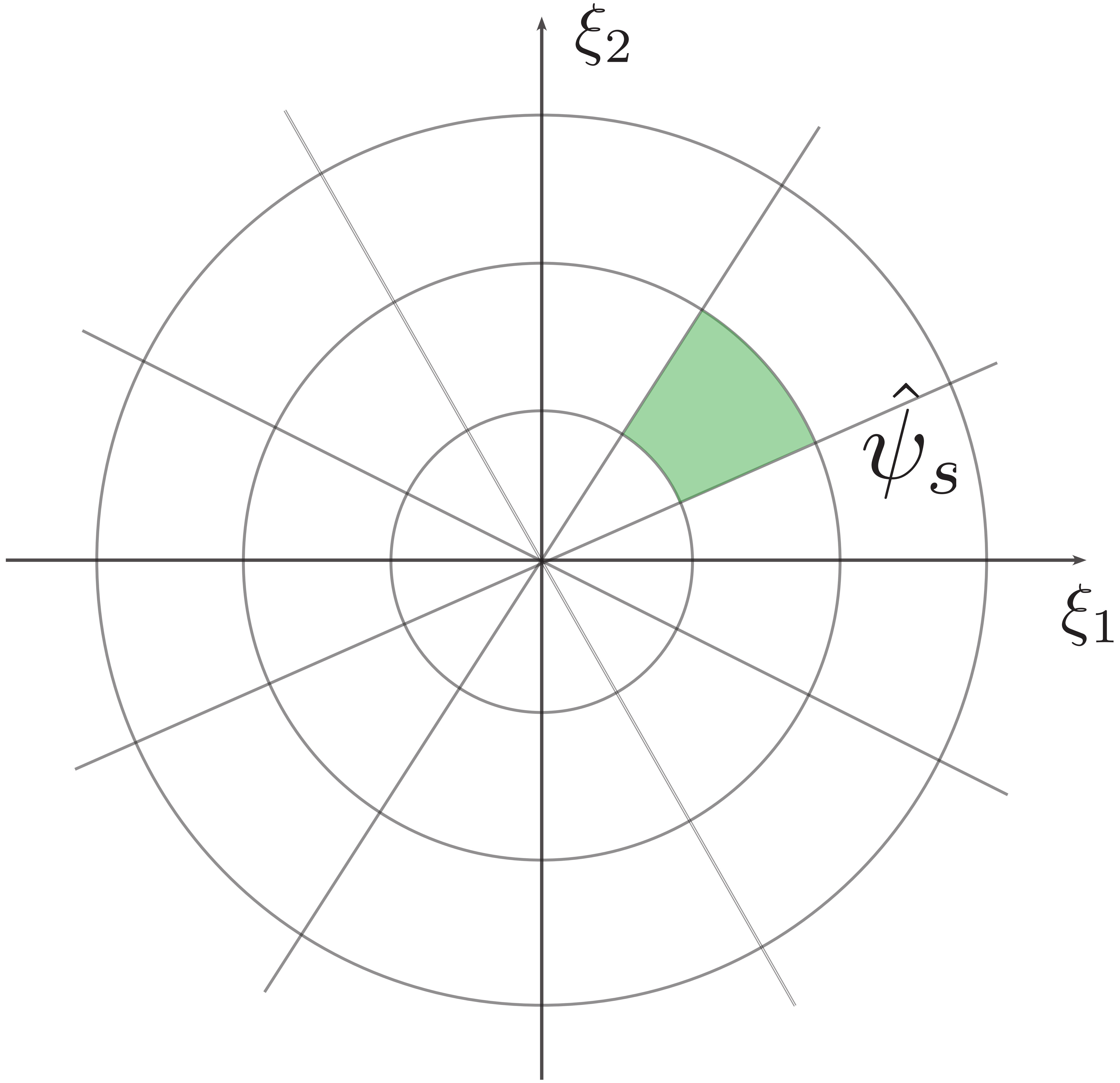
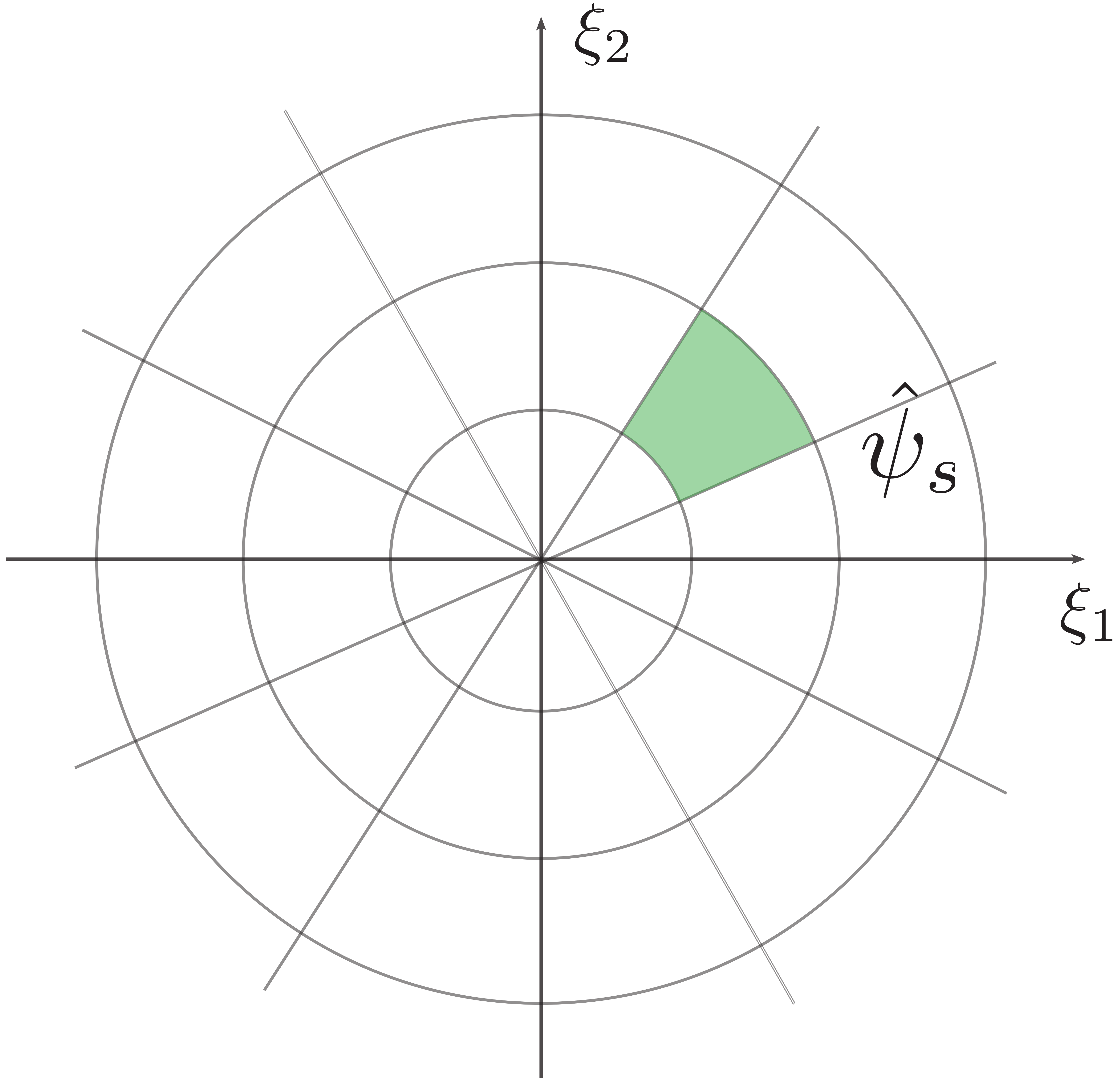
- Fefferman [1973]
- Perona [1991], Simoncelli & Freeman [1995]
- Candes [1999,2005]
- Do & Vetterli [2003], Kutyniok & Labate [2006]
- Unser et al. [2010,2012]

# Polar wavelets



$$\hat{\psi}_j(\xi_1, \xi_2) = \hat{h}_j(|\xi|)\, \hat{\gamma}_t(\theta_\xi)$$

# Polar wavelets



$$\hat{\psi}_j(\xi_1, \xi_2) = \hat{h}_j(|\xi|) \, \underbrace{\hat{\gamma}_t(\theta_\xi)}$$

$$\hat{\gamma}(\theta_\xi) = \sum_m \beta_m^{j,t} \, e^{im\theta_\xi}$$

# Polar wavelets

$$\psi_{j,t}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{\psi}_j(\xi) \, e^{i \langle \xi , x \rangle} \, d\xi$$

# Polar wavelets

$$\begin{aligned}\psi_{j,t}(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{\psi}_j(\xi)\, e^{i\langle \xi , x \rangle} \, d\xi \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}(|\xi|)\, \hat{\gamma}(\theta_\xi)\, e^{i\langle \xi , x \rangle} \, d\xi\end{aligned}$$

# Polar wavelets

$$\begin{aligned}
\psi_{j,t}(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{\psi}_j(\xi)\, e^{i\langle \xi , x \rangle}\, d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}(|\xi|)\, \hat{\gamma}(\theta_\xi)\, e^{i\langle \xi , x \rangle}\, d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}_j(|\xi|)\, \hat{\gamma}(\theta_\xi) \left( \sum_{n=-\infty}^{\infty} i^n\, e^{in(\theta_x - \theta_\xi)}\, J_n(|\xi|\,|x|) \right) d\xi
\end{aligned}$$

# Polar wavelets

$$\begin{aligned}\psi_{j,t}(x) &= \frac{1}{2\pi}\int_{\mathbb{R}^2_\xi} \hat{\psi}_j(\xi)\, e^{i\langle \xi , x\rangle}\, d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2_\xi} \hat{h}(|\xi|)\, \hat{\gamma}(\theta_\xi)\, e^{i\langle \xi , x\rangle}\, d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2_\xi} \hat{h}_j(|\xi|)\, \hat{\gamma}(\theta_\xi) \underbrace{\left( \sum_{n=-\infty}^{\infty} i^n\, e^{in(\theta_x - \theta_\xi)}\, J_n(|\xi|\,|x|) \right)}_{\text{Jacobi-Anger formula}} d\xi\end{aligned}$$

# Polar wavelets

$$
\begin{aligned}
\psi_{j,t}(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{\psi}_j(\xi)\, e^{i\langle \xi , x \rangle}\, d\xi \\
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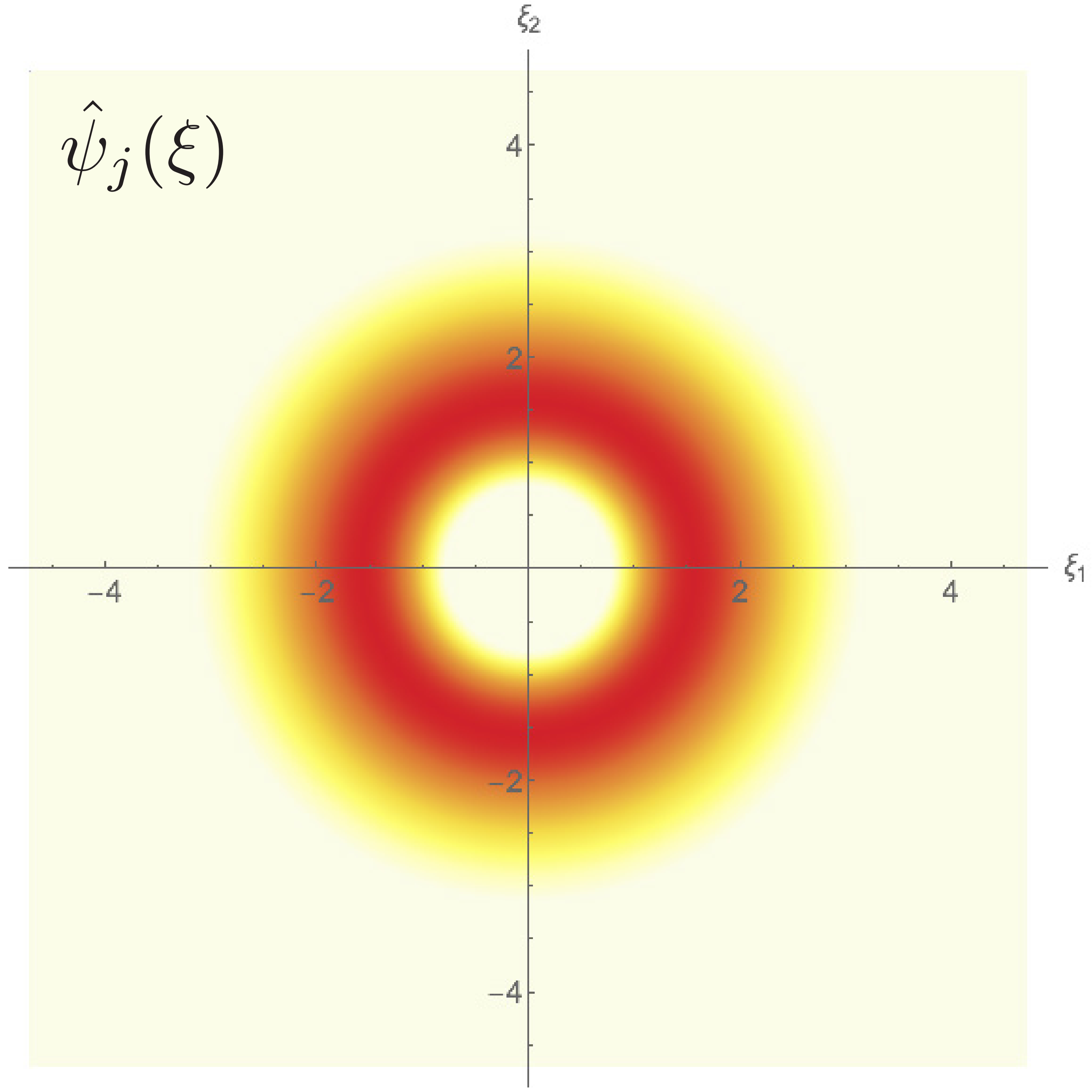
&= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}_j(|\xi|)\, \underbrace{\hat{\gamma}(\theta_\xi)}_{\hat{\gamma}(\theta_\xi) = \sum_m \beta_m^{j,t}\, e^{im\theta_\xi}} \left( \sum_{n=-\infty}^{\infty} i^n\, e^{in(\theta_x - \theta_\xi)}\, J_n(|\xi|\,|x|) \right) d\xi
\end{aligned}
$$

# Polar wavelets

$$\psi_{j,t}(x) = \sum_m i^m \, \beta_m^{j,t} \, e^{im\theta_x} \, h_m(|x|)$$



# Polar wavelets

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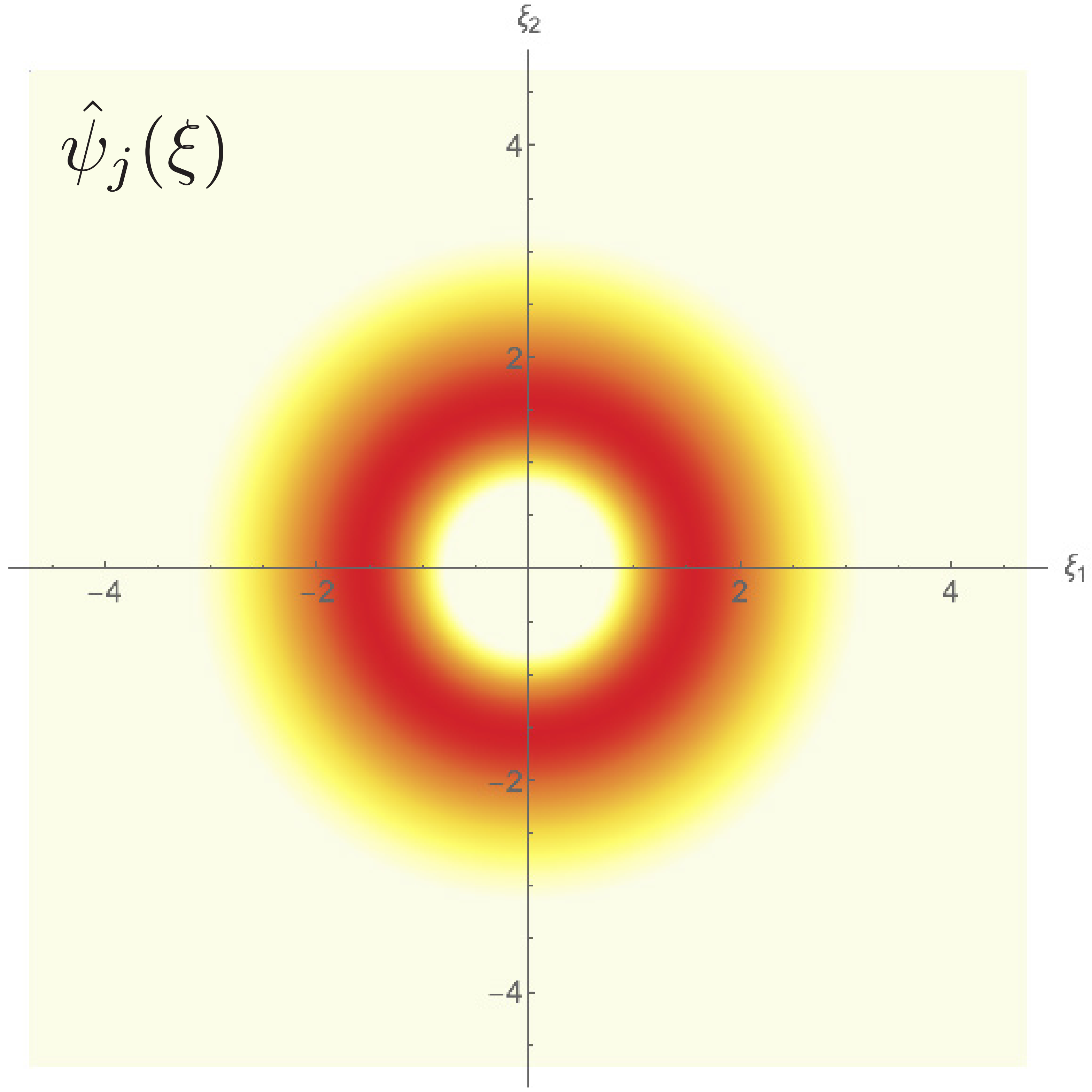

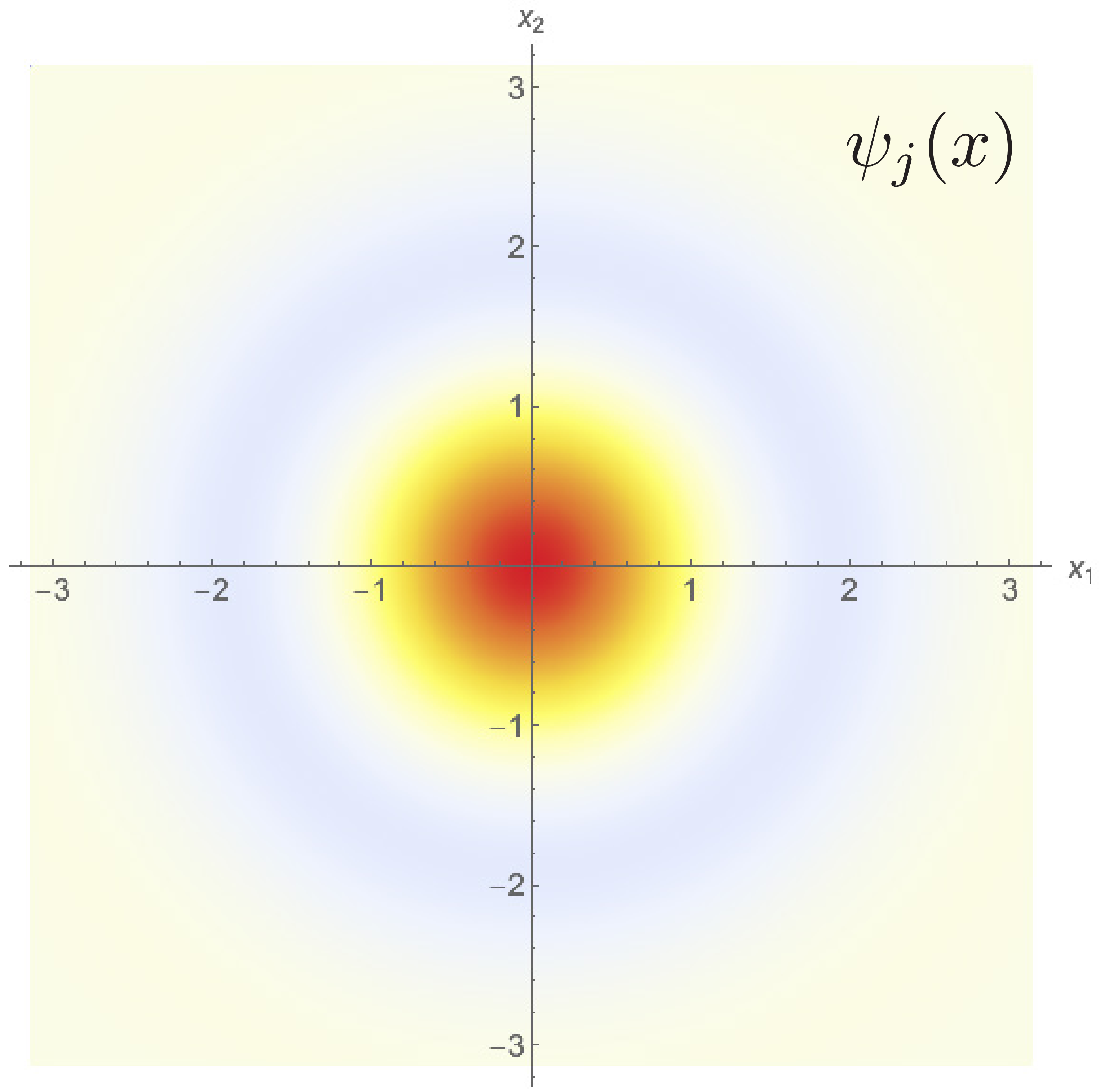

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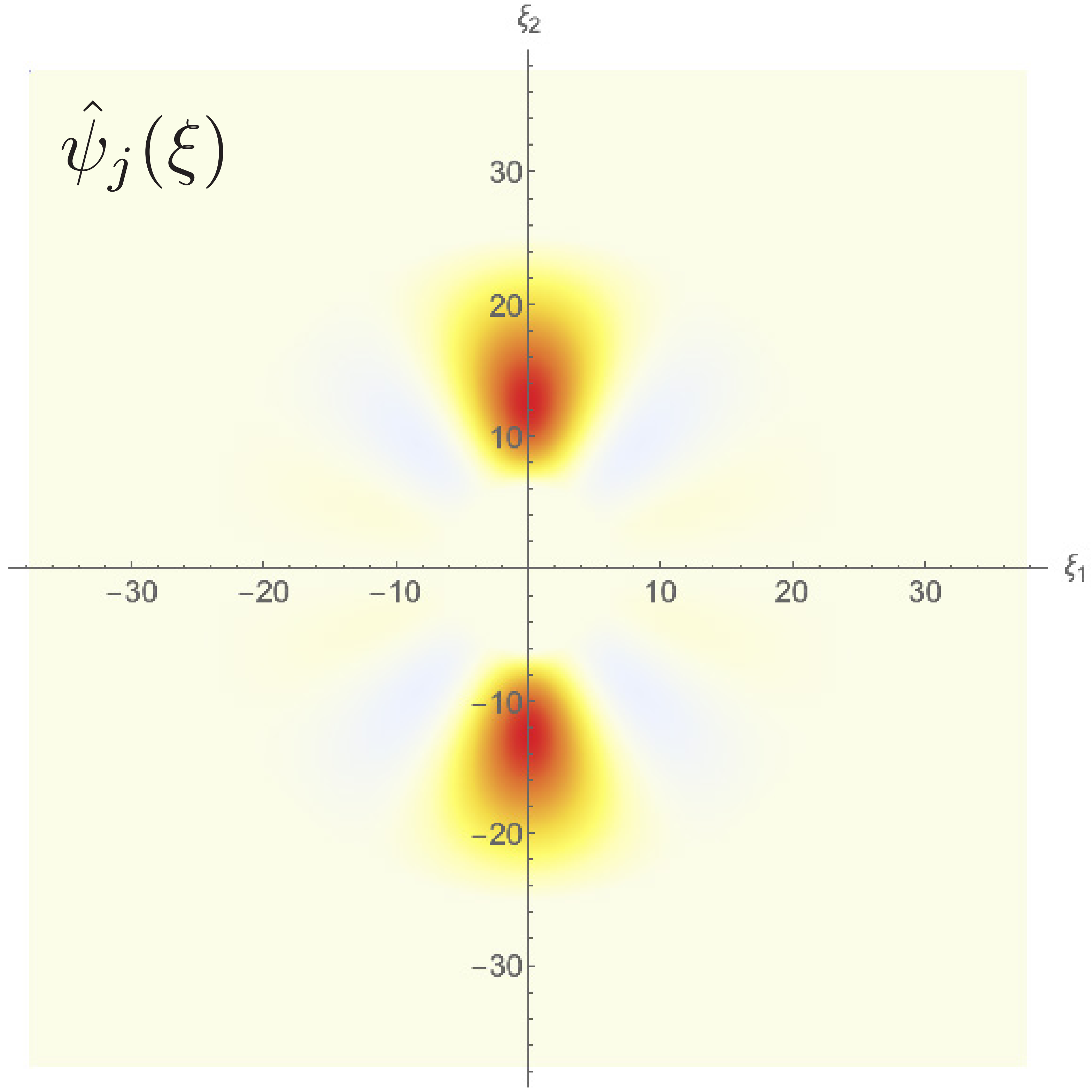

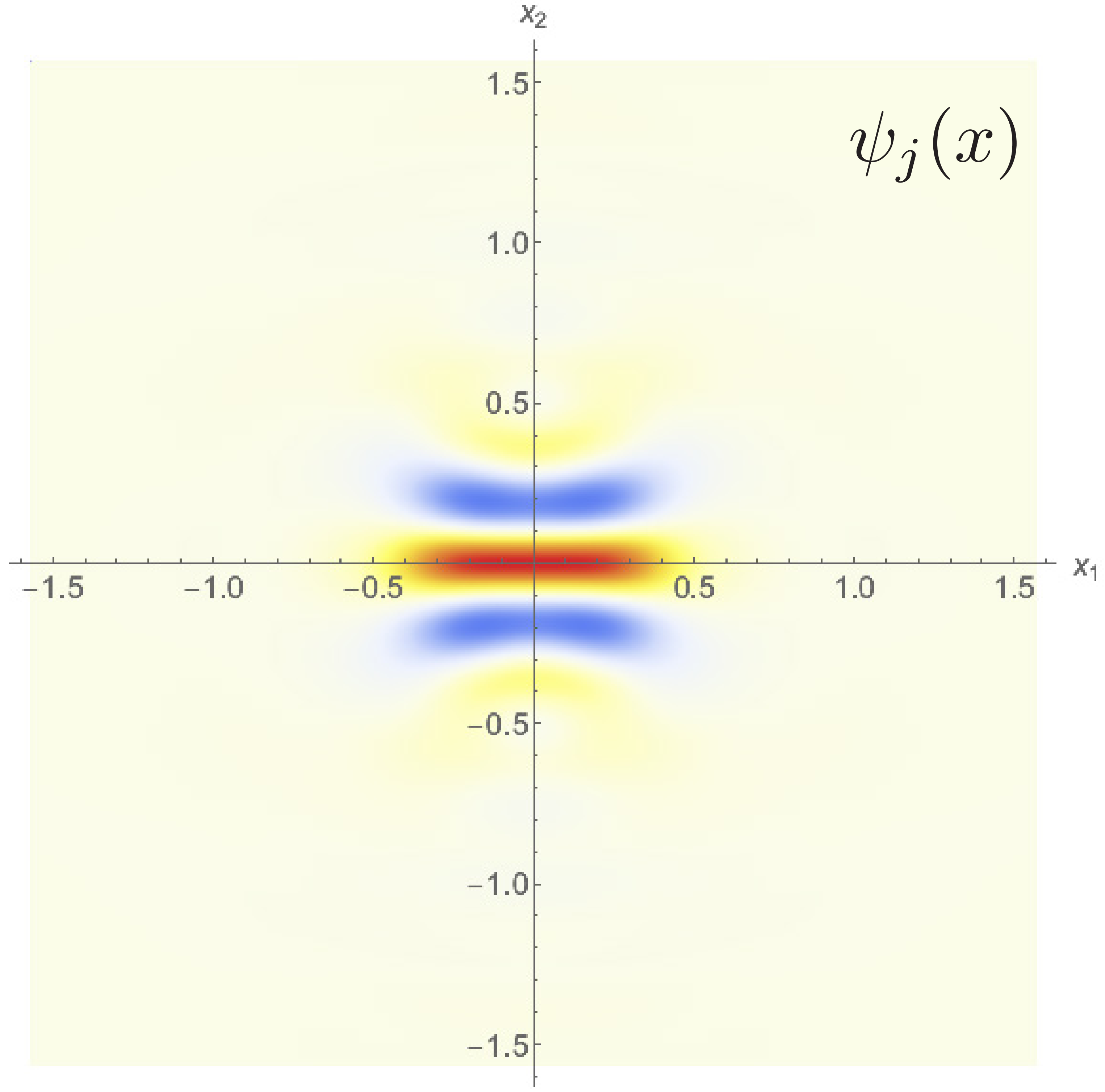

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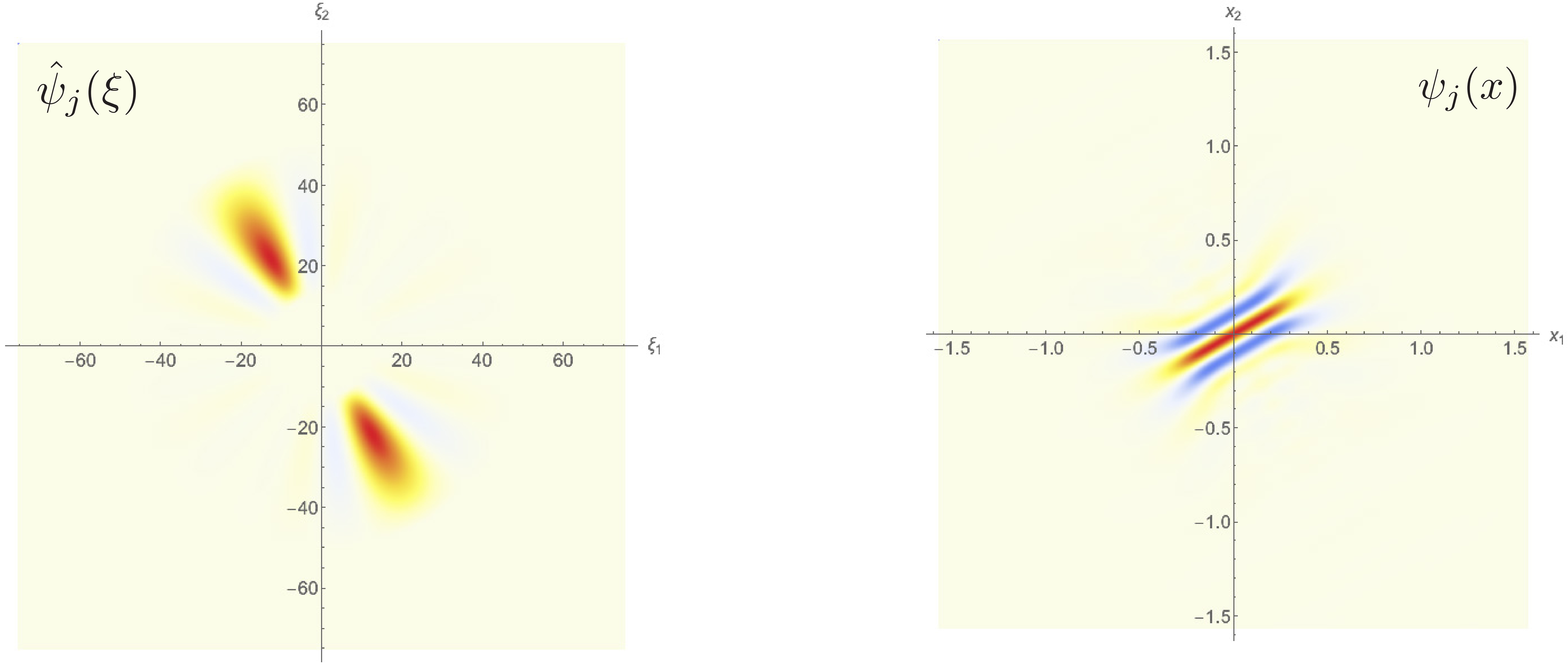
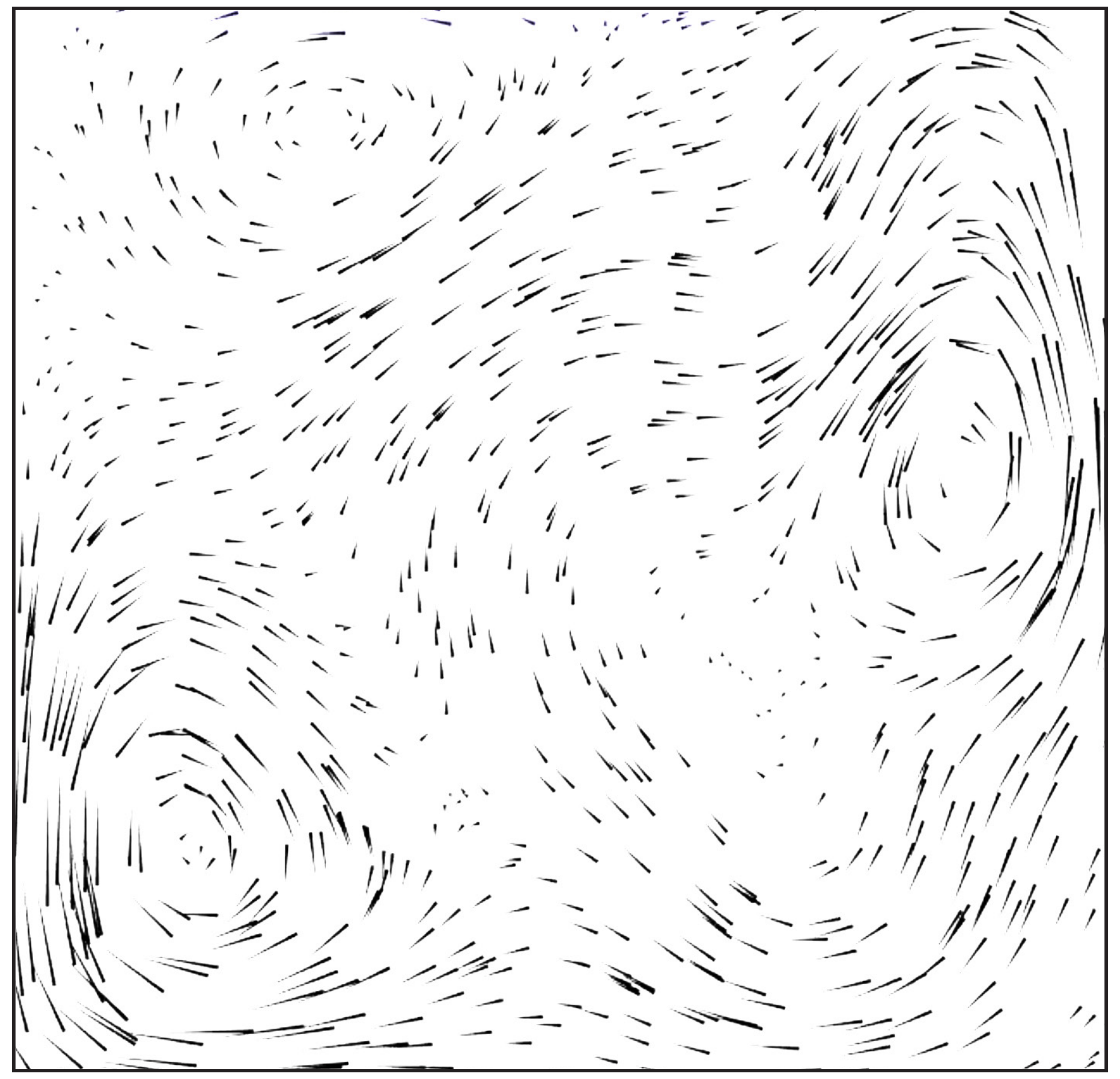


# II. Divergence free wavelets

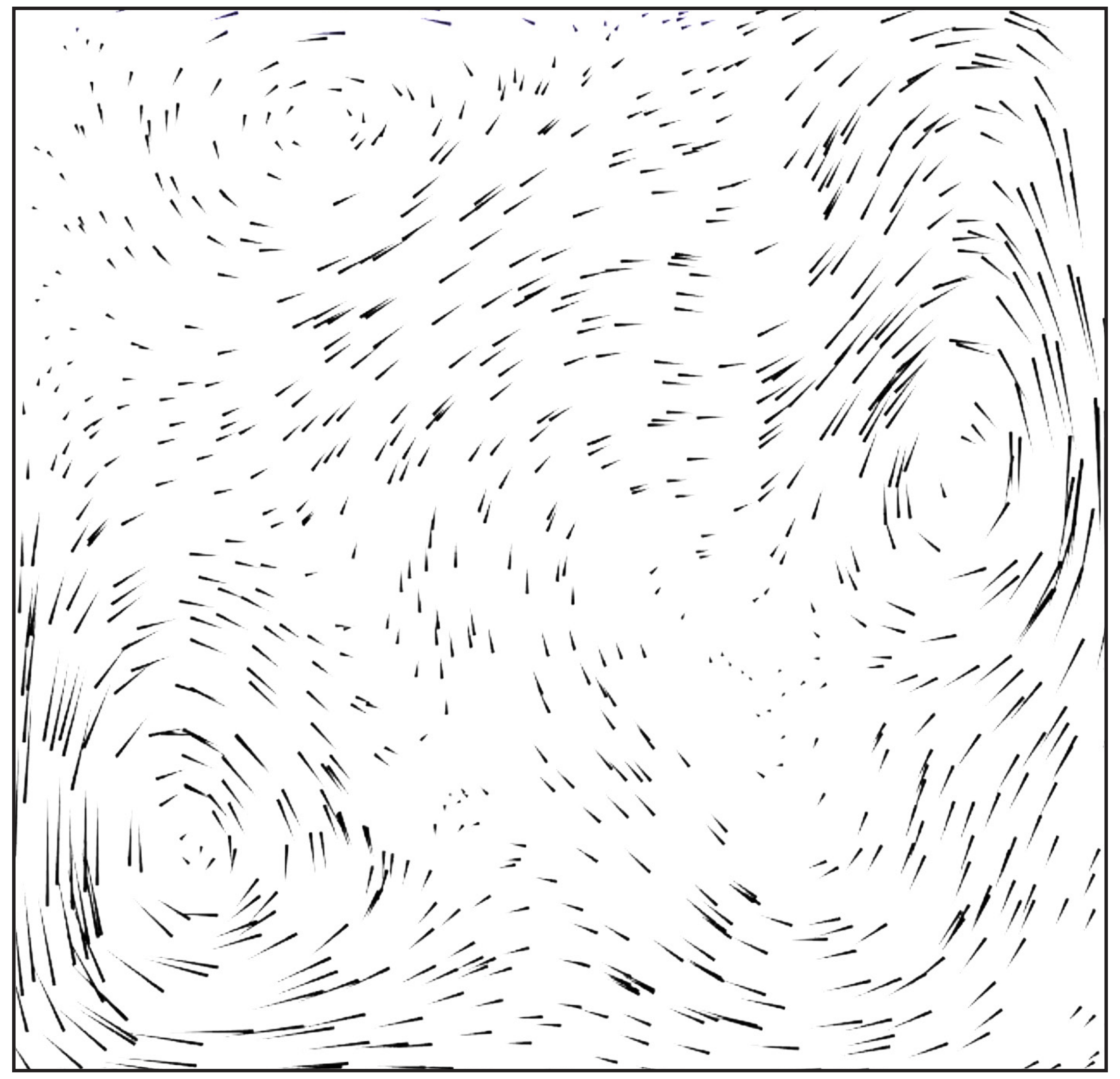
# Divergence freedom

$$\nabla \cdot \vec{u} = 0$$



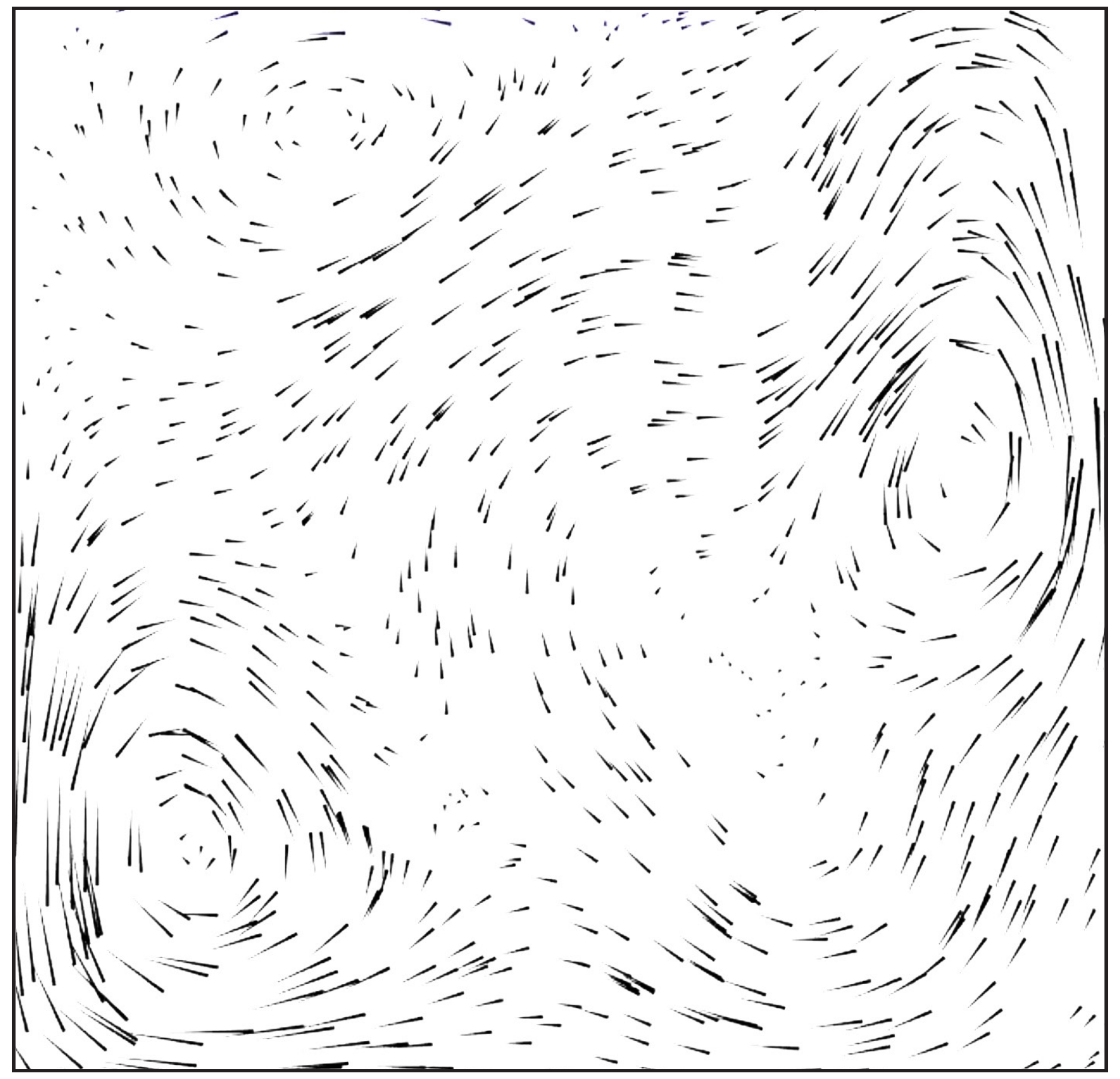
# Divergence freedom

$$\nabla \cdot \vec{u} = 0 \xrightarrow{\mathcal{F}}$$



# Divergence freedom

$$\nabla \cdot \vec{u} = 0 \xrightarrow{\mathcal{F}} \vec{\xi} \cdot \hat{\vec{u}} = 0$$



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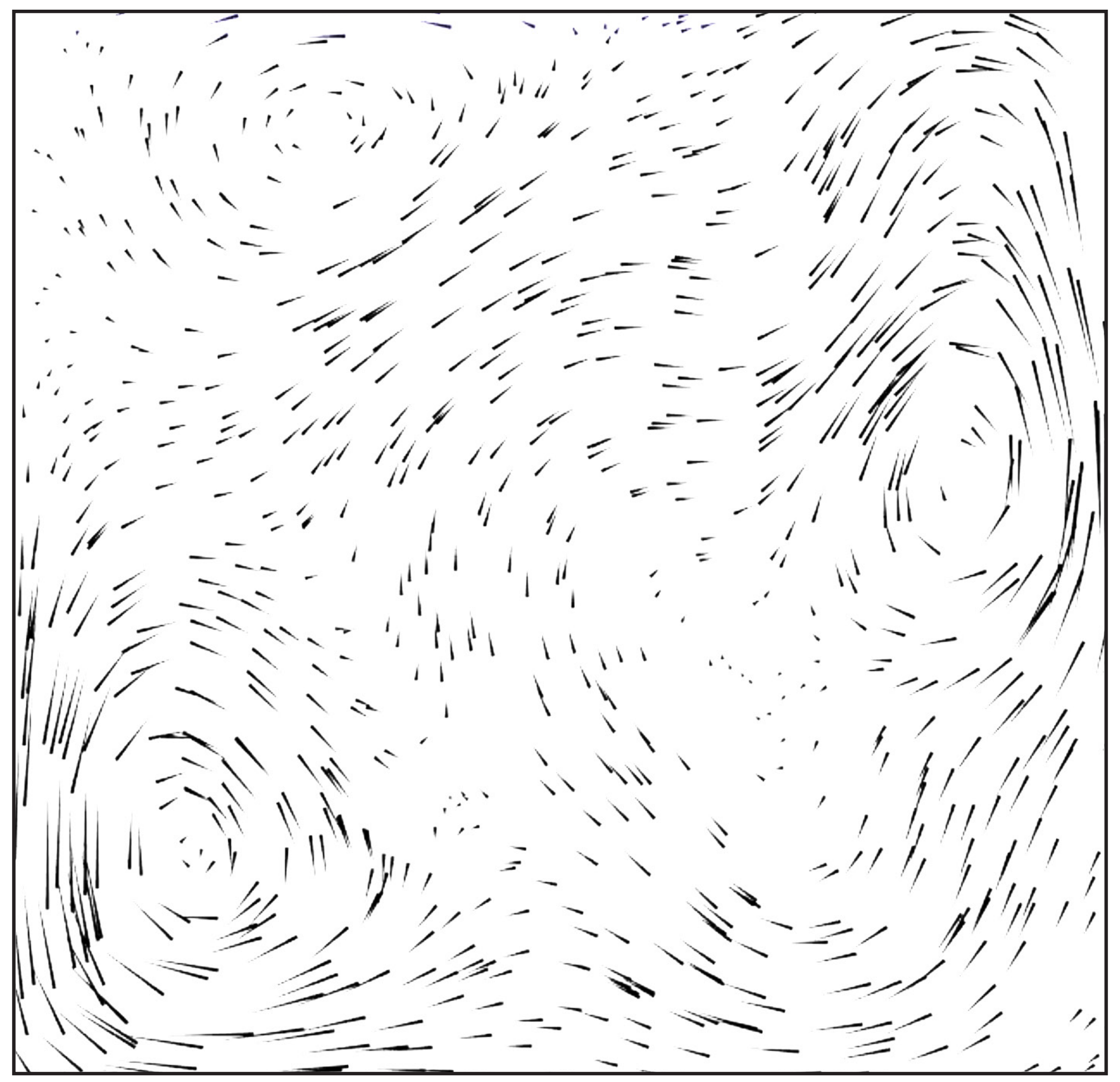
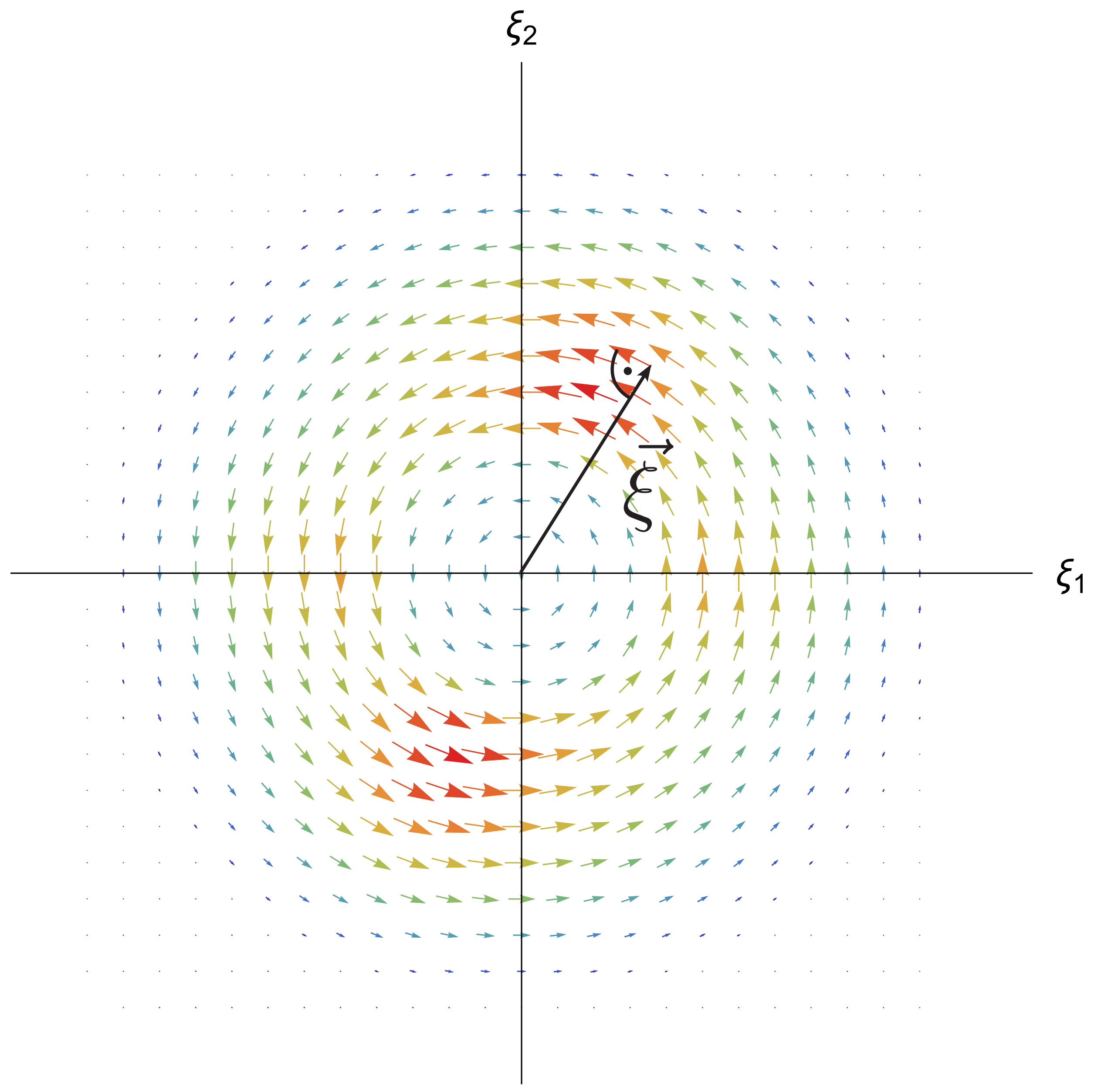
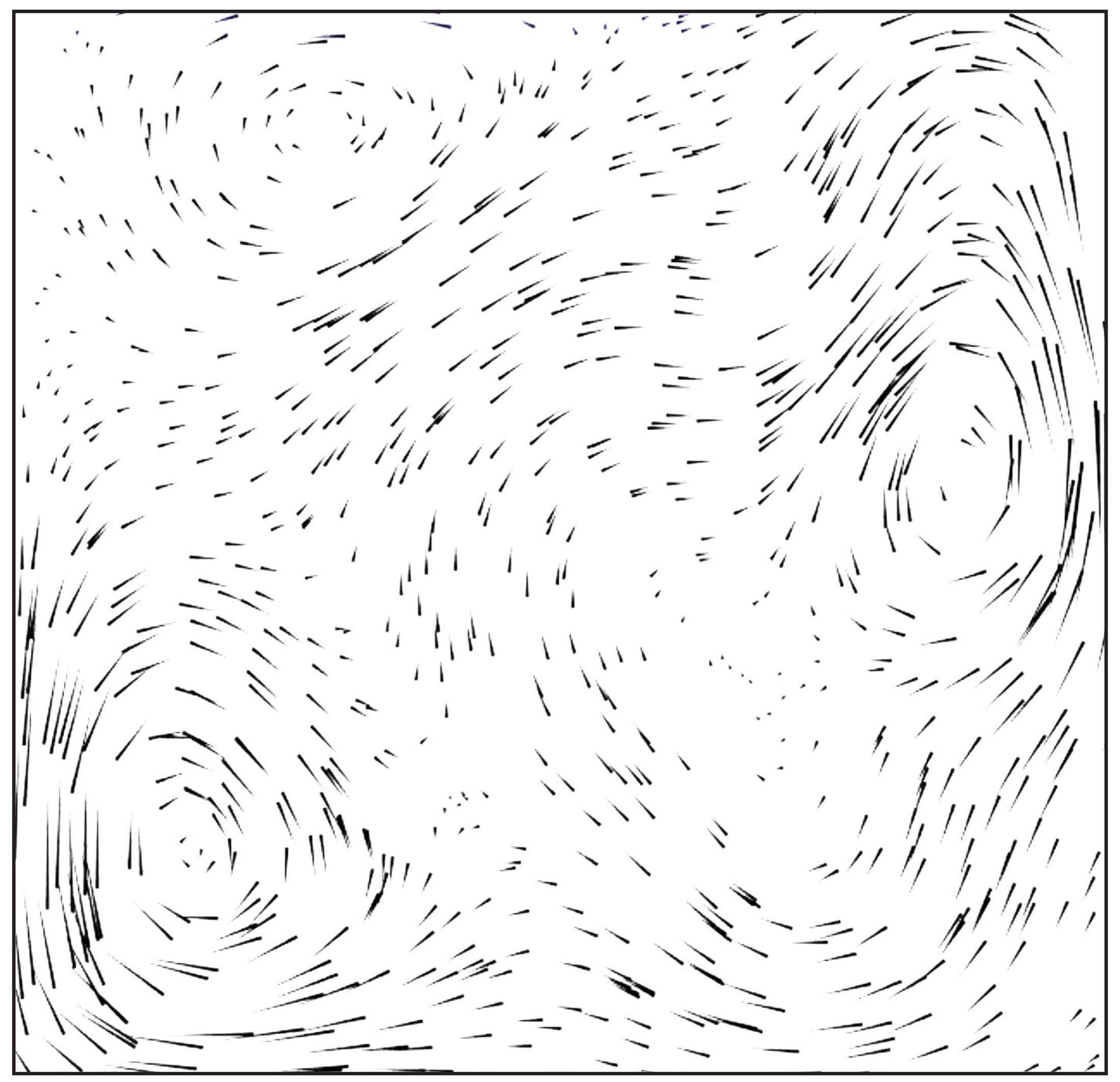
$$\nabla \cdot \vec{u} = 0 \xrightarrow{\mathcal{F}} \vec{\xi} \cdot \hat{\vec{u}} = 0$$

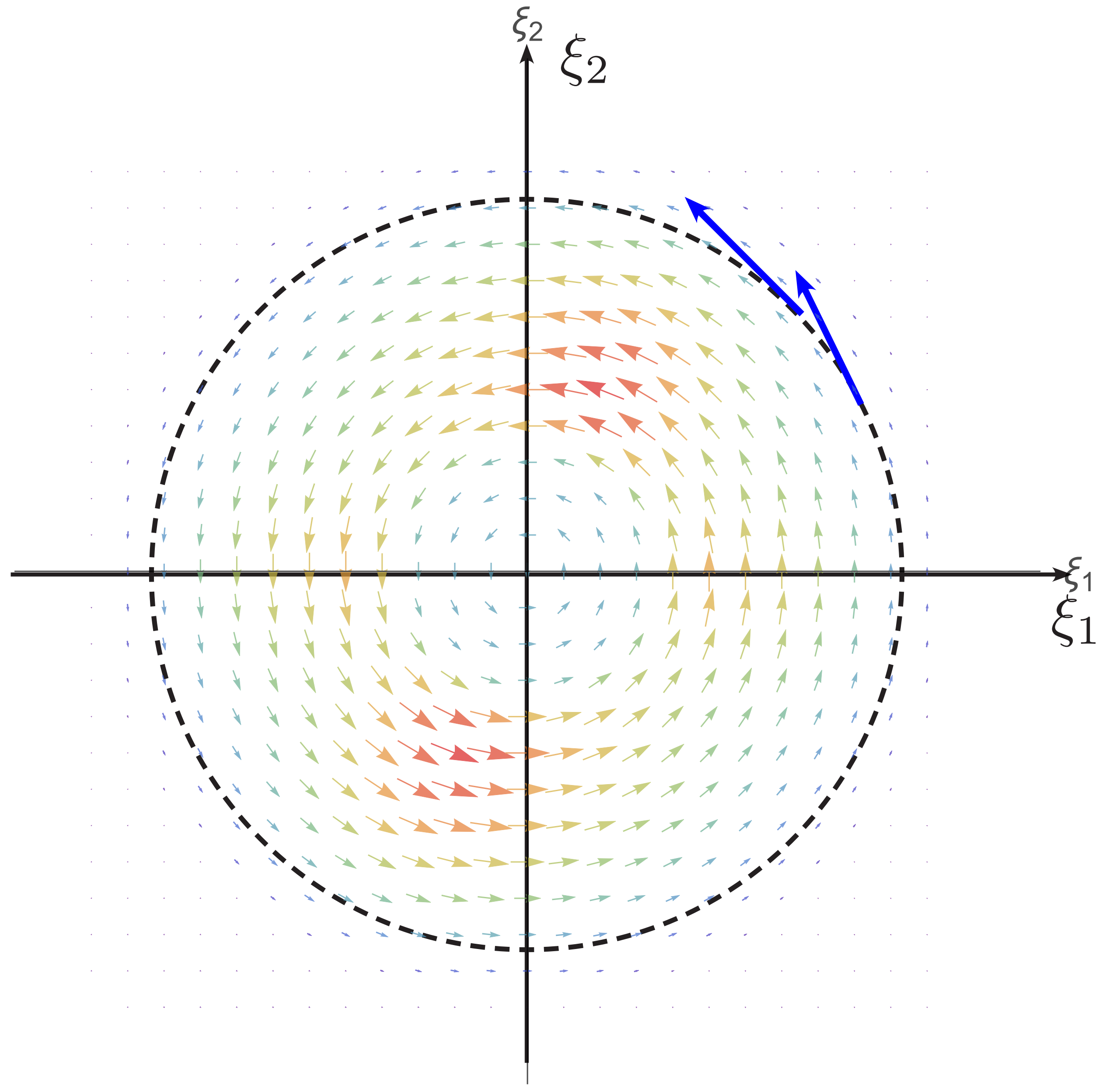




# Divergence freedom

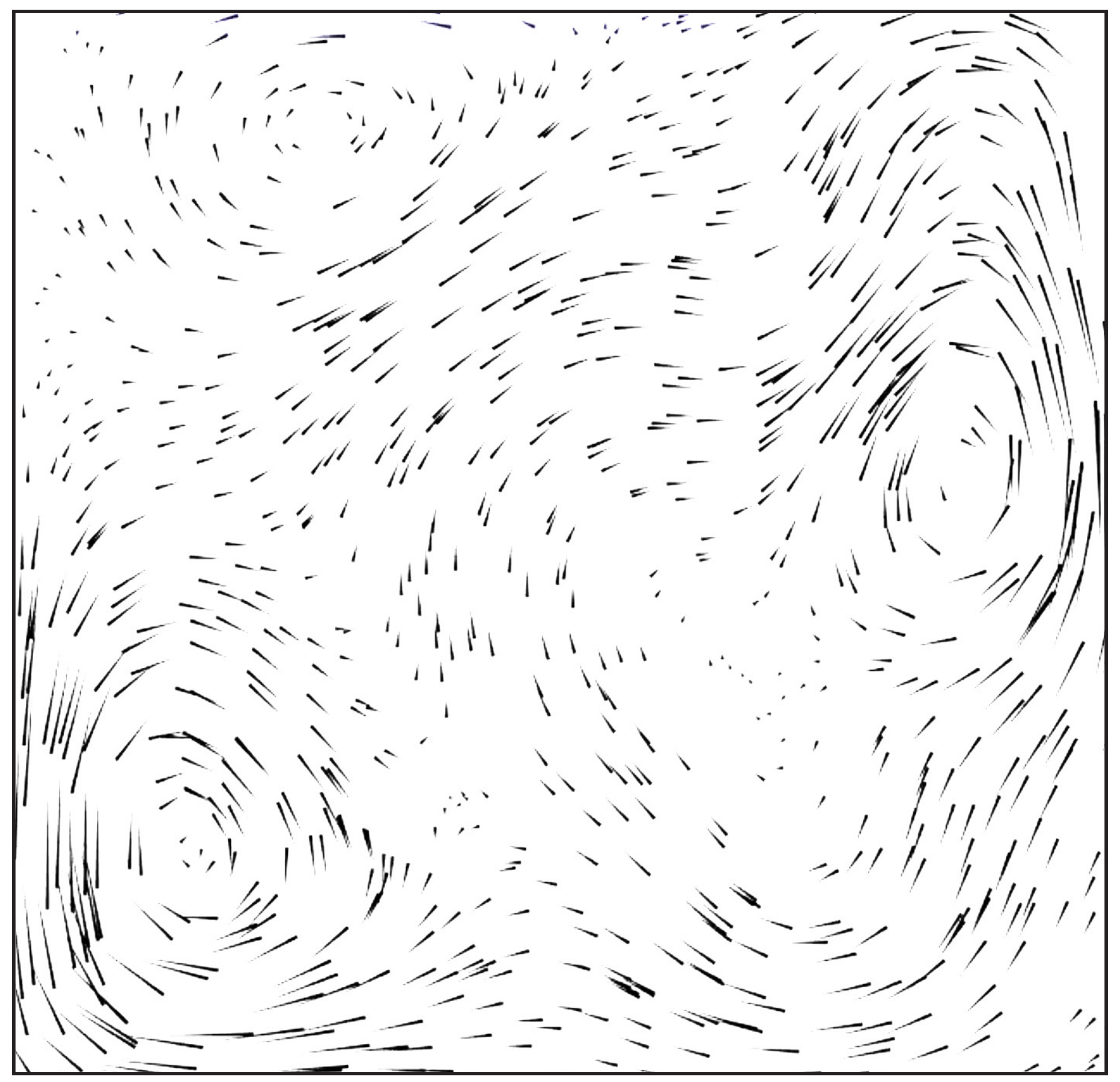
$$\nabla \cdot \vec{u} = 0 \xrightarrow{\mathcal{F}} \vec{\xi} \cdot \hat{\vec{u}} = 0$$

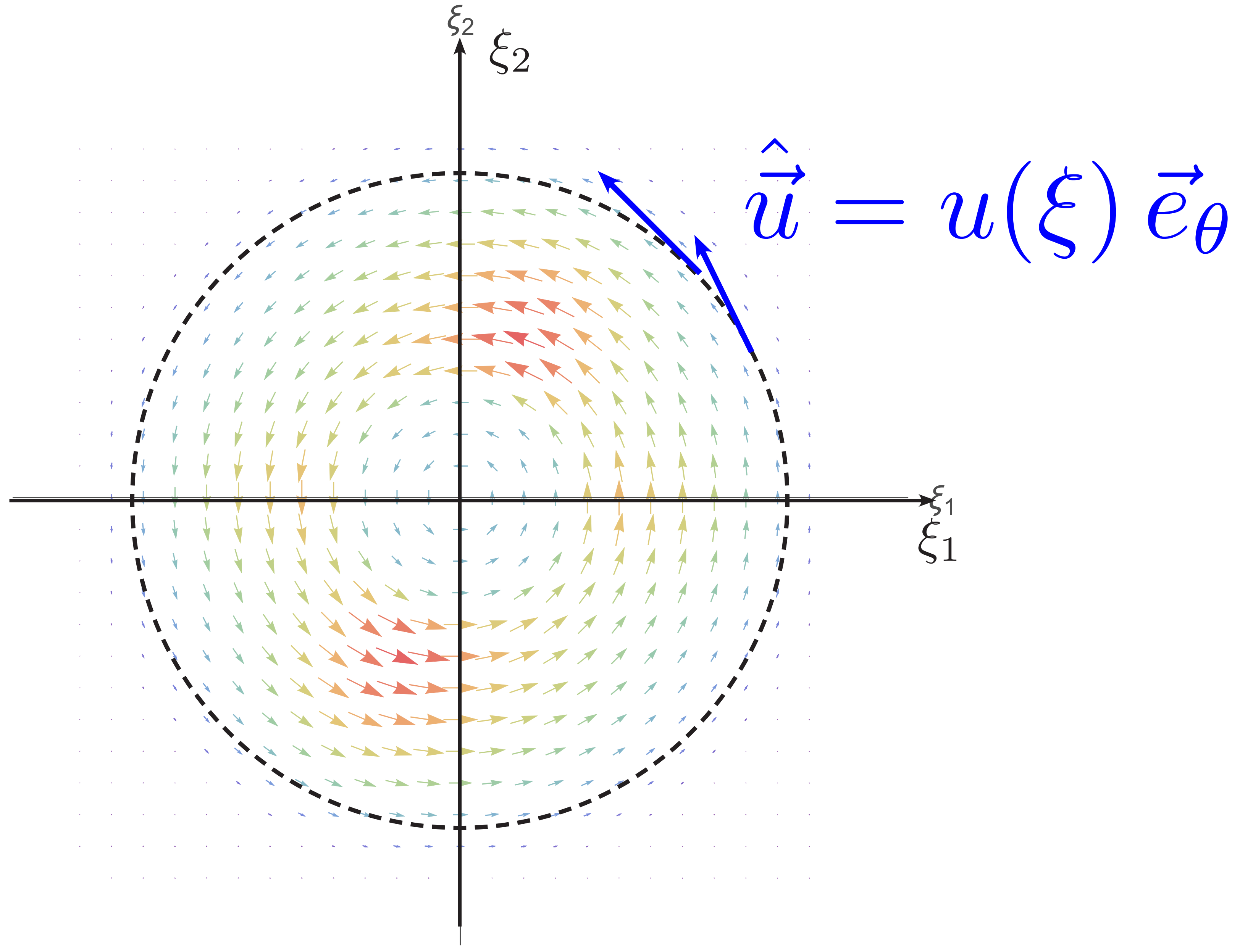




# Divergence freedom

$$\begin{tikzcd}\nabla \cdot \vec{u} = 0 \arrow[r, bend left, "\mathcal{F}"] & \vec{\xi} \cdot \hat{\vec{u}} = 0\end{tikzcd}$$

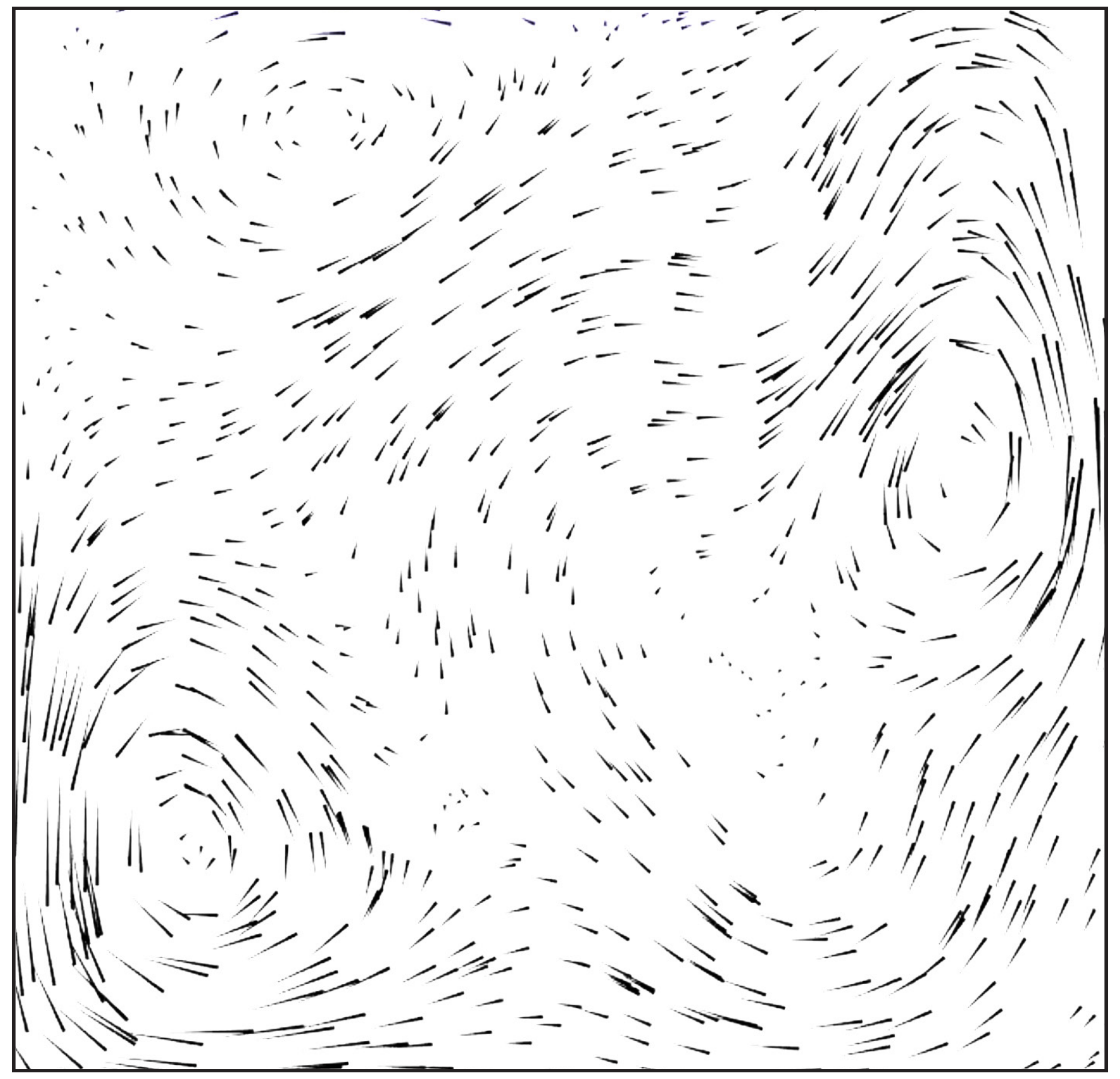


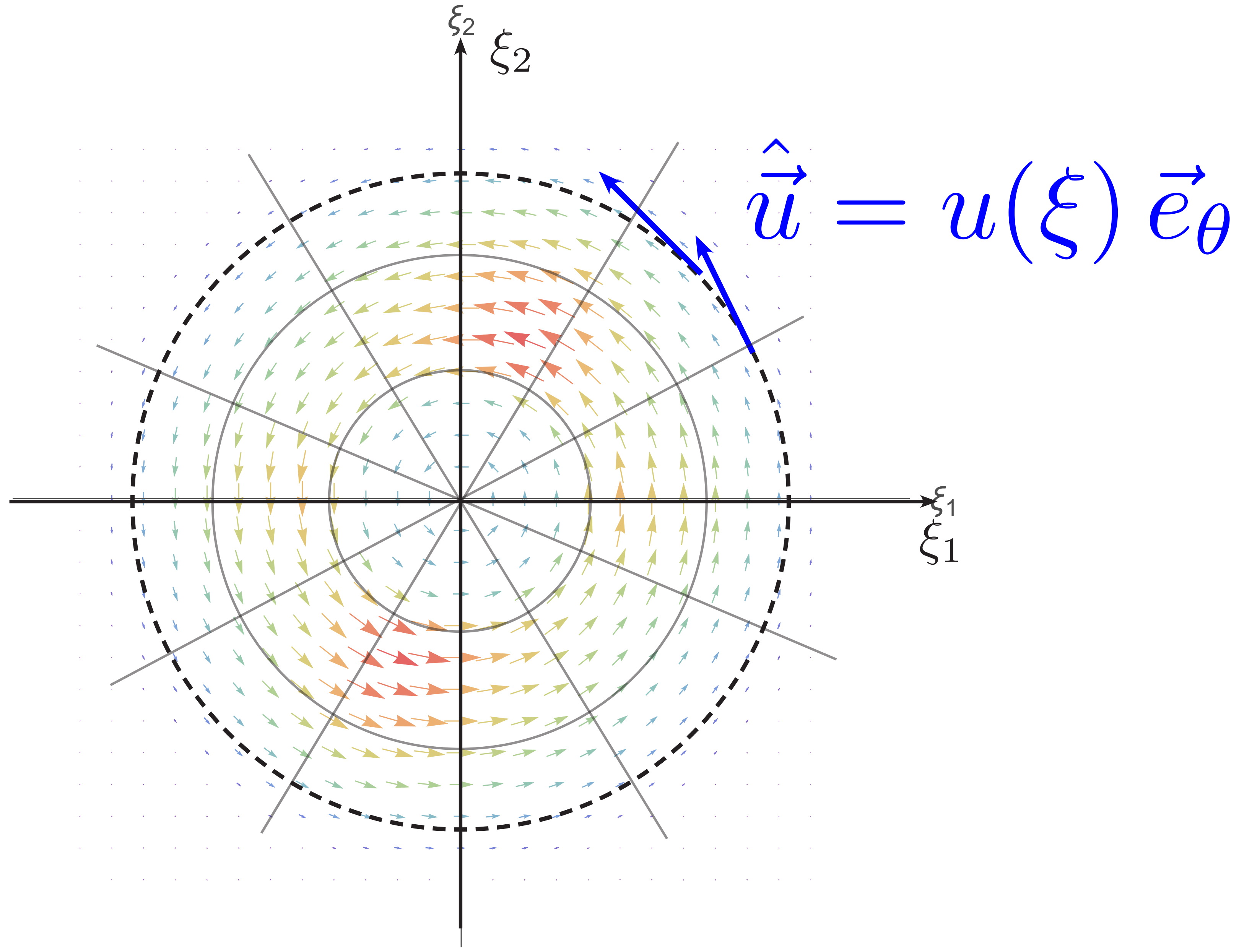


$\hat{\vec{u}} = u(\xi)\, \vec{e}_\theta$

# Divergence freedom

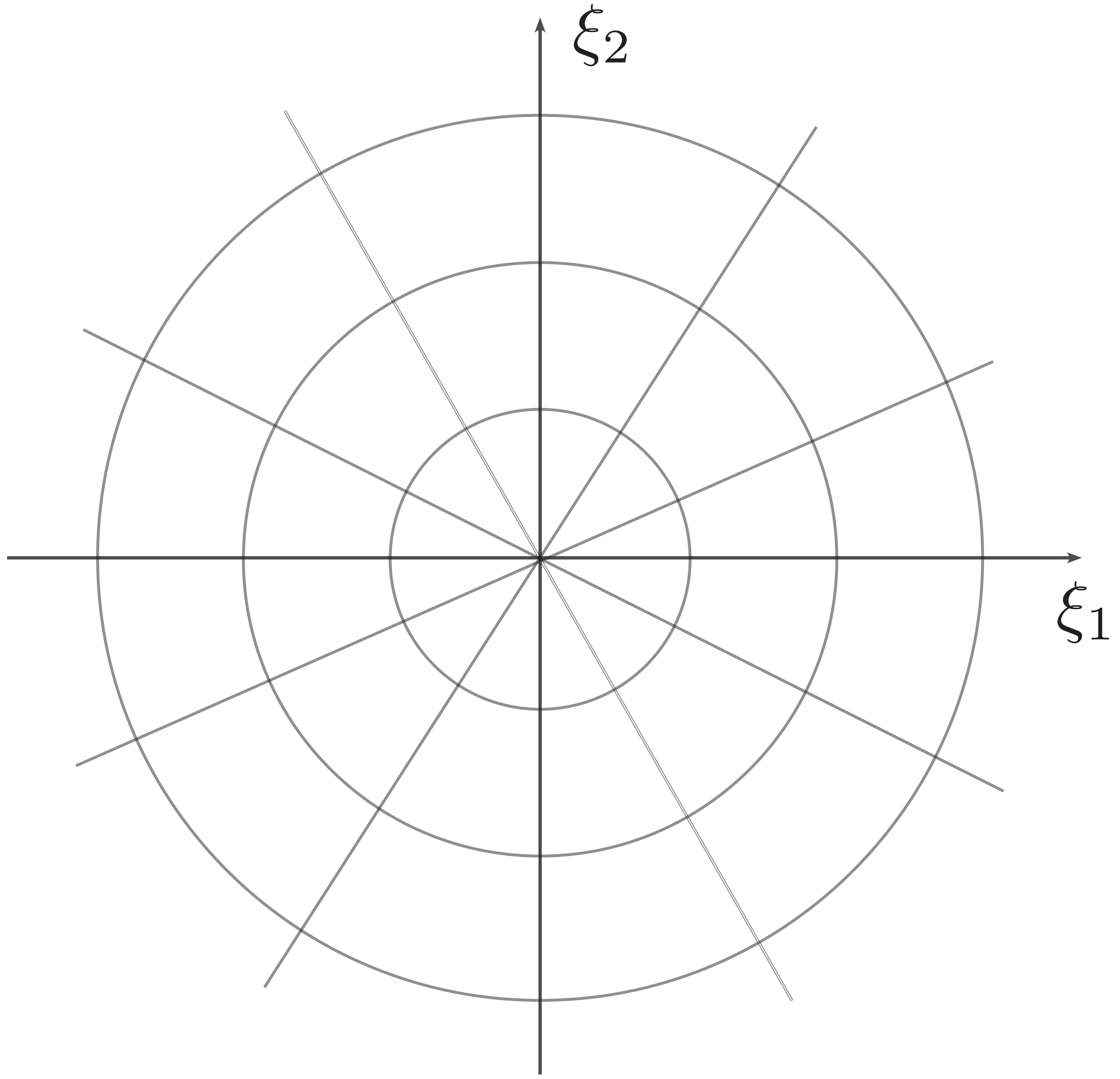
$$\begin{tikzcd} \nabla \cdot \vec{u} = 0 \arrow[r, bend left, "\mathcal{F}"] & \vec{\xi} \cdot \hat{\vec{u}} = 0 \end{tikzcd}$$





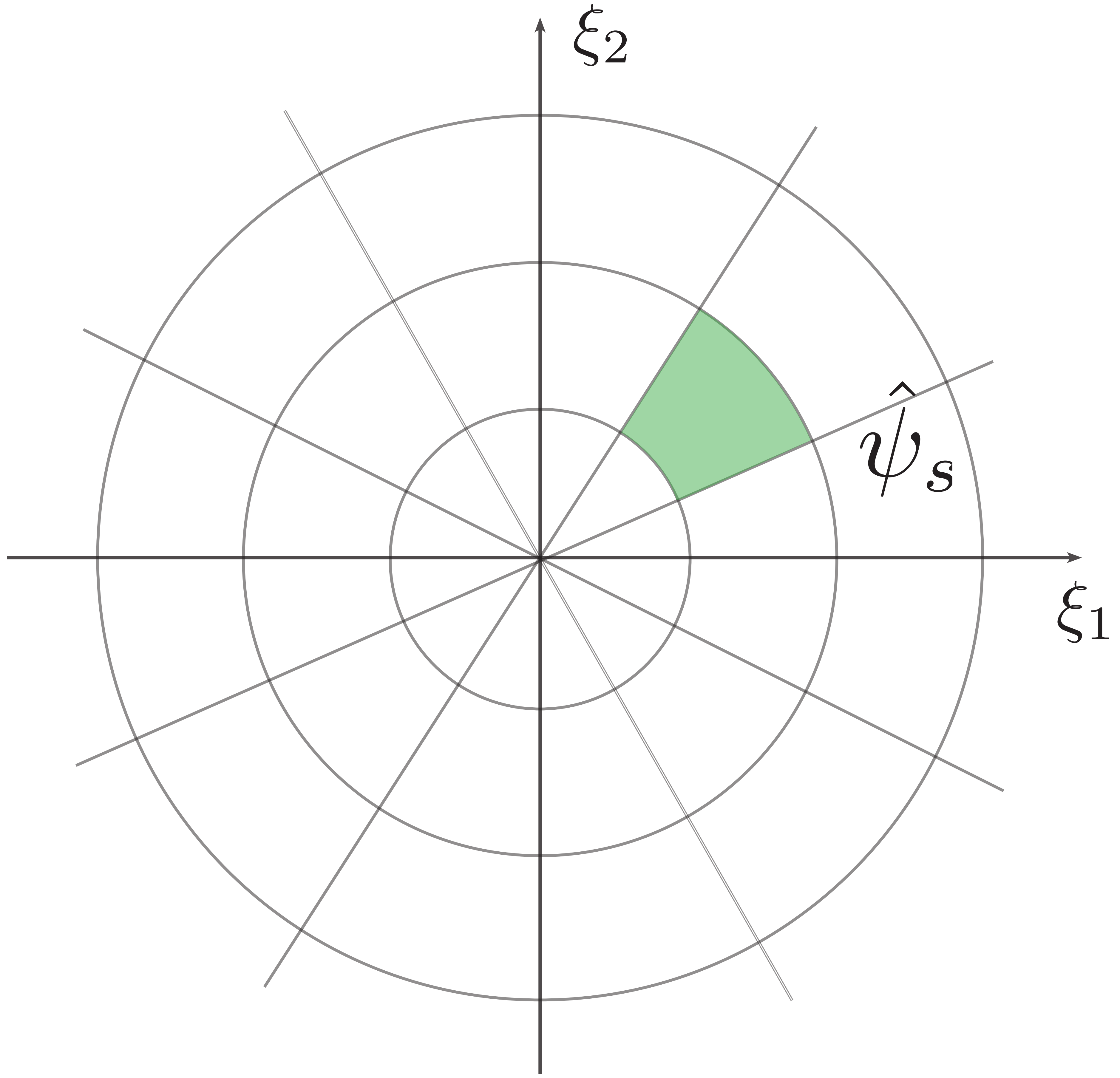
# Divergence free polar wavelets[3]

- Divergence free basis:



# Divergence free polar wavelets[3]

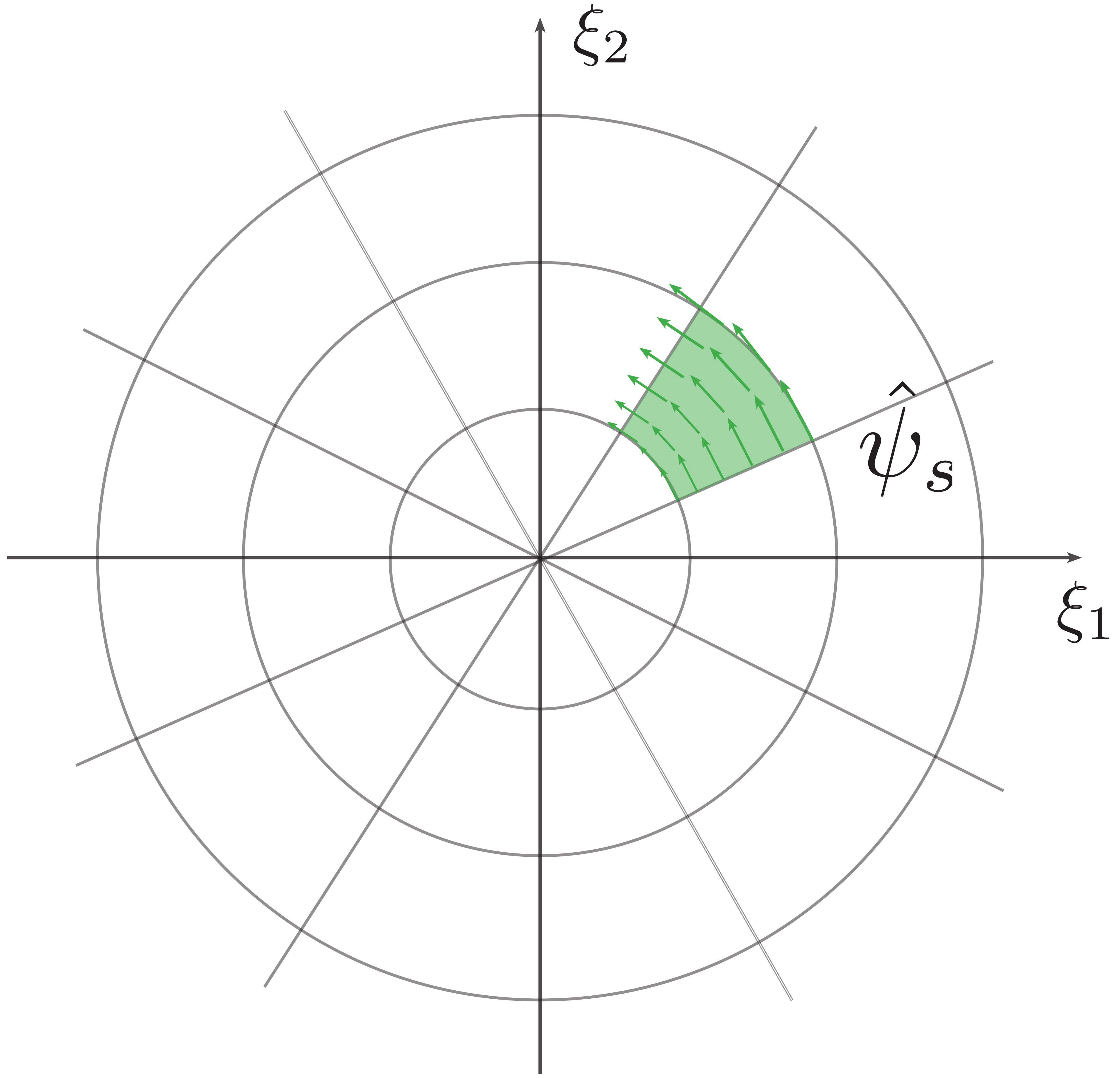
- Divergence free basis:



# Divergence free polar wavelets

- Divergence free basis:

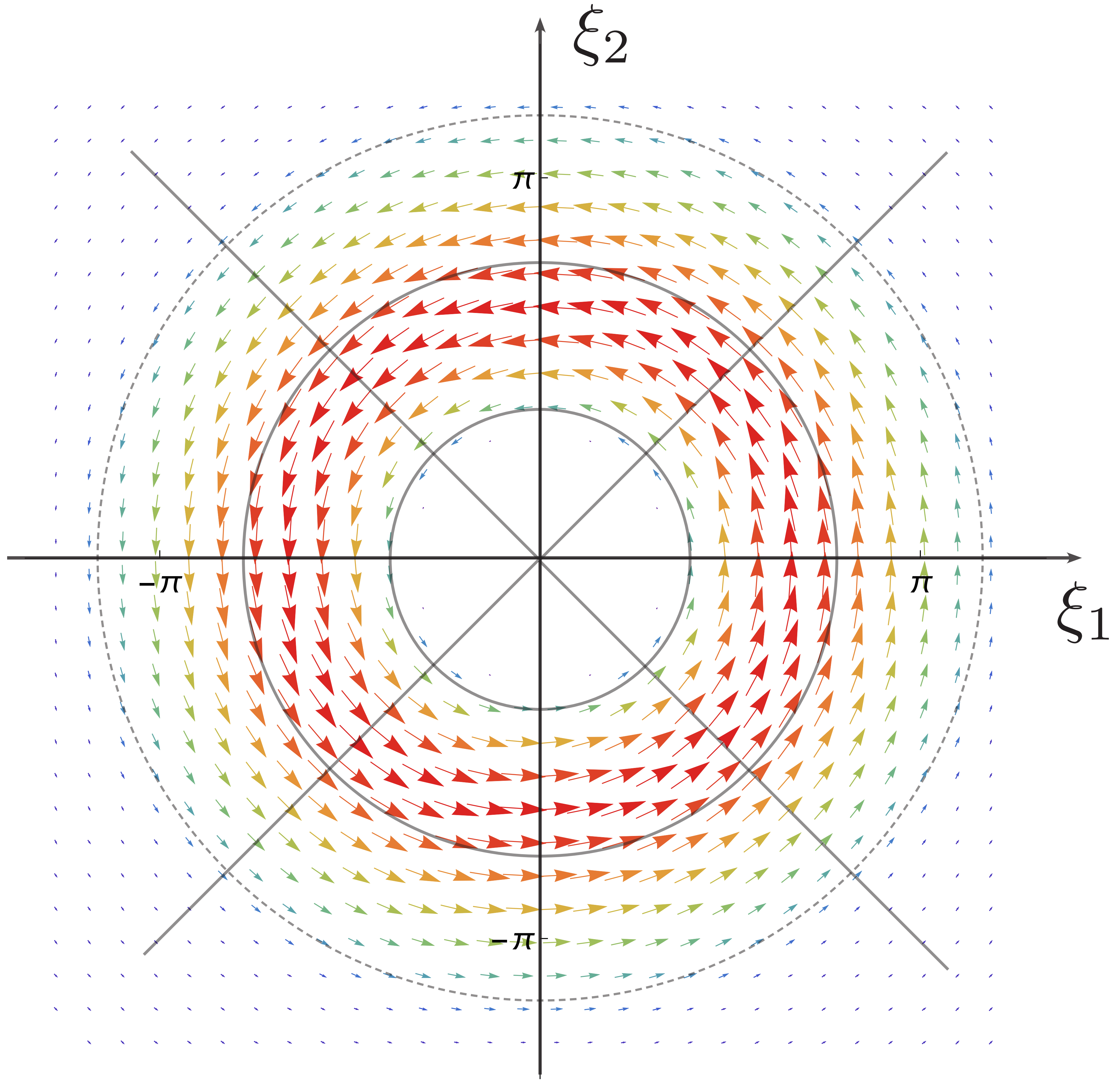
$$\hat{\vec{\psi}}_s(\xi) = \hat{\psi}_s(\xi)\,\vec{e}_\theta$$



# Divergence free polar wavelets

- Divergence free basis:

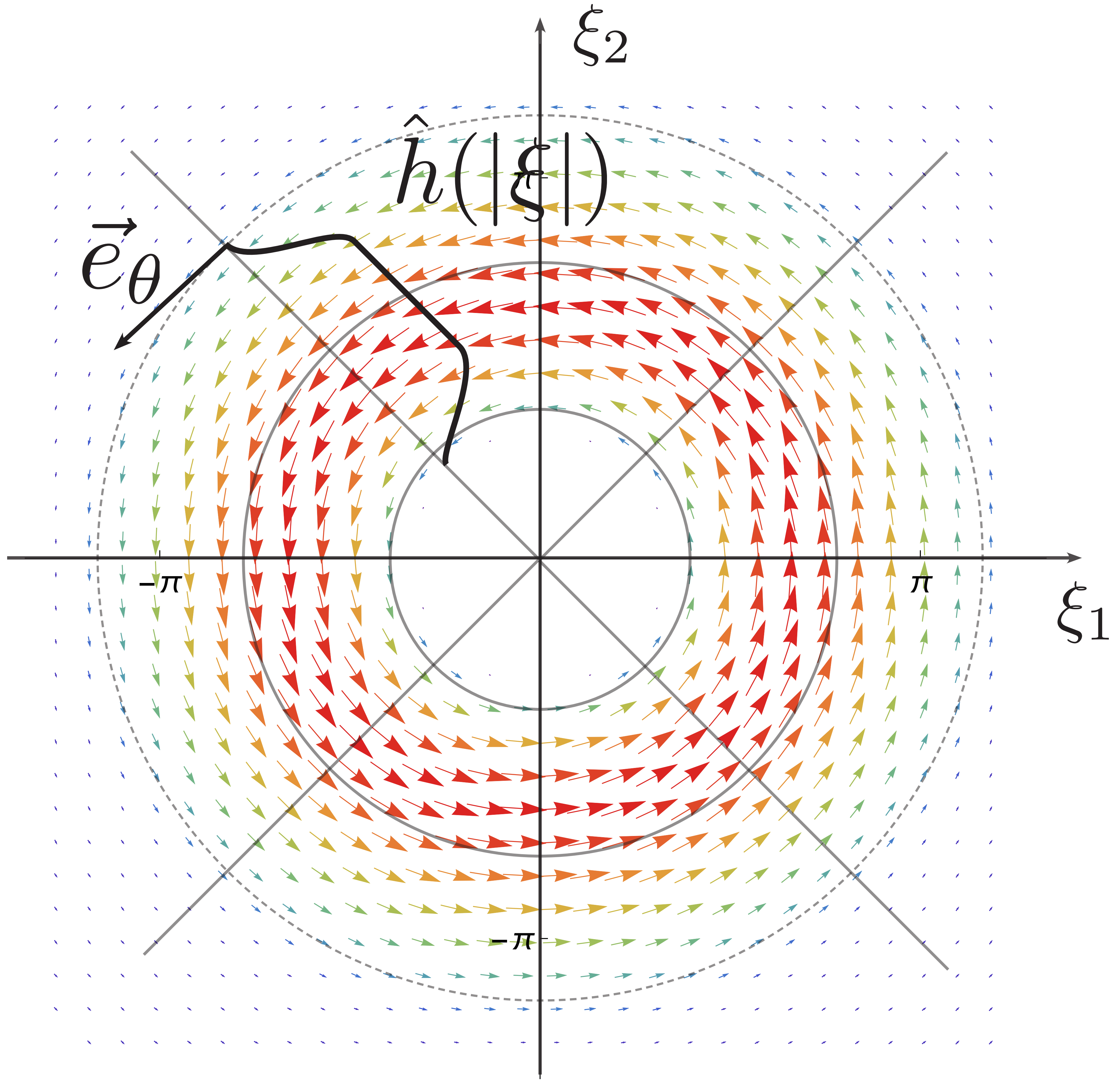
$$\hat{\vec{\psi}}_s(\xi) = \hat{\psi}(|\xi|)\, \vec{e}_\theta$$



# Divergence free polar wavelets

- Divergence free basis:

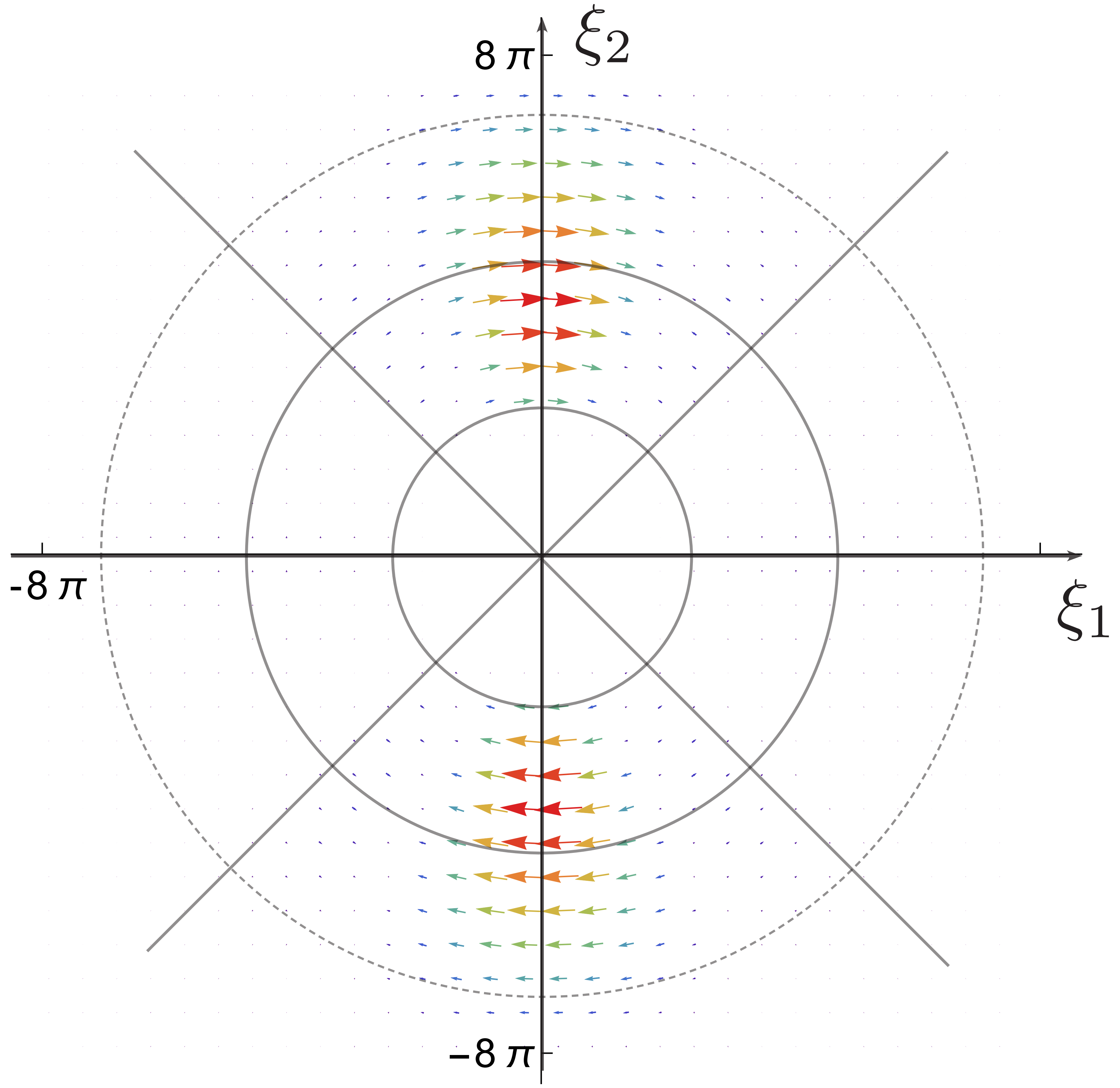
$$\hat{\vec{\psi}}_s(\xi) = \hat{\psi}(|\xi|)\,\vec{e}_\theta$$



# Divergence free polar wavelets

- Divergence free basis:

$$\hat{\vec{\psi}}_s(\xi) = \gamma(\theta_\xi)\,\hat{\psi}(|\xi|)\,\vec{e}_\theta$$



# Divergence free polar wavelets

**Proposition 1.** *Let $U_j$ be the $(M_j \times 2N_j + 1)$-dimensional matrix formed by the angular localization coefficients $\beta_n^{j,t} = \beta_n^j \, e^{-int(2\pi/M_j)}$ for the $M_j$ different orientations, and let $D_j$ be a diagonal matrix of size $(2N_j + 1) \times (2N_j + 1)$. When the Caldèron admissibility condition $\sum_{j \in \mathbb{Z}} \left| \hat{h}(2^{-j}|\xi|) \right|^2 = 1, \forall \xi \in \mathbb{R}^2$ is satisfied and $U_j^H U_j = D_j$ with $\mathrm{tr}(D_j) = 1$ for all levels $j$, then any divergence free vector field $\vec{u}(x) \in L_2^{\mathrm{div}}(\mathbb{R}^{2,2})$ has the representation*

$$\vec{u}(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} \sum_{t=1}^{M_j} \left\langle \vec{u}(y), \vec{\psi}_{j,k,t}(y) \right\rangle \vec{\psi}_{j,k,t}(x) \tag{5a}$$

*with frame functions*

$$\vec{\psi}_{j,k,t}(x) = \frac{2^j}{2\pi} \, \vec{\psi}\big(R_{2\pi t/M_j}(2^j x - k)\big) \tag{5b}$$

*where $\vec{\psi}(x)$ is given by Eq. 4 and $R_{2\pi t/M_j}$ is the rotation by $2\pi t/M_j$.*

# Scalar free polar wavelets

$$
\begin{aligned}
\psi_{j,t}(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{\psi}_j(\xi)\, e^{i\langle \xi, x \rangle}\, d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}(|\xi|)\, \hat{\gamma}(\theta_\xi)\, e^{i\langle \xi, x \rangle}\, d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}_j(|\xi|)\, \hat{\gamma}(\theta_\xi) \left( \sum_{n=-\infty}^{\infty} i^n\, e^{in(\theta_x - \theta_\xi)}\, J_n(|\xi|\, |x|) \right) d\xi
\end{aligned}
$$

# Scalar free polar wavelets

$$
\begin{aligned}
\psi_{j,t}(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{\psi}_j(\xi)\, e^{i\langle \xi , x \rangle}\, d\xi \\
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&= \frac{1}{2\pi} \int_{\mathbb{R}^2_\xi} \hat{h}_j(|\xi|)\, \hat{\gamma}(\theta_\xi) \left( \sum_{n=-\infty}^{\infty} i^n\, e^{in(\theta_x - \theta_\xi)} J_n(|\xi|\,|x|) \right) d\xi
\end{aligned}
$$

$$\vec{e}_\theta = (-\sin\theta_\xi , \cos\theta_\xi)^T$$

# Divergence free polar wavelets

$$\hat{\gamma}(\xi)\,\vec{e}_{\theta} = \sum_{m} \beta_m \, e^{i m \theta_{\xi}} \begin{pmatrix} -\sin\theta_{\xi} \\ \cos\theta_{\xi} \end{pmatrix}$$

# Divergence free polar wavelets

$$\hat{\gamma}(\xi)\,\vec{e}_{\theta} = \sum_m \beta_m\, e^{im\theta_\xi} \begin{pmatrix} -\sin\theta_\xi \\ \cos\theta_\xi \end{pmatrix}$$

$$= \sum_m \beta_m\, e^{im\theta_\xi} \begin{pmatrix} -\frac{i}{2}(e^{-i\theta} - e^{i\theta}) \\ \frac{1}{2}(e^{-i\theta} + e^{i\theta}) \end{pmatrix}$$

# Divergence free polar wavelets

$$\hat{\gamma}(\xi)\,\vec{e}_{\theta} = \sum_m \beta_m\, e^{im\theta_\xi} \begin{pmatrix} -\sin\theta_\xi \\ \cos\theta_\xi \end{pmatrix}$$

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$$= \begin{pmatrix} -\frac{i}{2}\big(\sum_m \beta_m\, e^{i(m-1)\theta_\xi} - \sum_m \beta_m\, e^{i(m+1)\theta_\xi}\big) \\ \frac{1}{2}\big(\sum_m \beta_m\, e^{i(m-1)\theta_\xi} + \sum_m \beta_m\, e^{i(m+1)\theta_\xi}\big) \end{pmatrix}$$

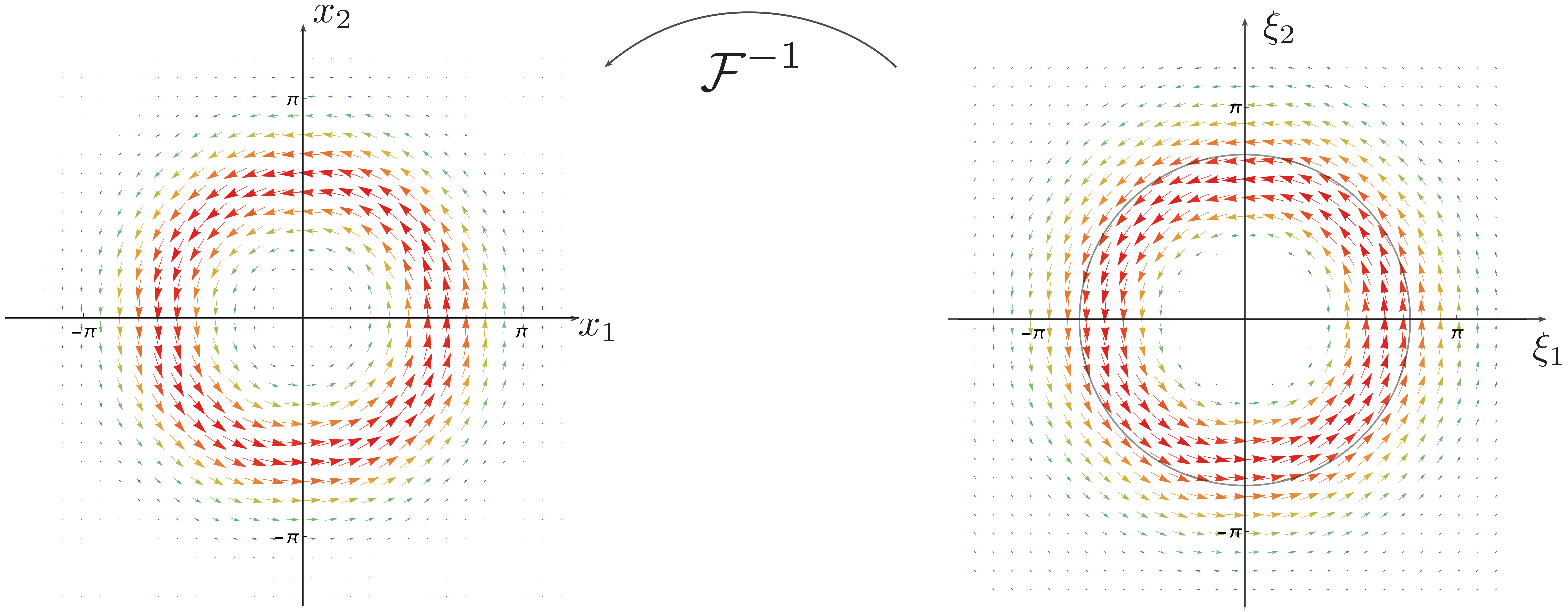
# Divergence free polar wavelets

$$\vec{\psi}(\xi) = \sum_m \beta_m \, e^{im\theta_\xi} \, \hat{h}(|\xi|) \, \vec{e}_{\theta_\xi}$$

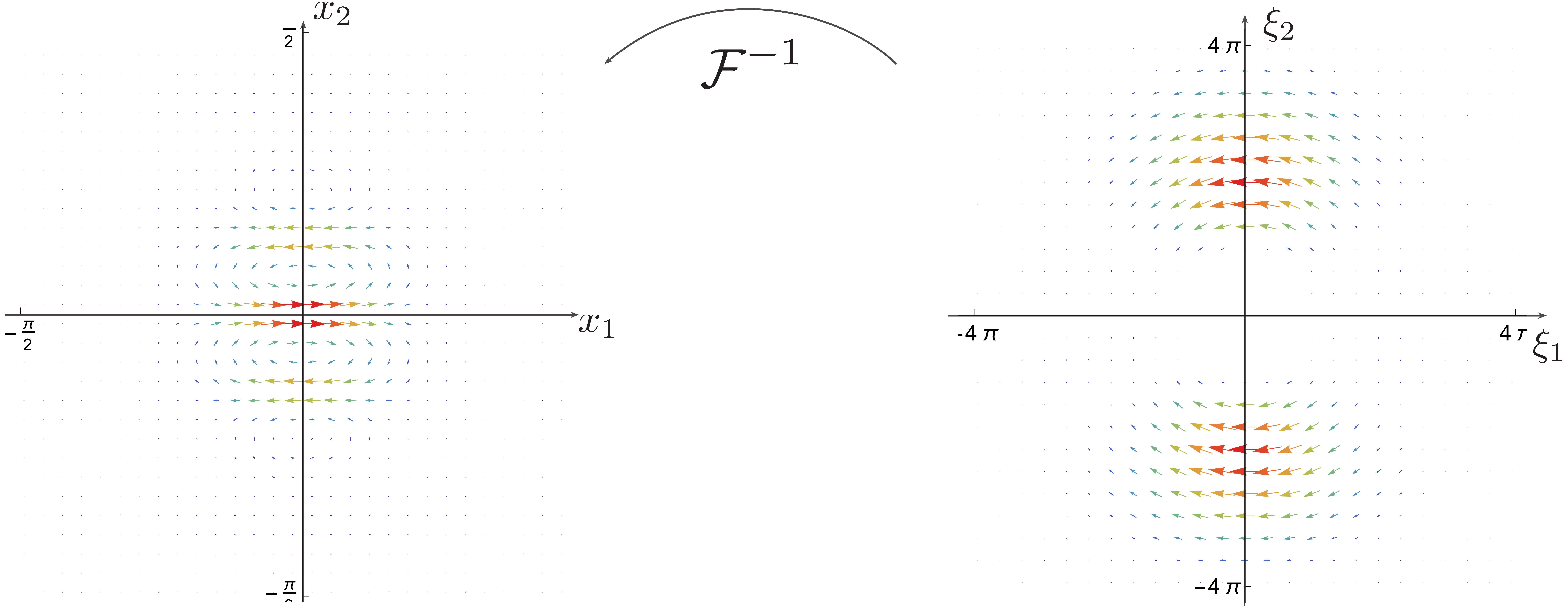
$$\Big\downarrow \quad \mathcal{F}^{-1}$$

$$\vec{\psi}(x) = \frac{1}{2} \sum_{\sigma \in \{-1,1\}} \sum_m i^{m+\sigma} \beta_m \, e^{i(m+\sigma)\theta_x} h_{m+\sigma}(|x|) \begin{pmatrix} -\sigma \\ i \end{pmatrix}$$

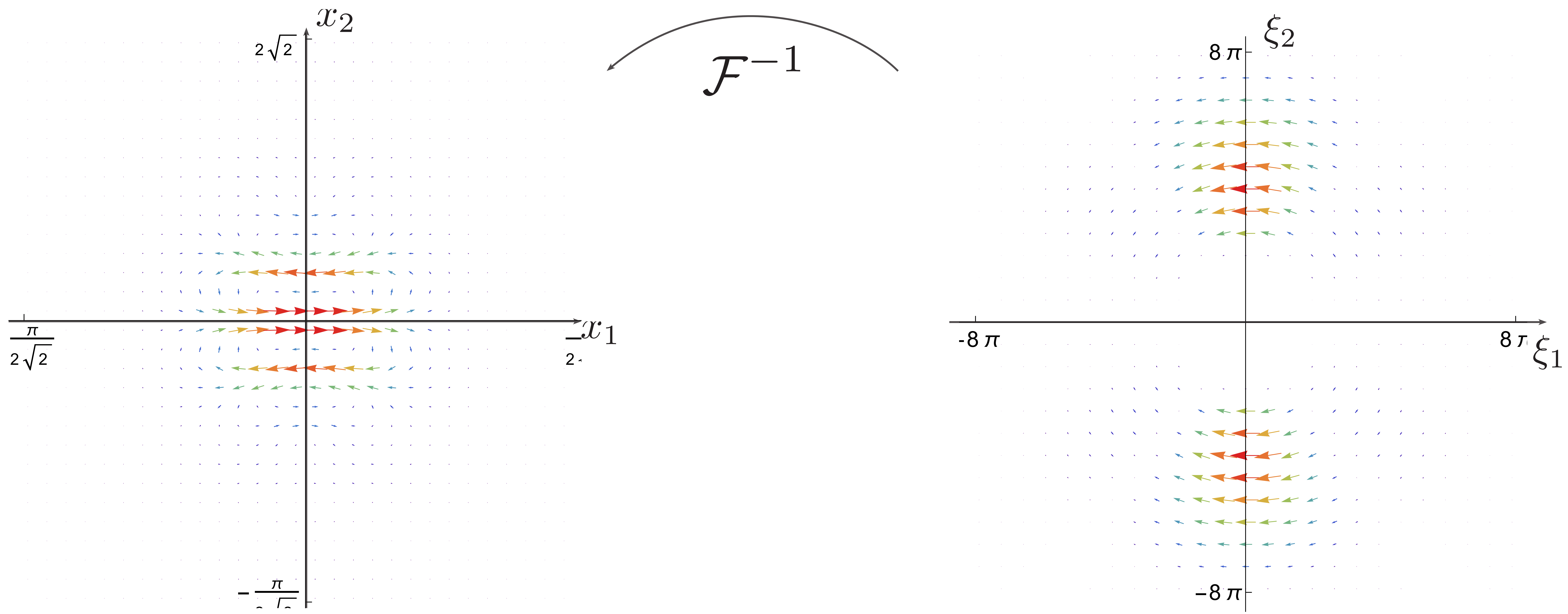
# Divergence free polar wavelets



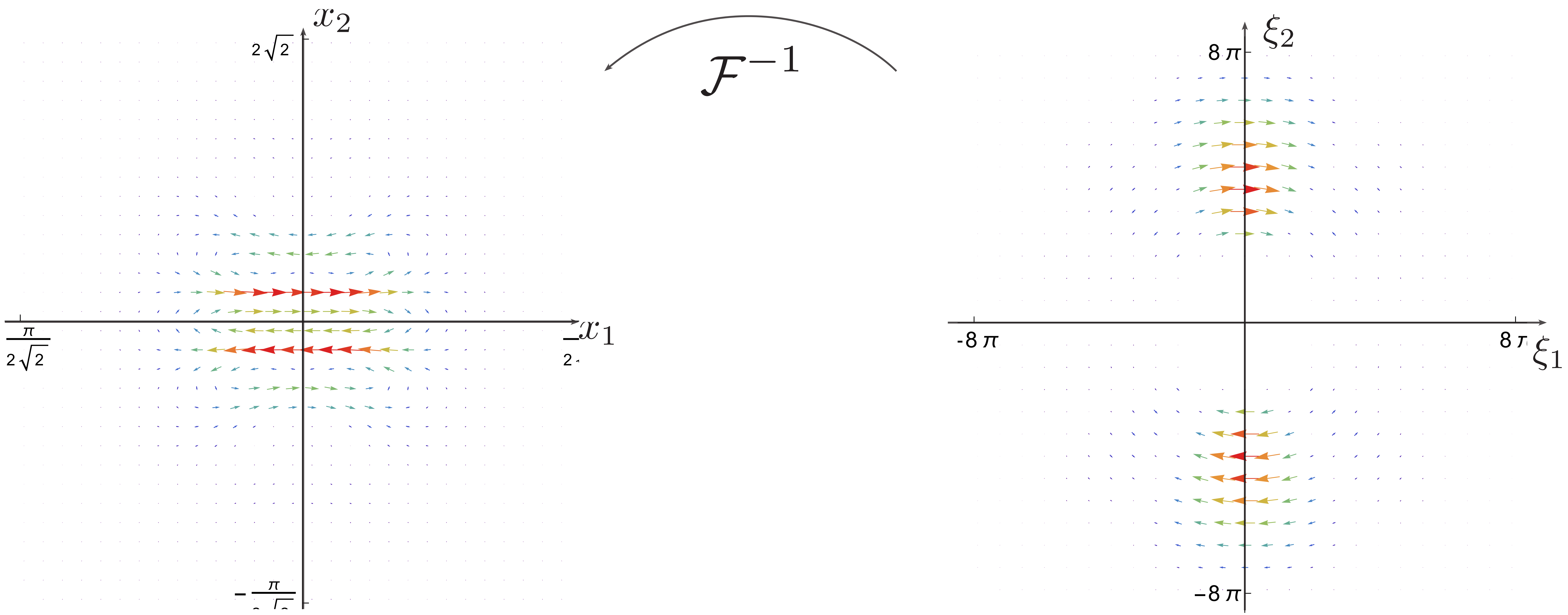
# Divergence free polar wavelets



# Divergence free polar wavelets



# Divergence free polar wavelets



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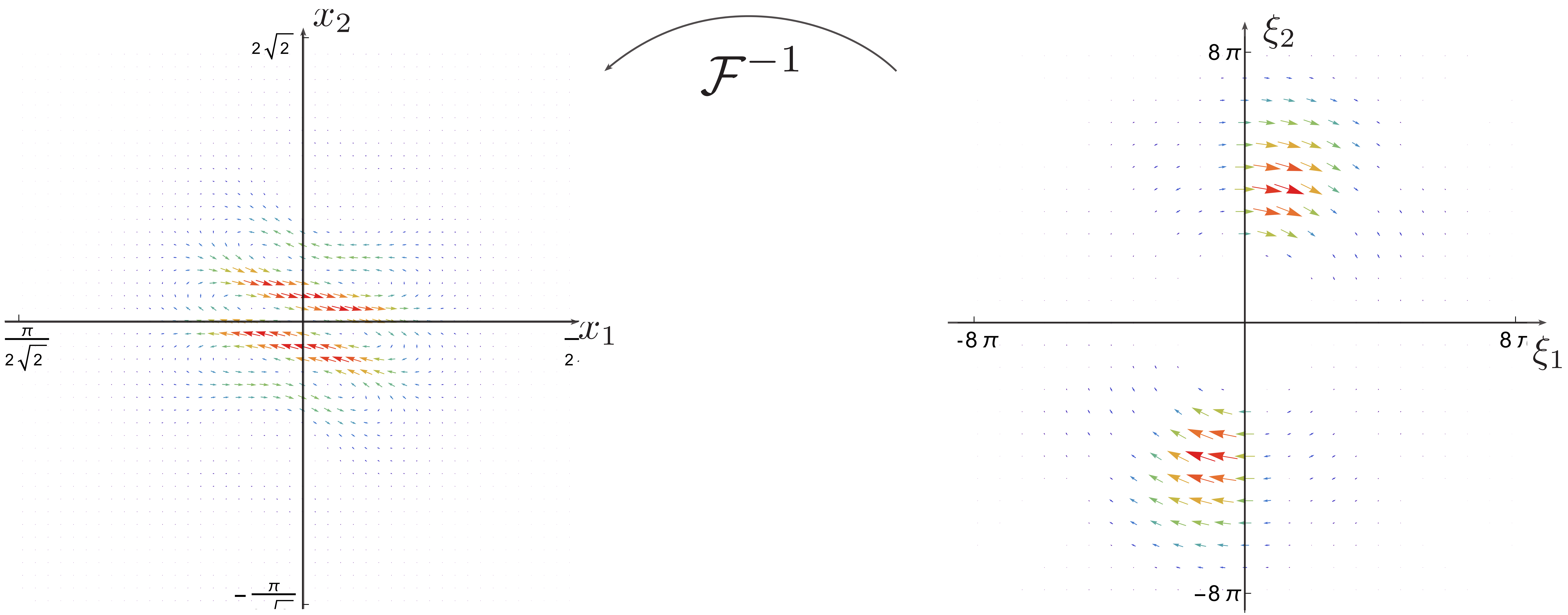
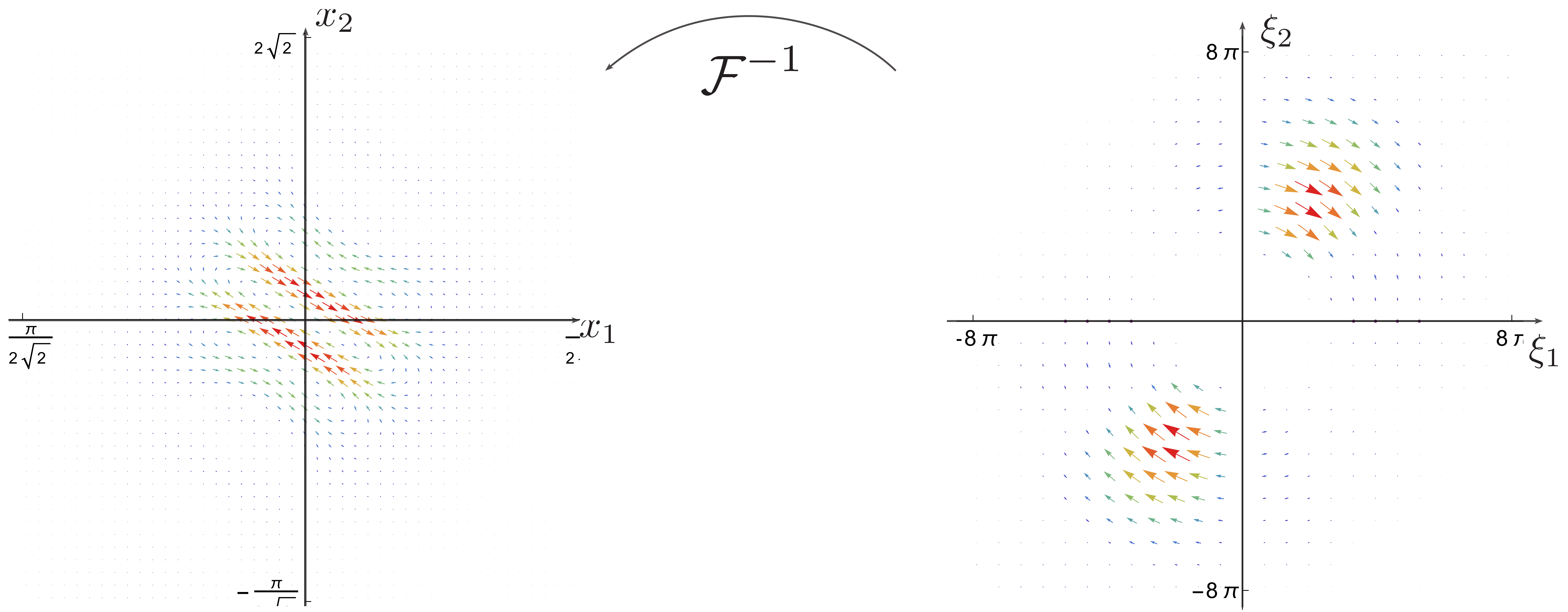
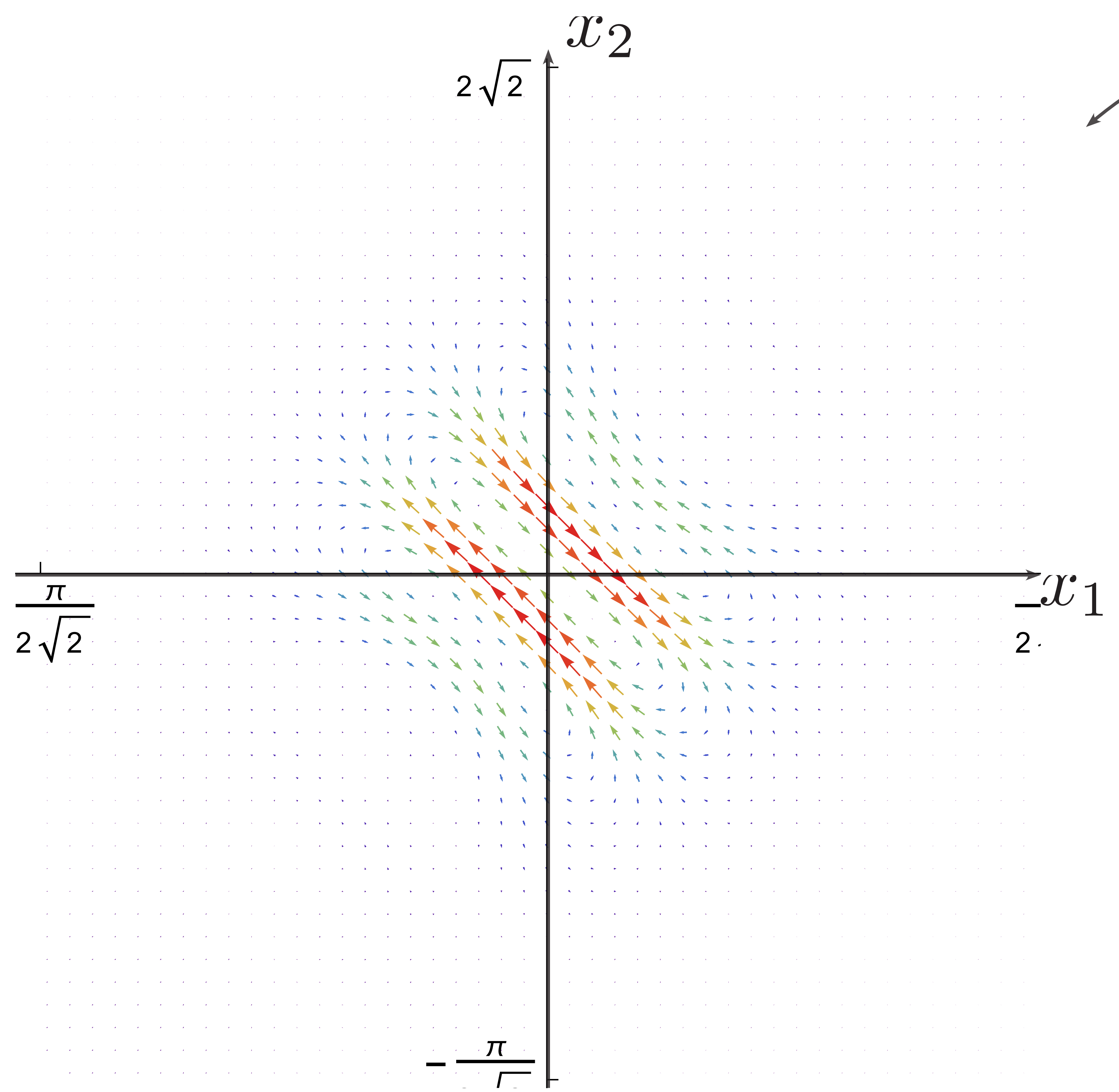


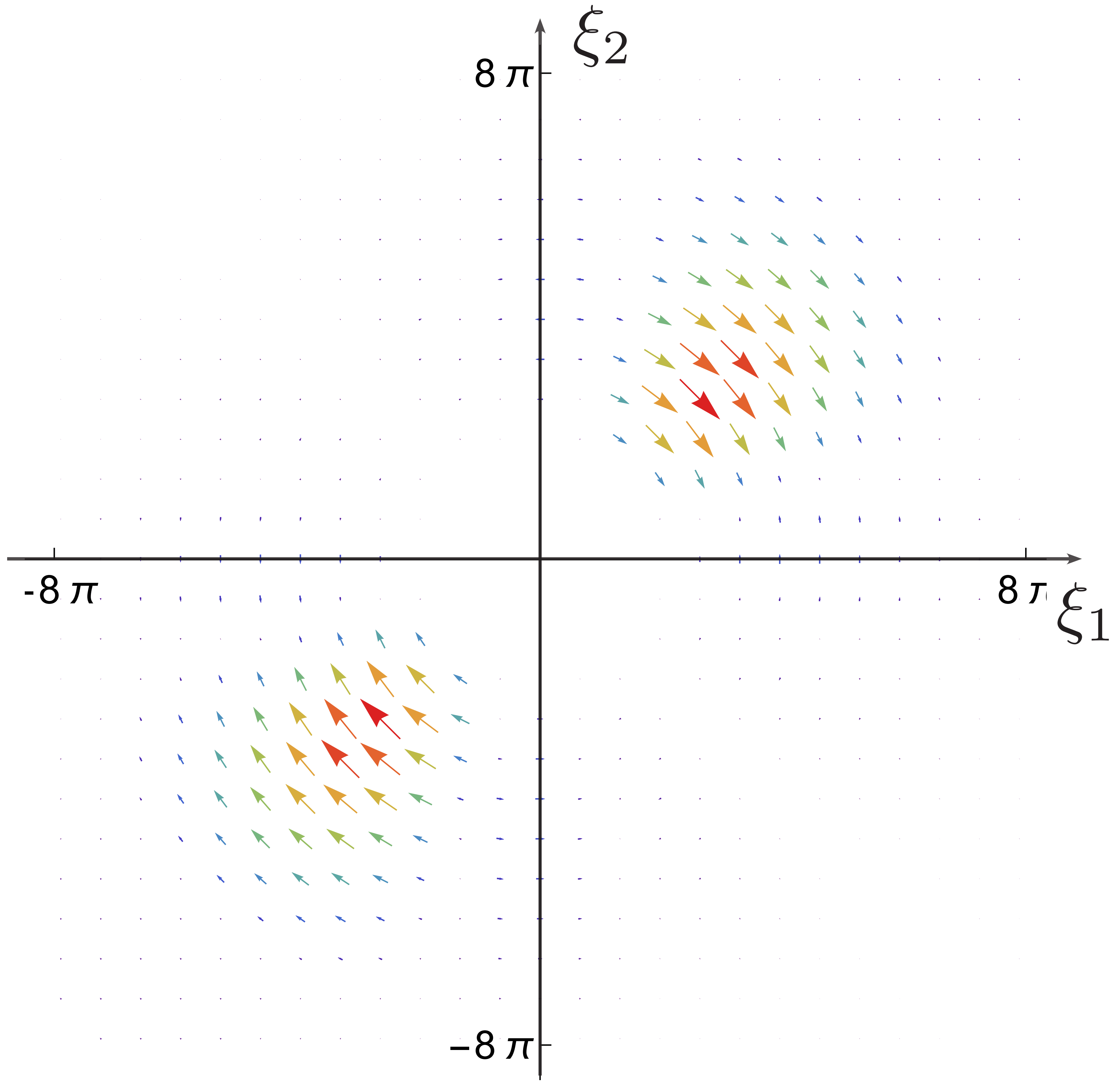
# Divergence free polar wavelets



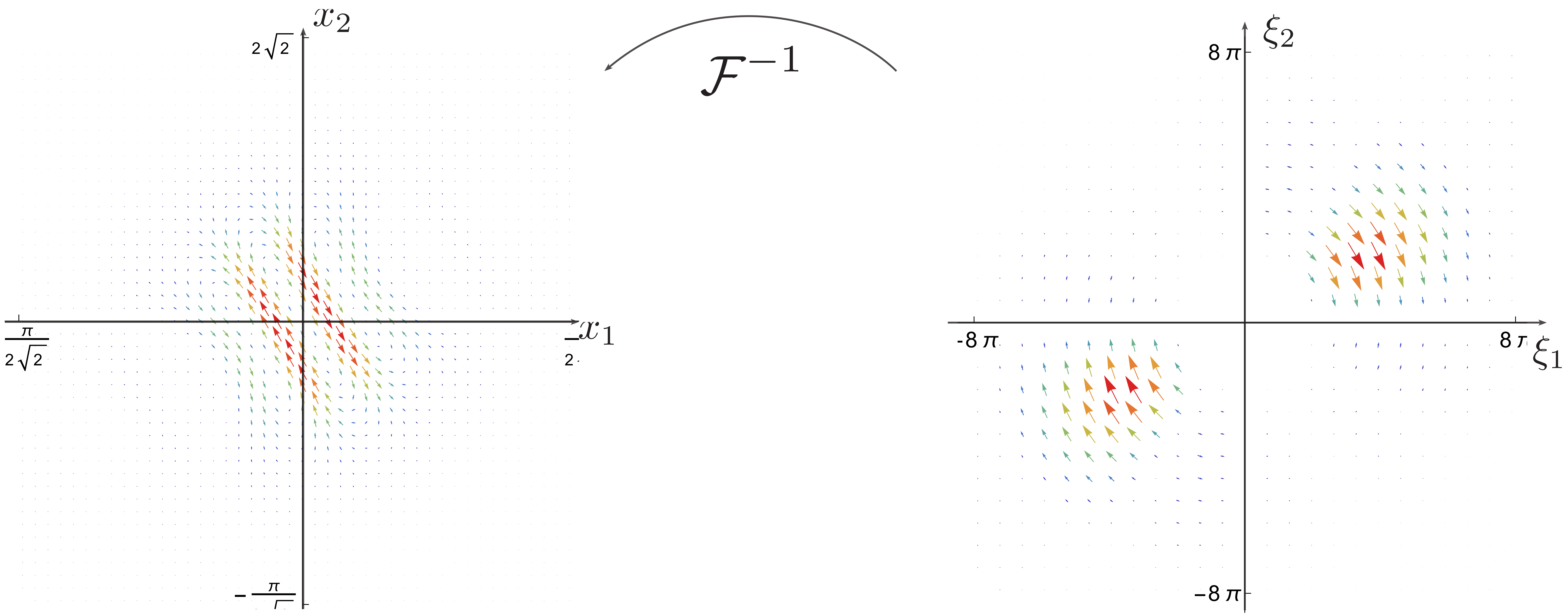
# Divergence free polar wavelets



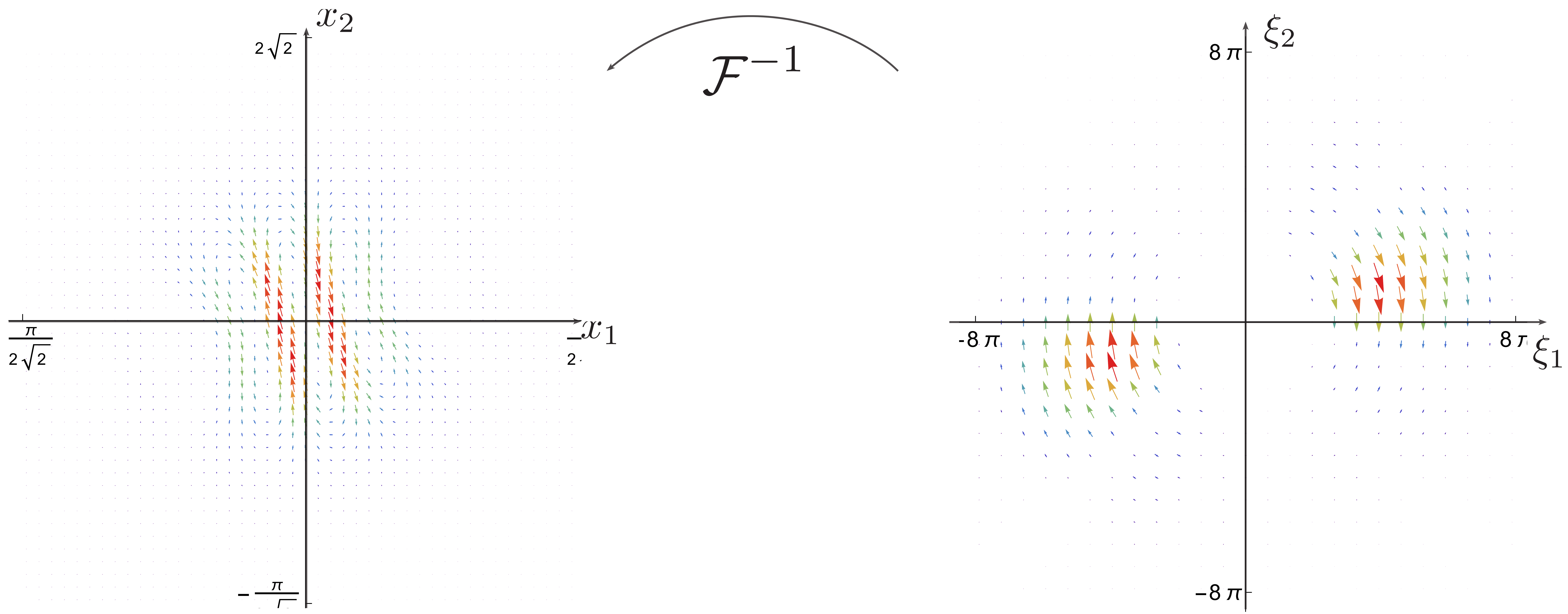
$\mathcal{F}^{-1}$



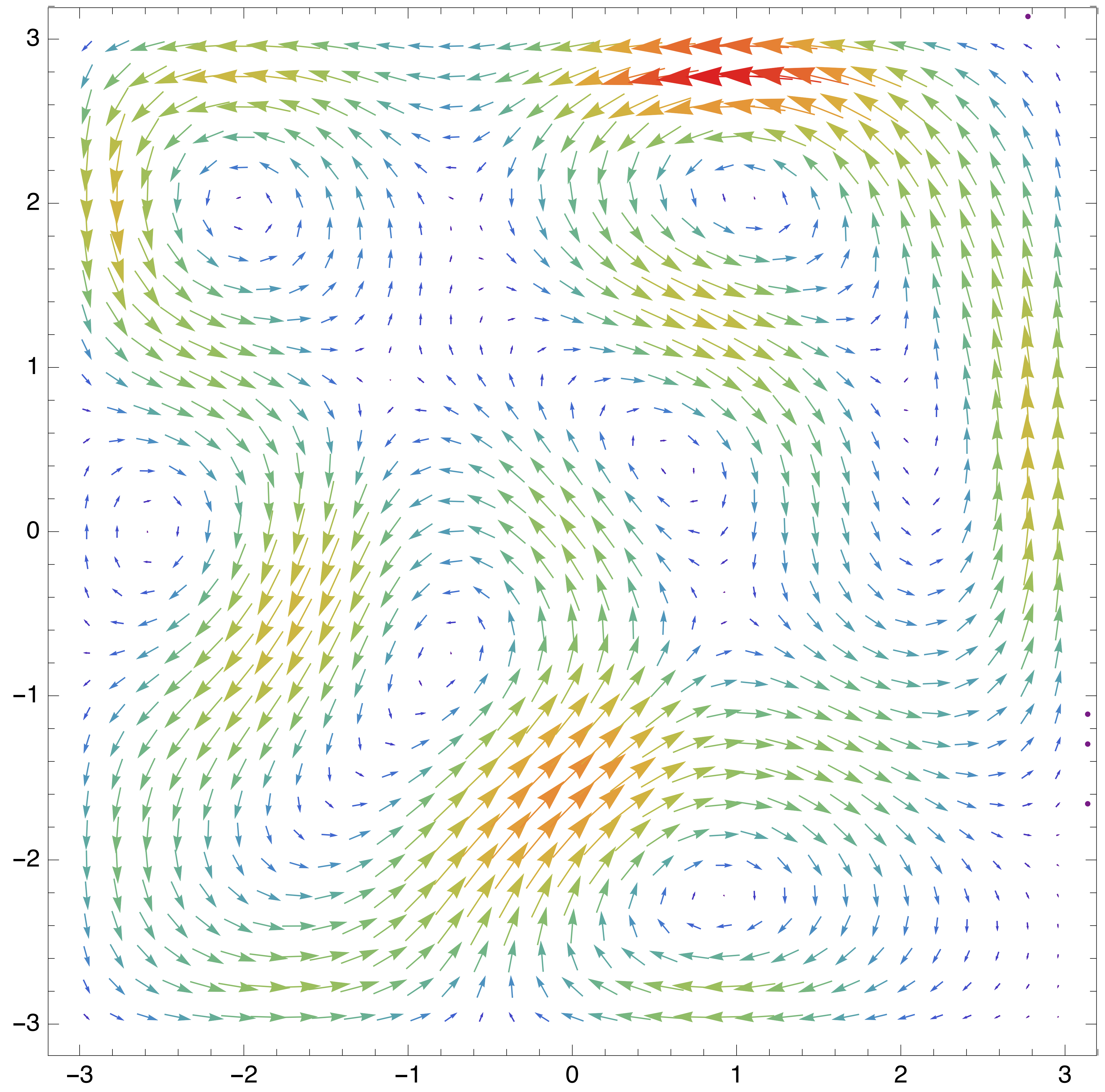
# Divergence free polar wavelets



# Divergence free polar wavelets



# Divergence free polar wavelets



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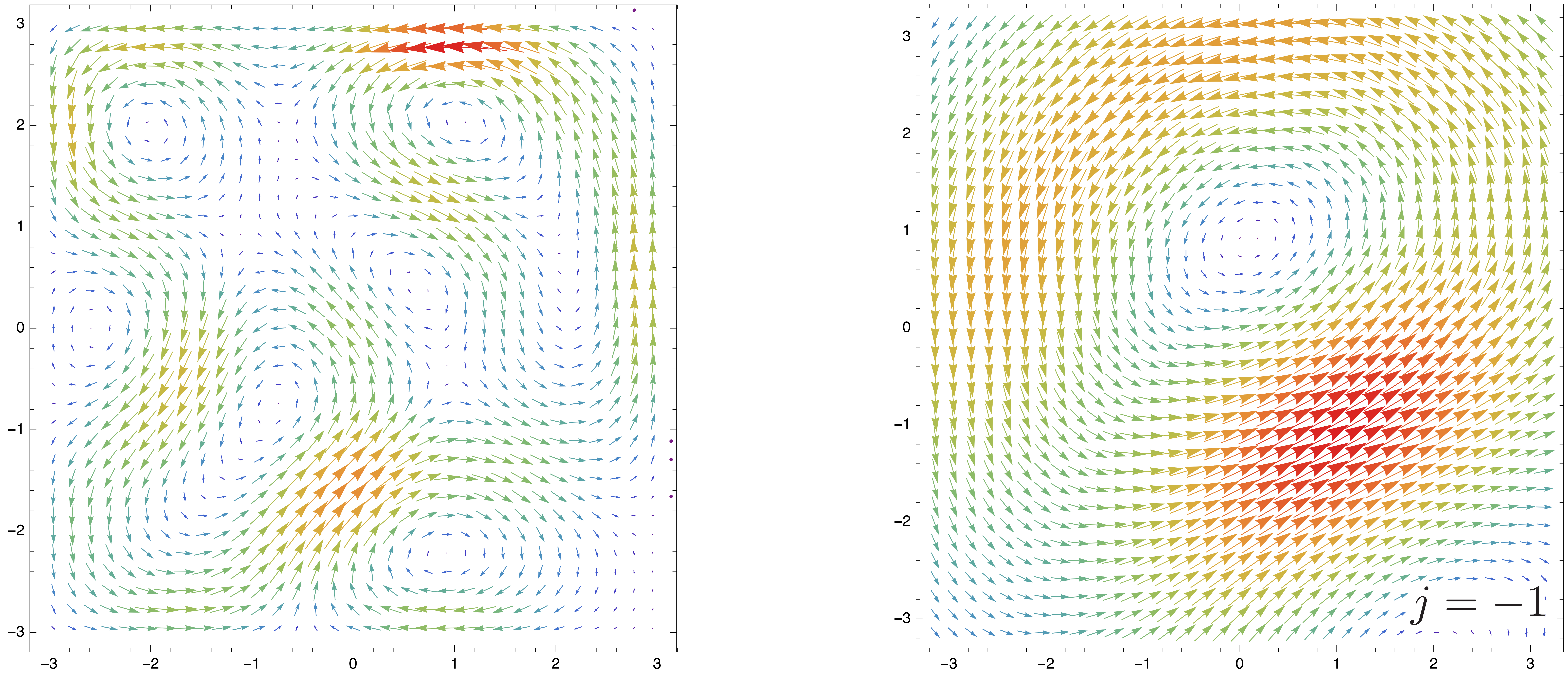
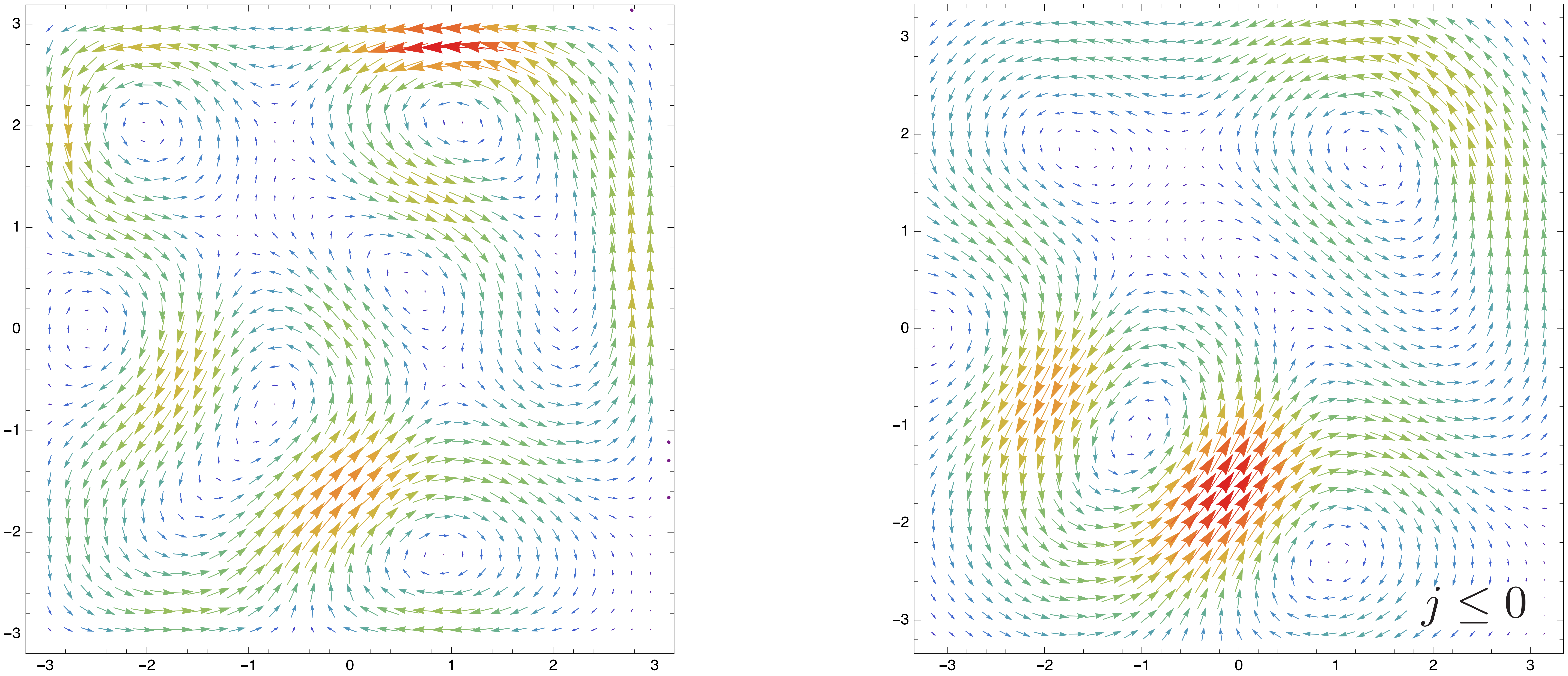


# Divergence free polar wavelets



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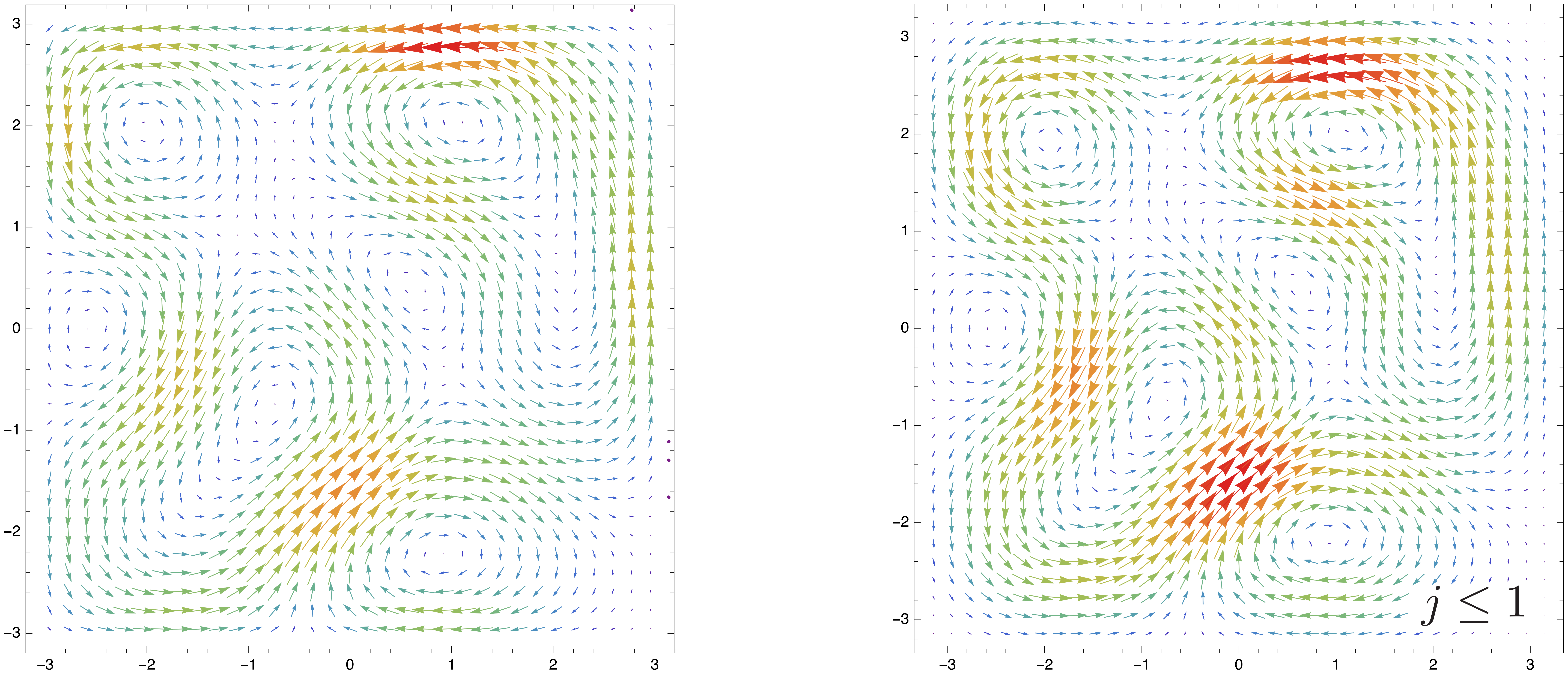

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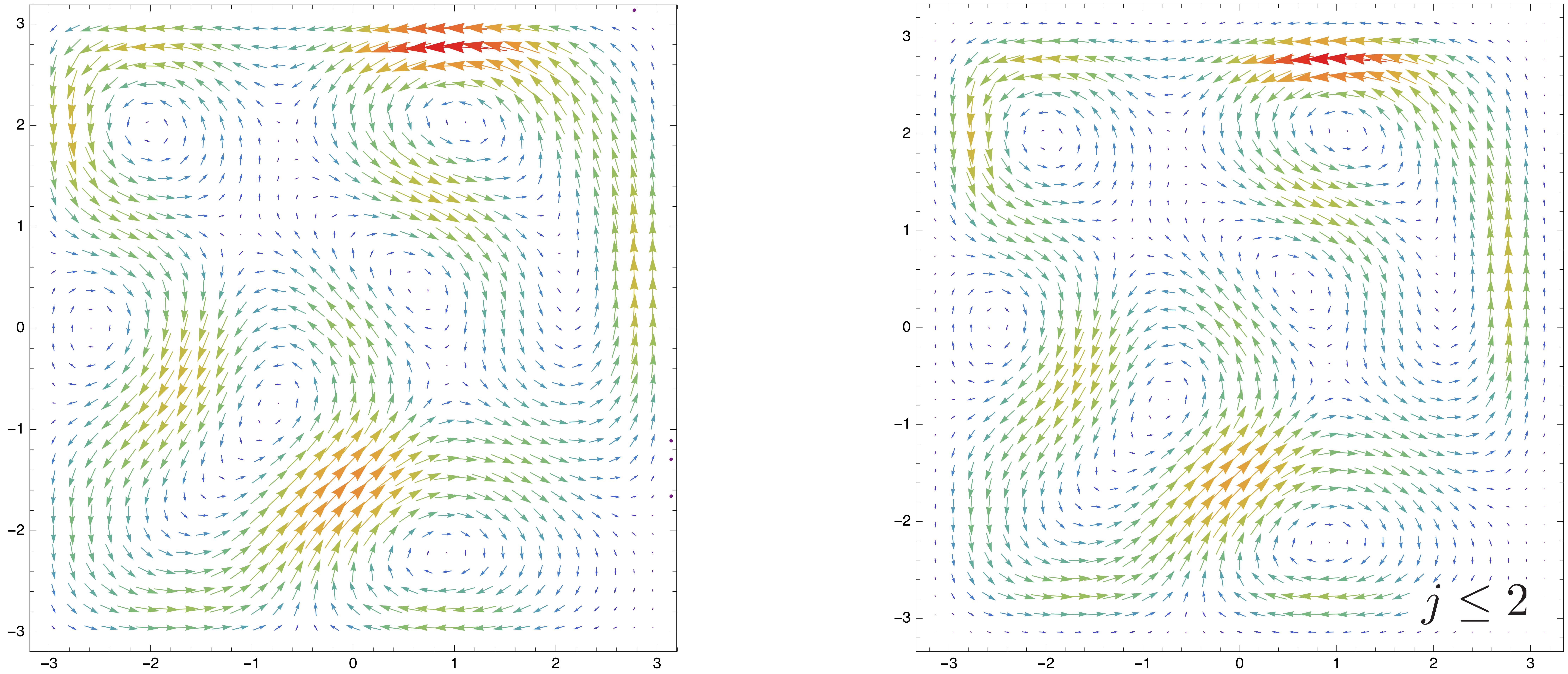

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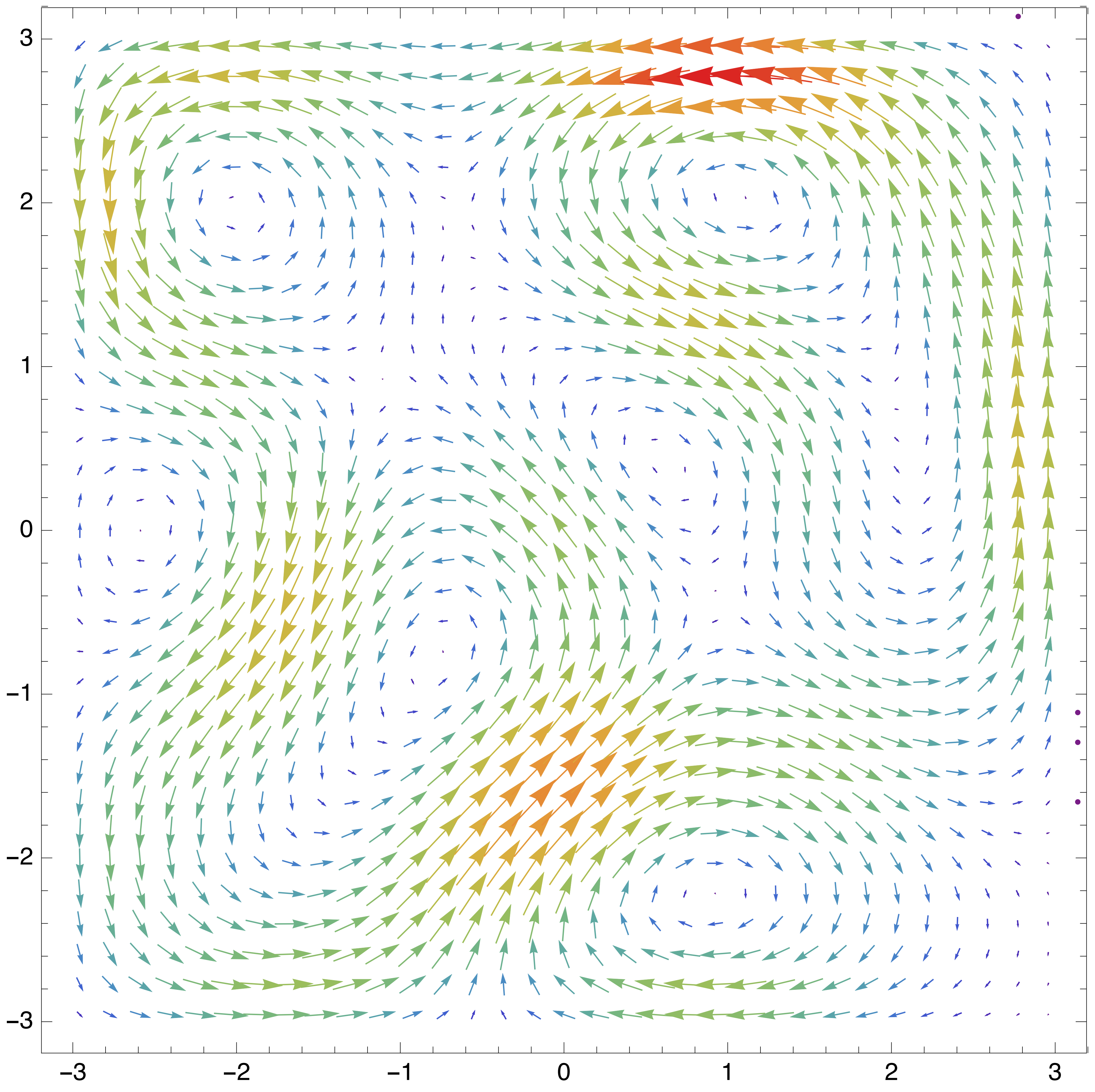
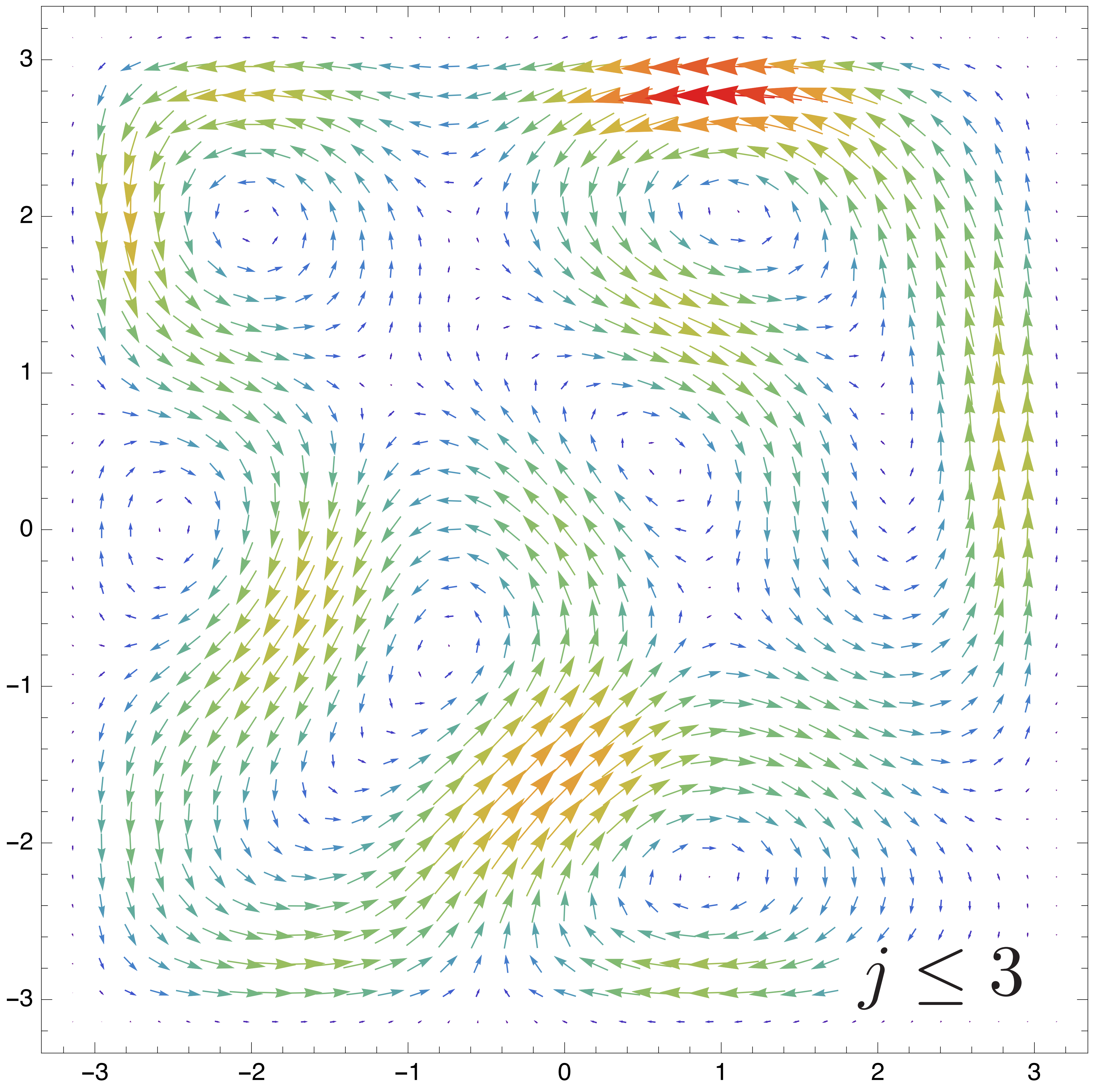
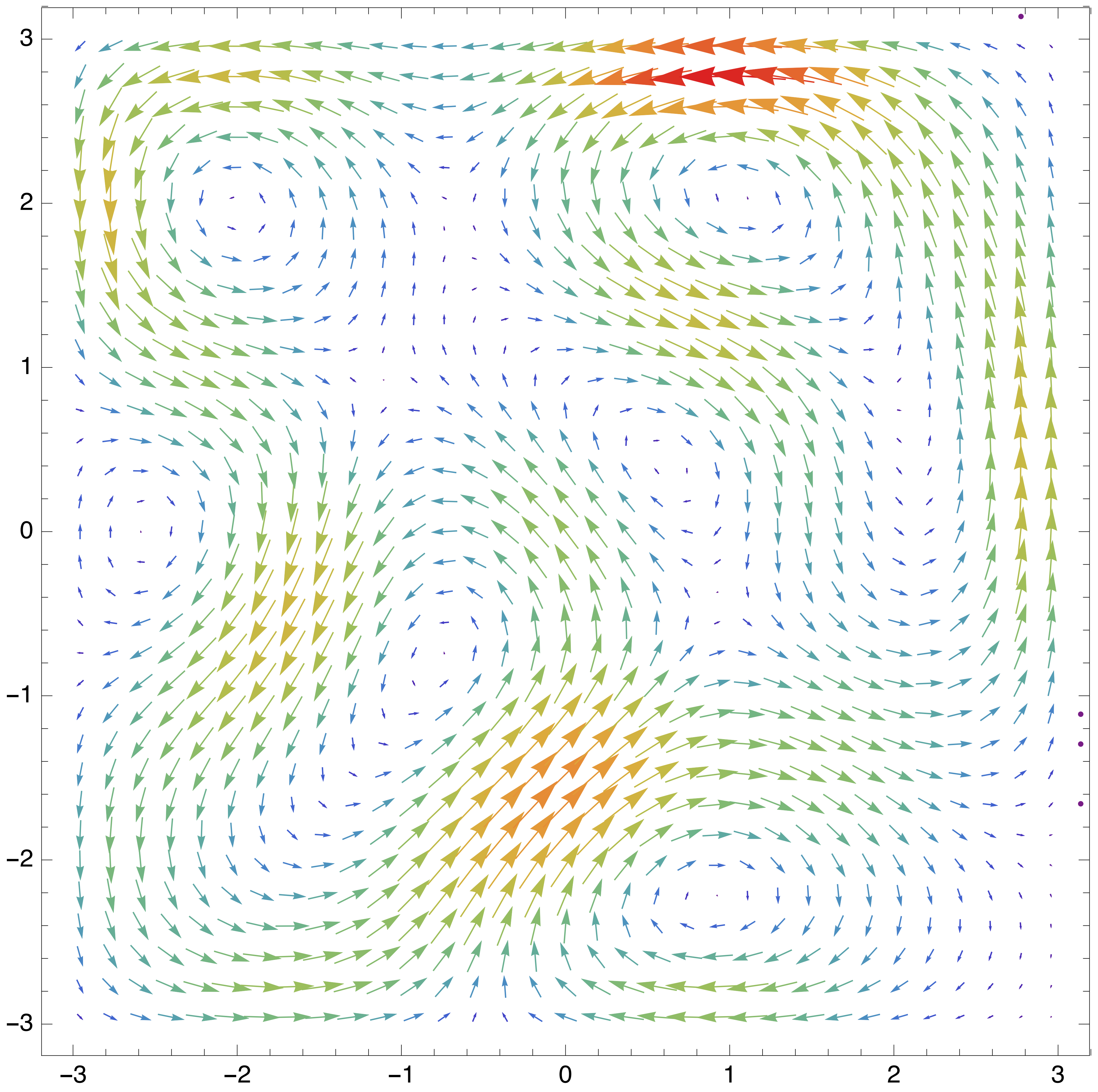
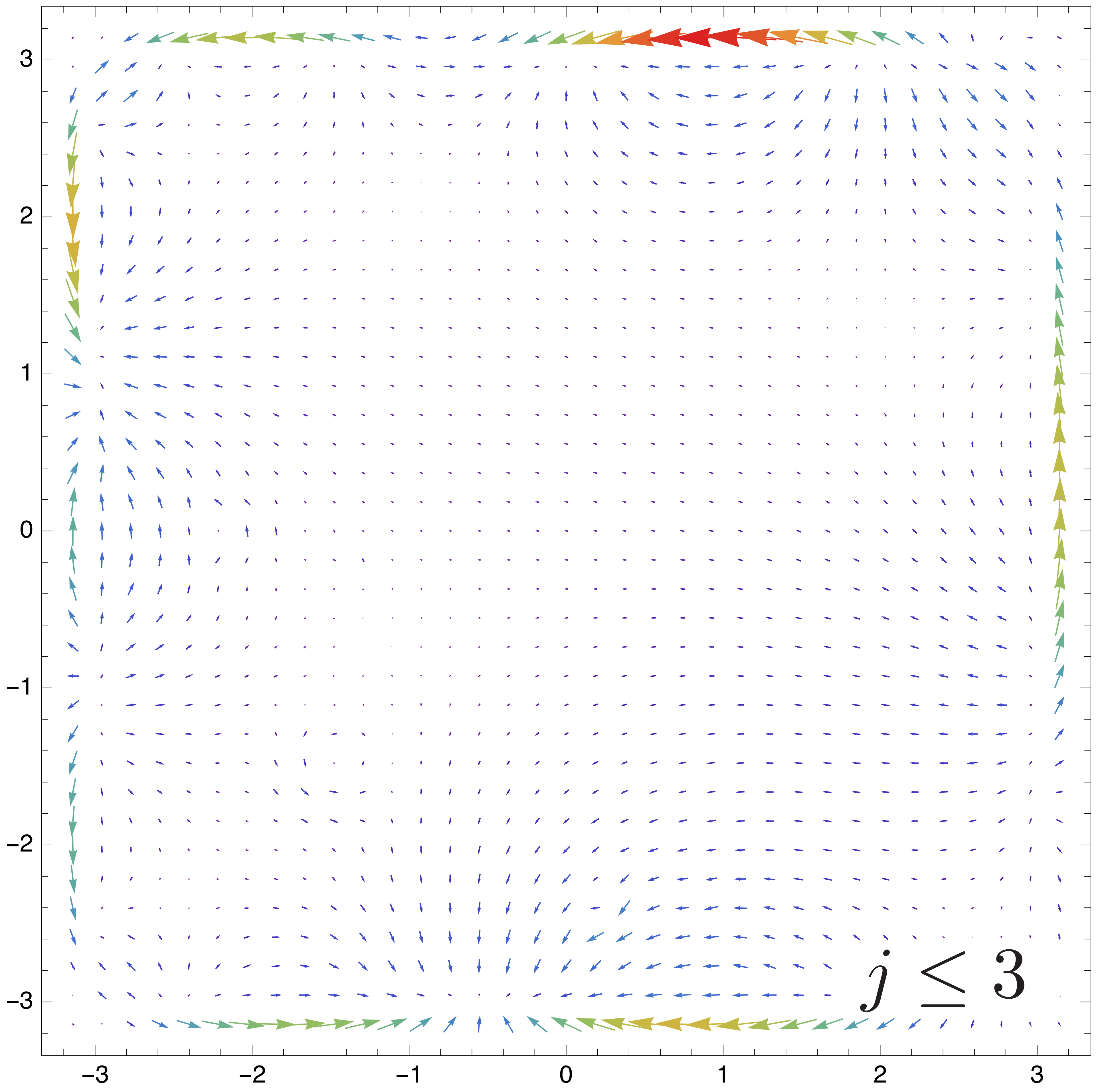




# Divergence free polar wavelets

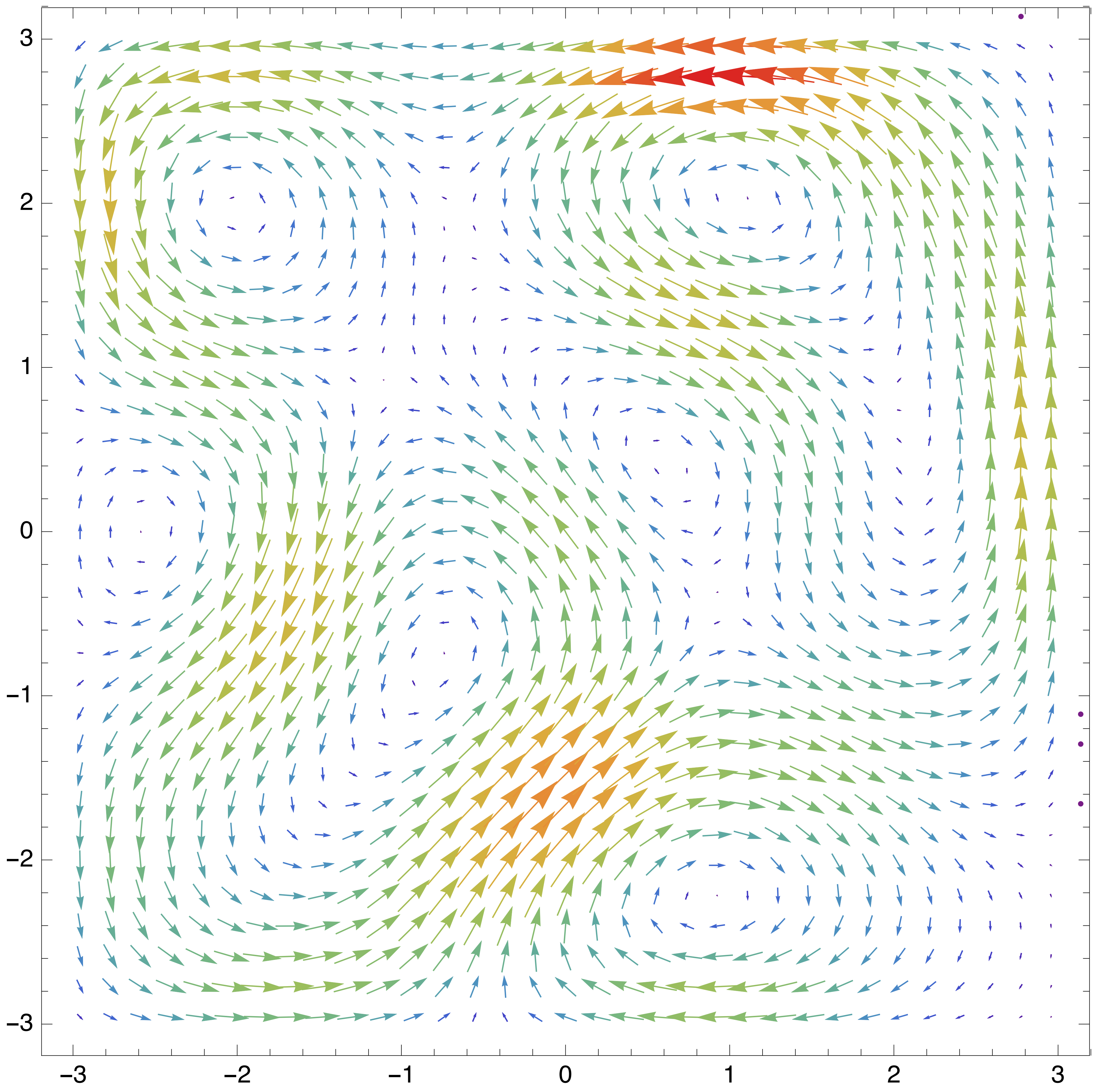


error

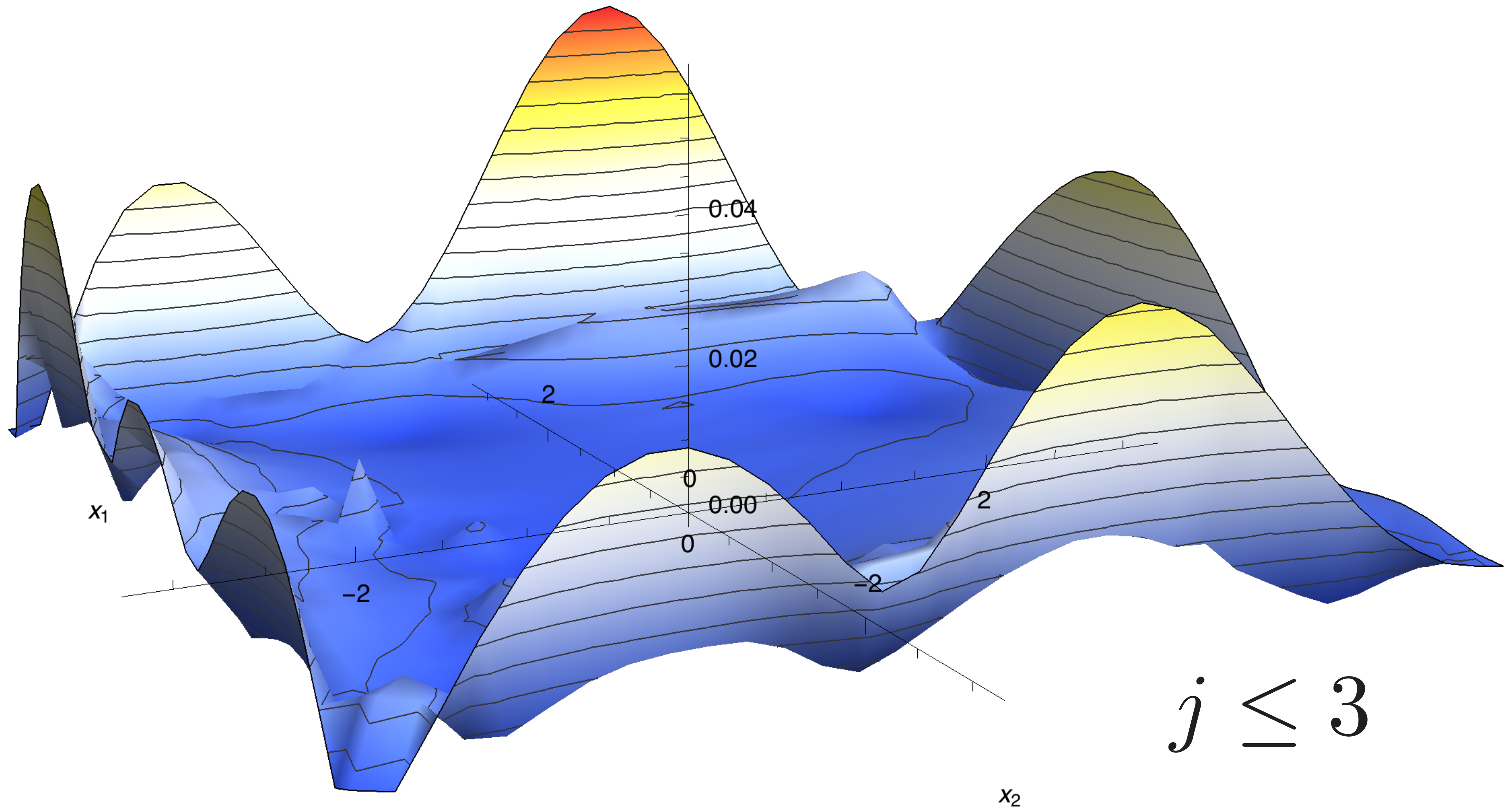


$j \leq 3$

# Divergence free polar wavelets

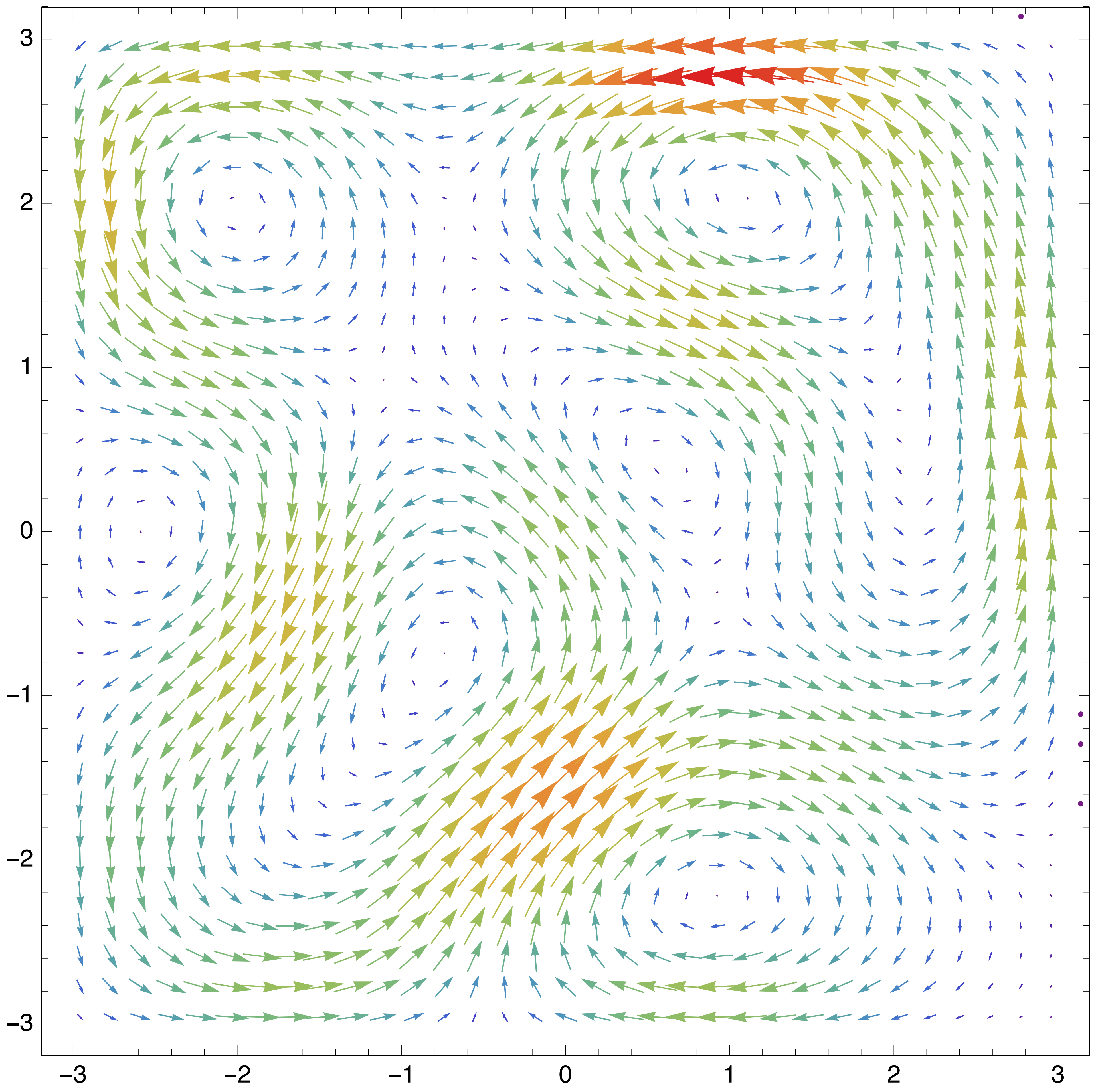


error with only isotropic wavelets

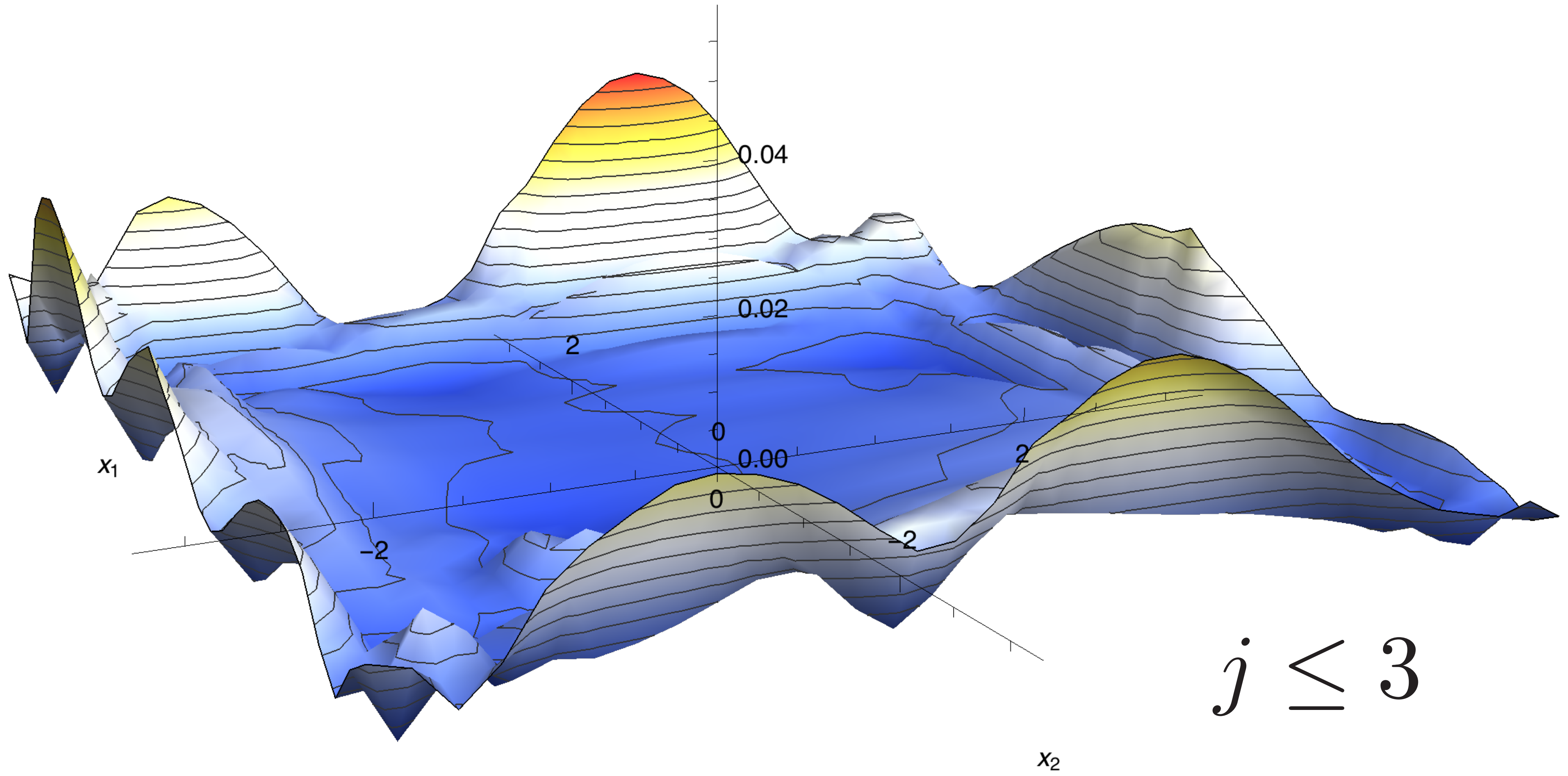


$j \leq 3$

# Divergence free polar wavelets



error with anisotropic wavelets



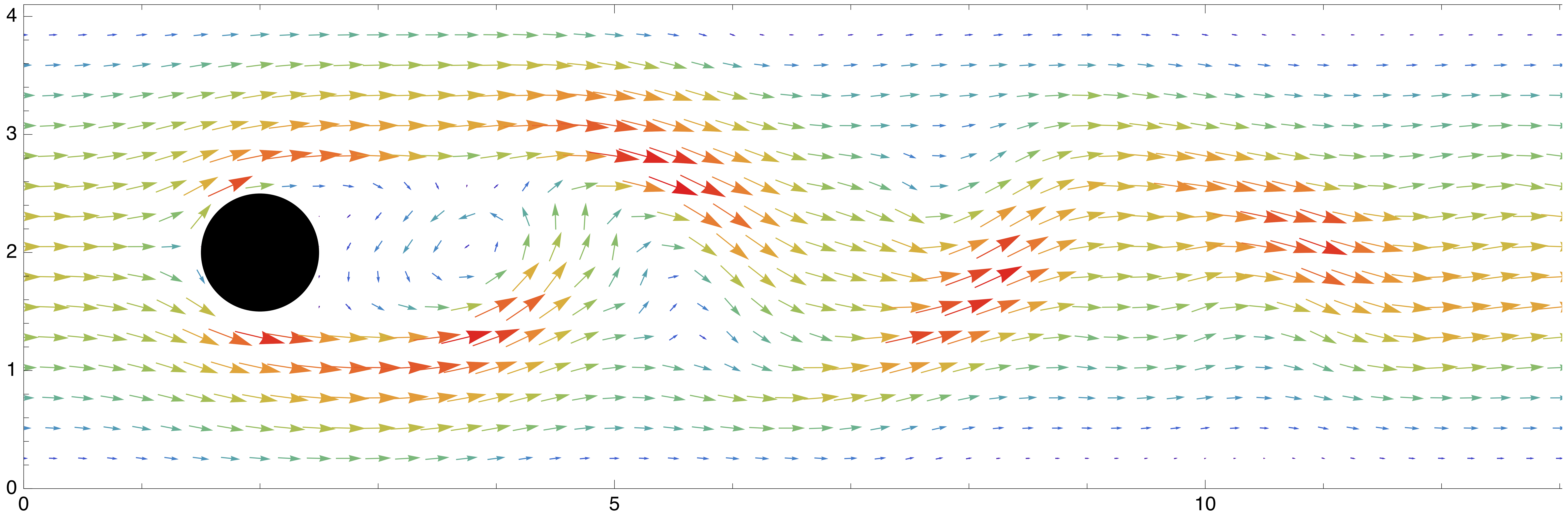
$j \leq 3$

# Divergence free polar wavelets

**Proposition 2.** *Let* $\vec{u}(x) \in L_2^{\mathrm{div}}(\mathbb{R}^{2,2})$ *be a* $C^2$*-smooth divergence free vector field away from* $C^2$ *discontinuities with* $\mathcal{F}^{-1}(\hat{u}) \in \mathcal{E}^2(A)$ [5, Def. 1], *which we write as* $\vec{u} \in \mathcal{E}^2_{\mathrm{div}}(A)$. *When the windows* $\hat{\gamma}_j(\theta_\xi)$ *and* $\hat{h}(|\xi|)$ *satisfy the admissibility conditions of second generation curvelets* [5, Sec. 2], *then the* $n$*-largest coefficient* $|u_s|_n$ *in the coefficient sequence* $(|u_s|)_n$ *satisfies*

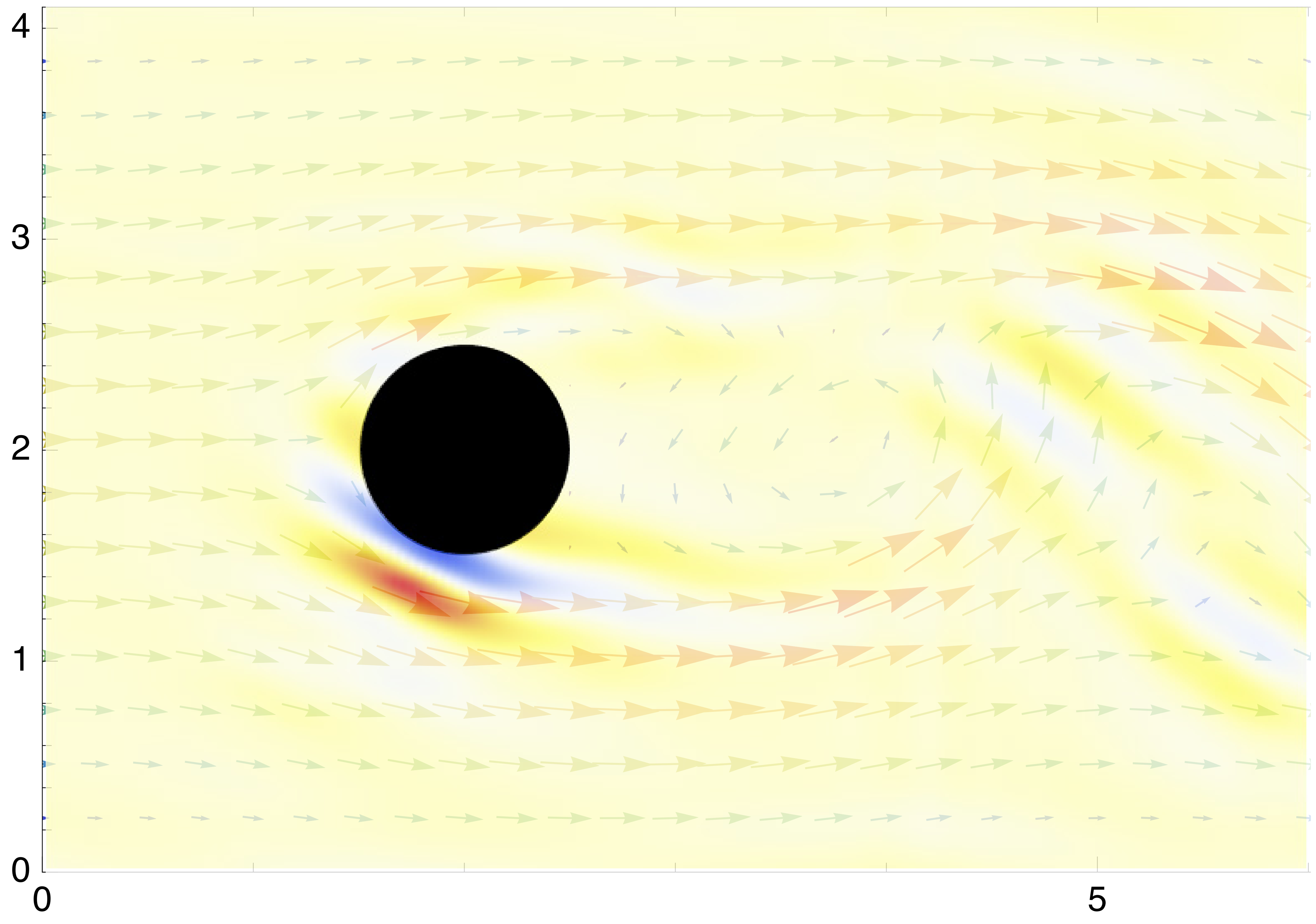
$$\sup_{\vec{u} \in \mathcal{E}^2_{\mathrm{div}}(A)} |u_s|_n \leq C \cdot n^{-3/2} \, (\log n)^{3/2}. \tag{8}$$

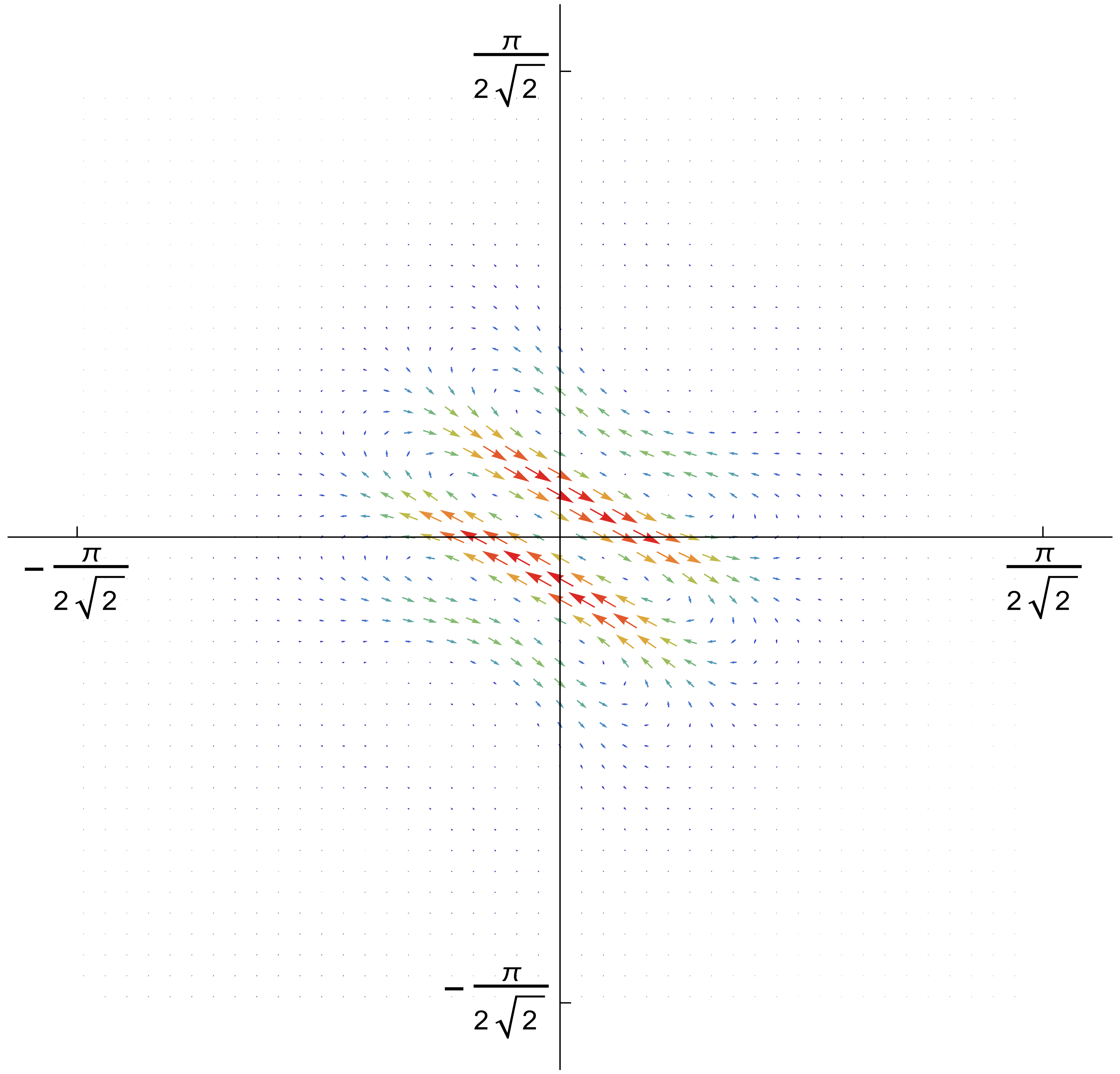
# Divergence free polar wavelets



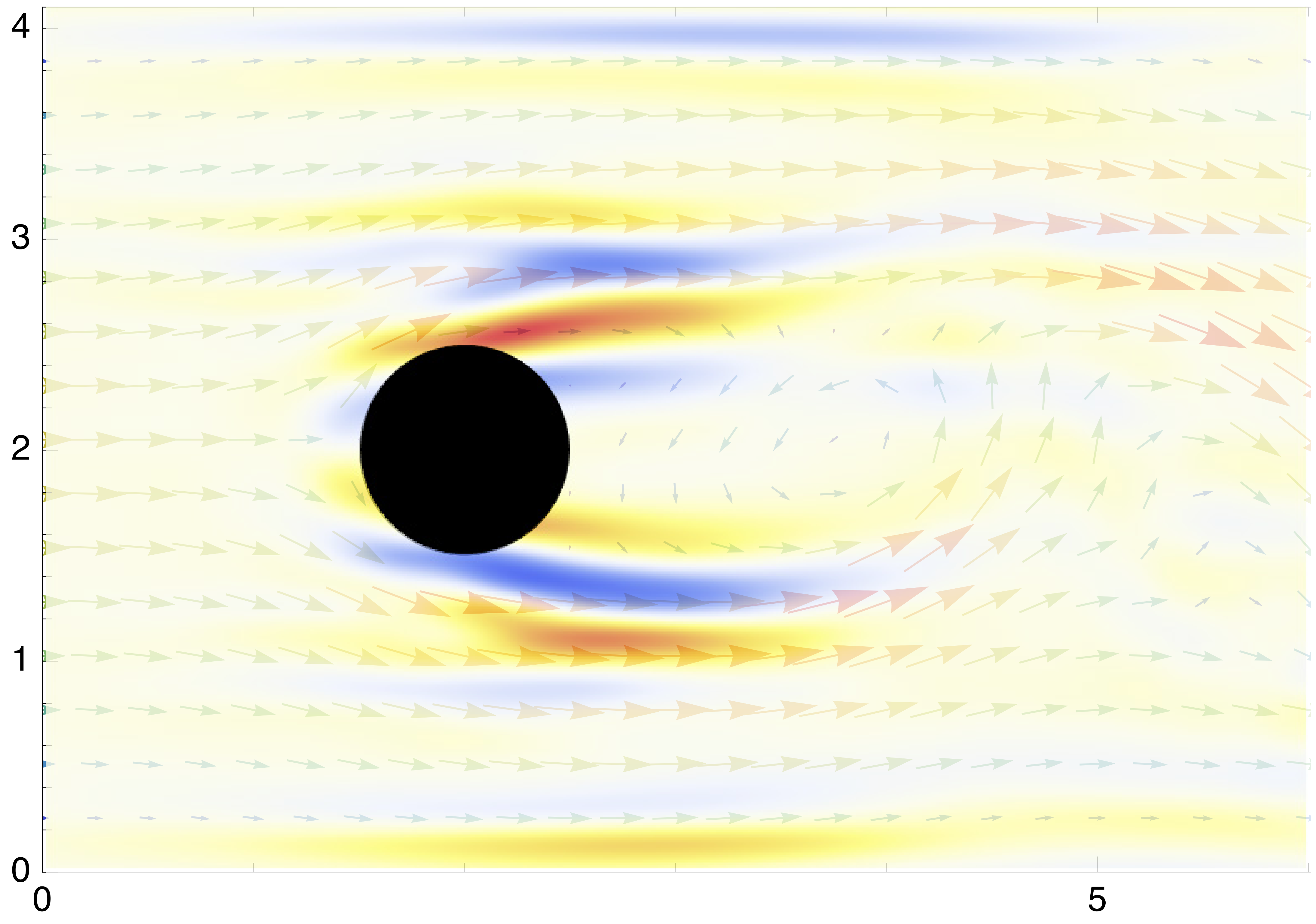
Courtesy Thomas Richter

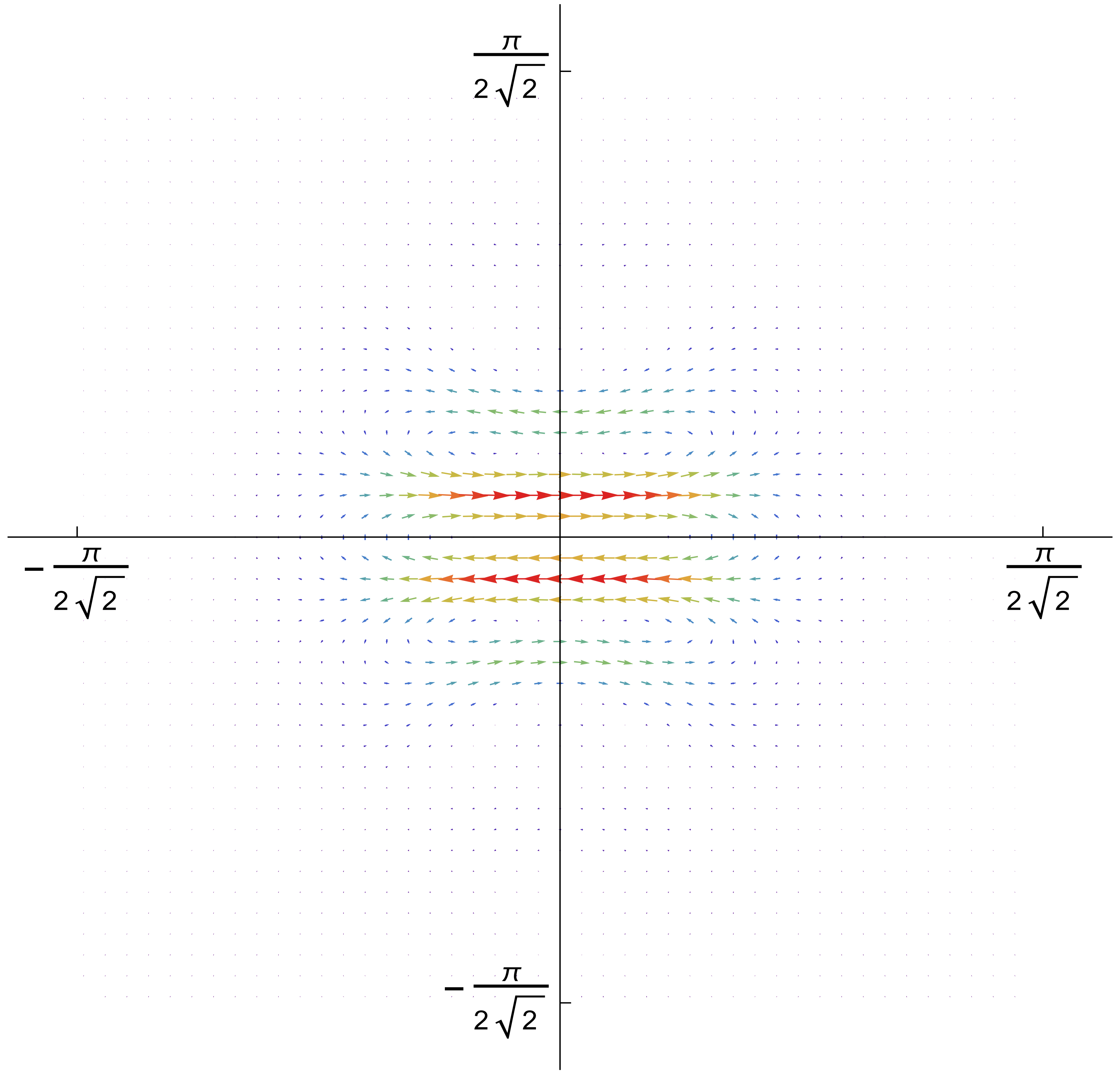
# Divergence free polar wavelets



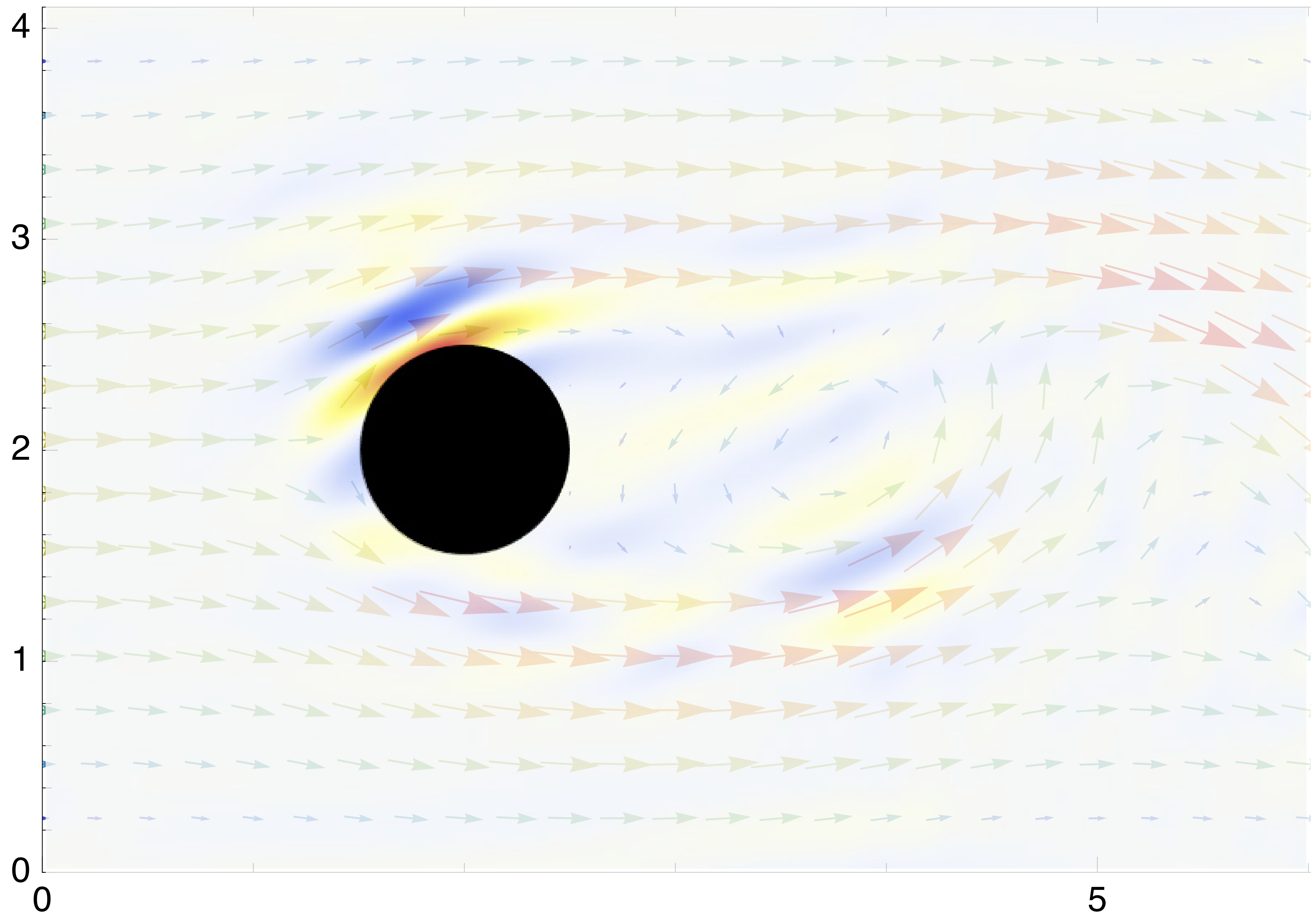


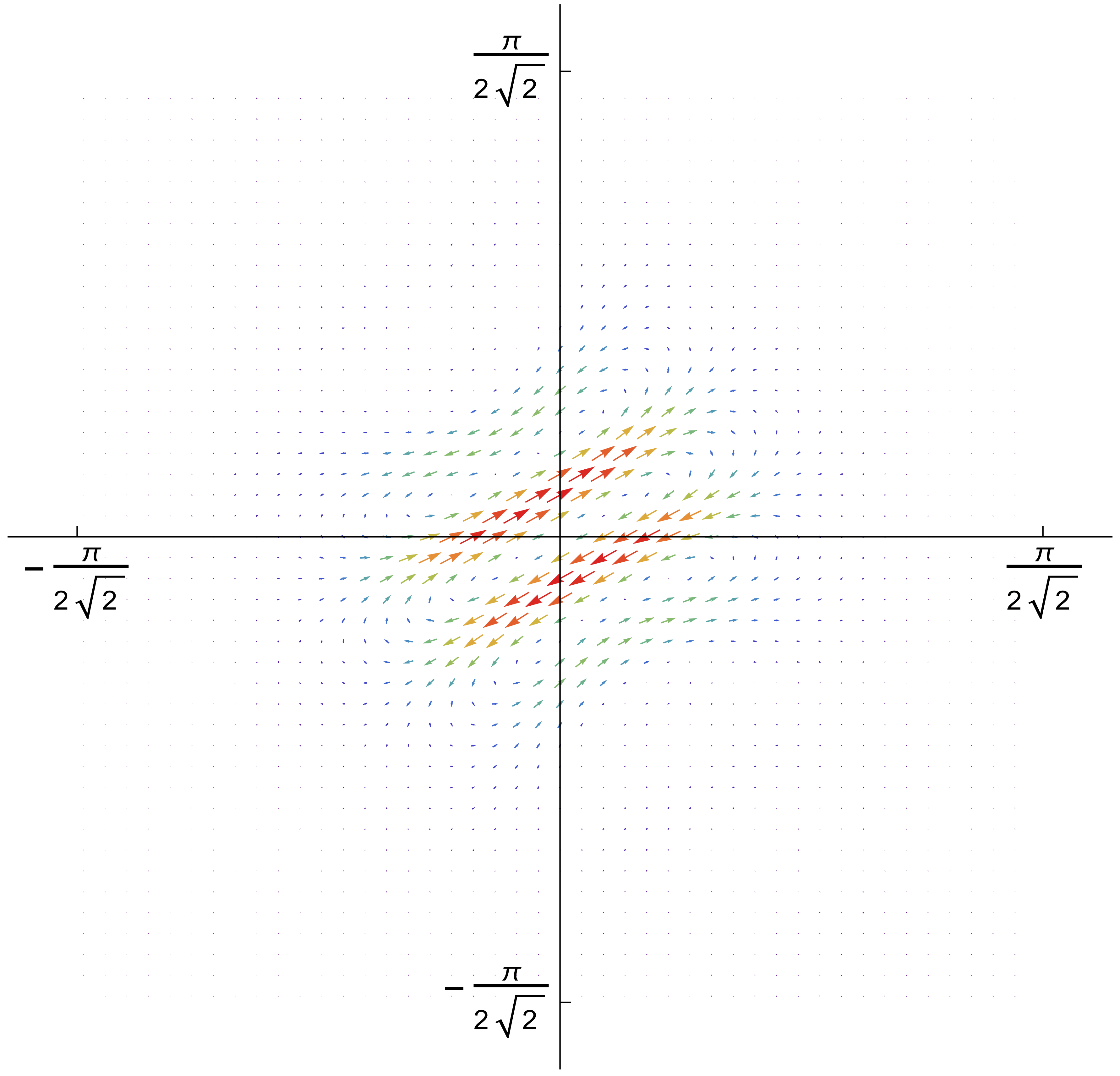
# Divergence free polar wavelets





# Divergence free polar wavelets





# Divergence free polar wavelets in $\mathbb{R}^3$

# Divergence free polar wavelets in $\mathbb{R}^3$

- Divergence freedom:

$$\mathrm{div}(\vec{u}) = 0 \iff \hat{\vec{u}} \in TS^{n-1}$$

# “Polar” wavelets in $\mathbb{R}^3$

$$\hat{\psi}(\xi) = \hat{h}(|\xi|)\,\hat{\gamma}(\bar{\xi})$$

# “Polar” wavelets in $\mathbb{R}^3$

$$\begin{aligned}\hat{\psi}(\xi) &= \hat{h}(|\xi|)\,\hat{\gamma}(\bar{\xi}) \\ &= \hat{h}(|\xi|)\left(\sum_{l,m} \beta_{lm}\, y_{lm}(\bar{\xi})\right)\end{aligned}$$

# "Polar" wavelets in $\mathbb{R}^3$

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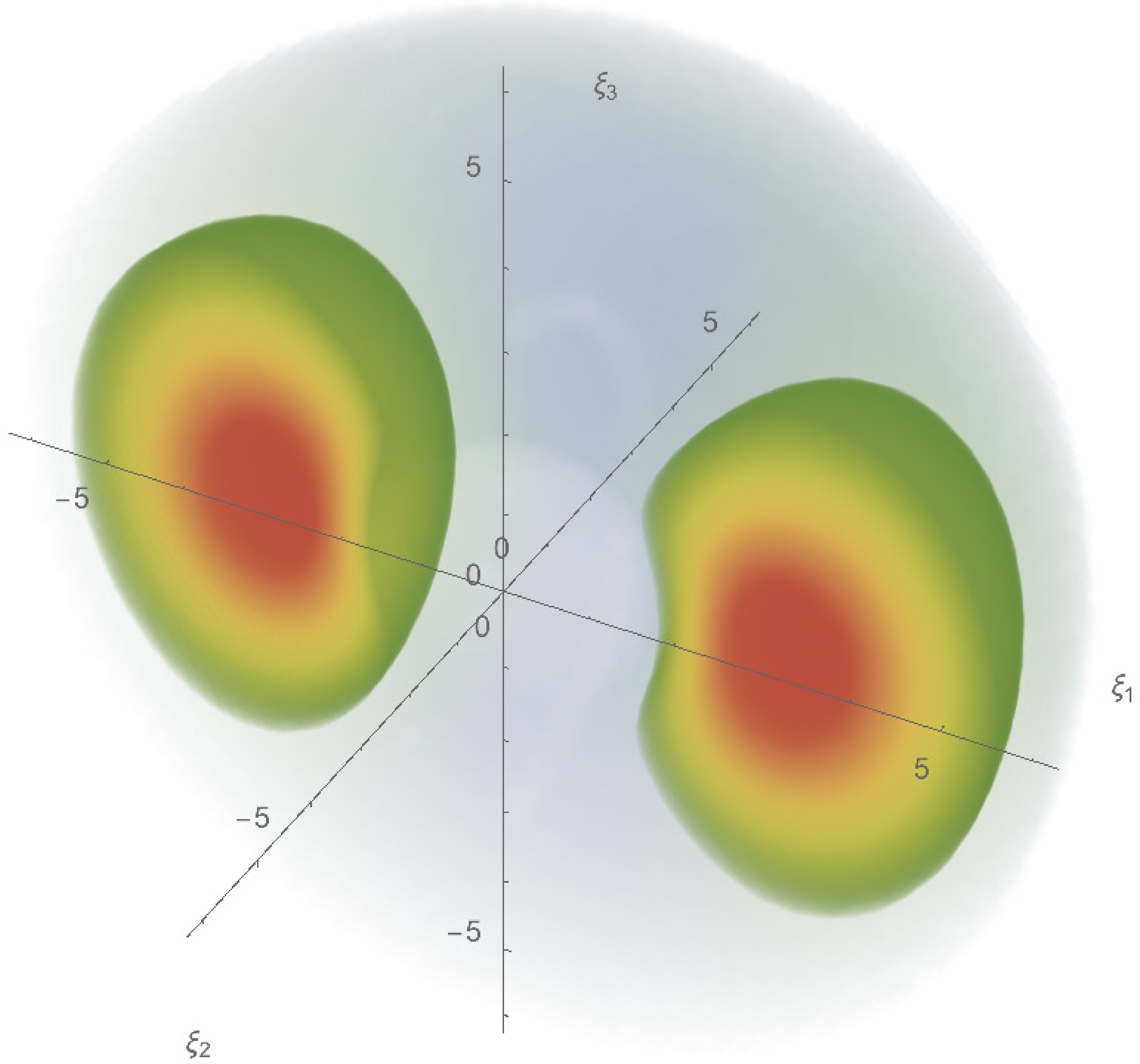

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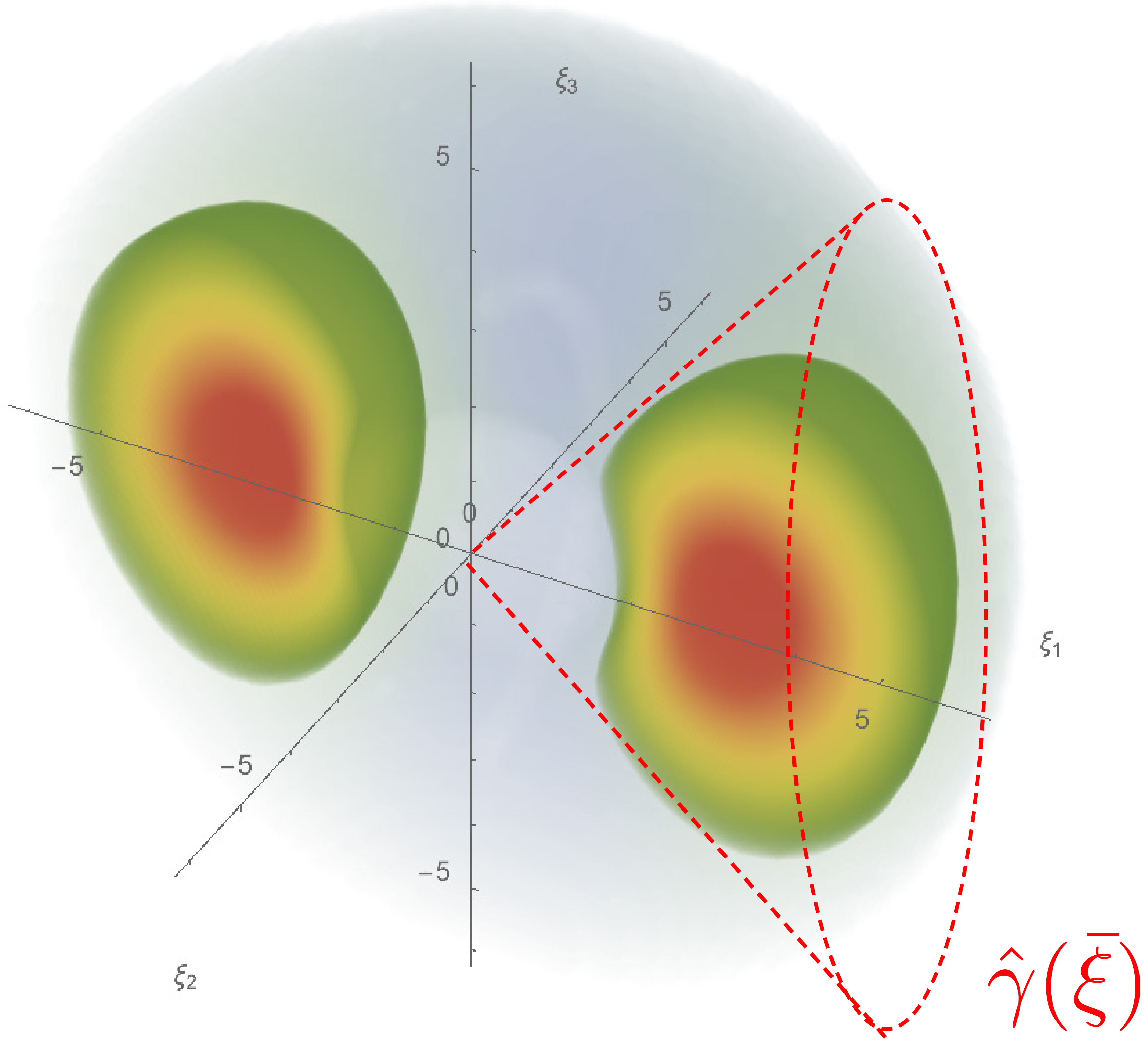

# “Polar” wavelets in $\mathbb{R}^3$

$$\psi_{j,t}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3_\xi} \hat{\psi}_j(\xi)\, e^{i\langle \xi , x \rangle}\, d\xi$$

# “Polar” wavelets in $\mathbb{R}^3$

$$
\begin{aligned}
\psi_{j,t}(x) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3_\xi} \hat{\psi}_j(\xi)\, e^{i\langle \xi , x \rangle}\, d\xi \\
&= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3_\xi} \hat{h}(|\xi|)\, \hat{\gamma}(\bar{\xi})\, e^{i\langle \xi , x \rangle}\, d\xi
\end{aligned}
$$

# “Polar” wavelets in $\mathbb{R}^3$

$$\begin{aligned}
\psi_{j,t}(x) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3_\xi} \hat{\psi}_j(\xi)\, e^{i\langle \xi , x \rangle}\, d\xi \\
&= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3_\xi} \hat{h}(|\xi|)\, \hat{\gamma}(\bar{\xi})\, e^{i\langle \xi , x \rangle}\, d\xi \\
&= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2_\xi} \hat{h}_j(|\xi|)\, \hat{\gamma}(\bar{\xi}) \left( 4\pi \sum_{l,m} i^l\, y_{lm}(\bar{\xi})\, y_{lm}(\bar{x})\, j_l(|\xi|\,|x|) \right) d\xi
\end{aligned}$$

# "Polar" wavelets in $\mathbb{R}^3$

$$\psi_{j,t}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3_\xi} \hat{\psi}_j(\xi)\, e^{i\langle \xi , x \rangle}\, d\xi$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3_\xi} \hat{h}(|\xi|)\, \hat{\gamma}(\bar{\xi})\, e^{i\langle \xi , x \rangle}\, d\xi$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2_\xi} \hat{h}_j(|\xi|)\, \hat{\gamma}(\bar{\xi}) \underbrace{\left( 4\pi \sum_{l,m} i^l\, y_{lm}(\bar{\xi})\, y_{lm}(\bar{x})\, j_l(|\xi|\,|x|) \right)}_{\text{Rayleigh formula}} d\xi$$

# "Polar" wavelets in $\mathbb{R}^3$

$$\psi_{j,t}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3_\xi} \hat{\psi}_j(\xi)\, e^{i\langle \xi , x \rangle}\, d\xi$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3_\xi} \hat{h}(|\xi|)\, \hat{\gamma}(\bar{\xi})\, e^{i\langle \xi , x \rangle}\, d\xi$$

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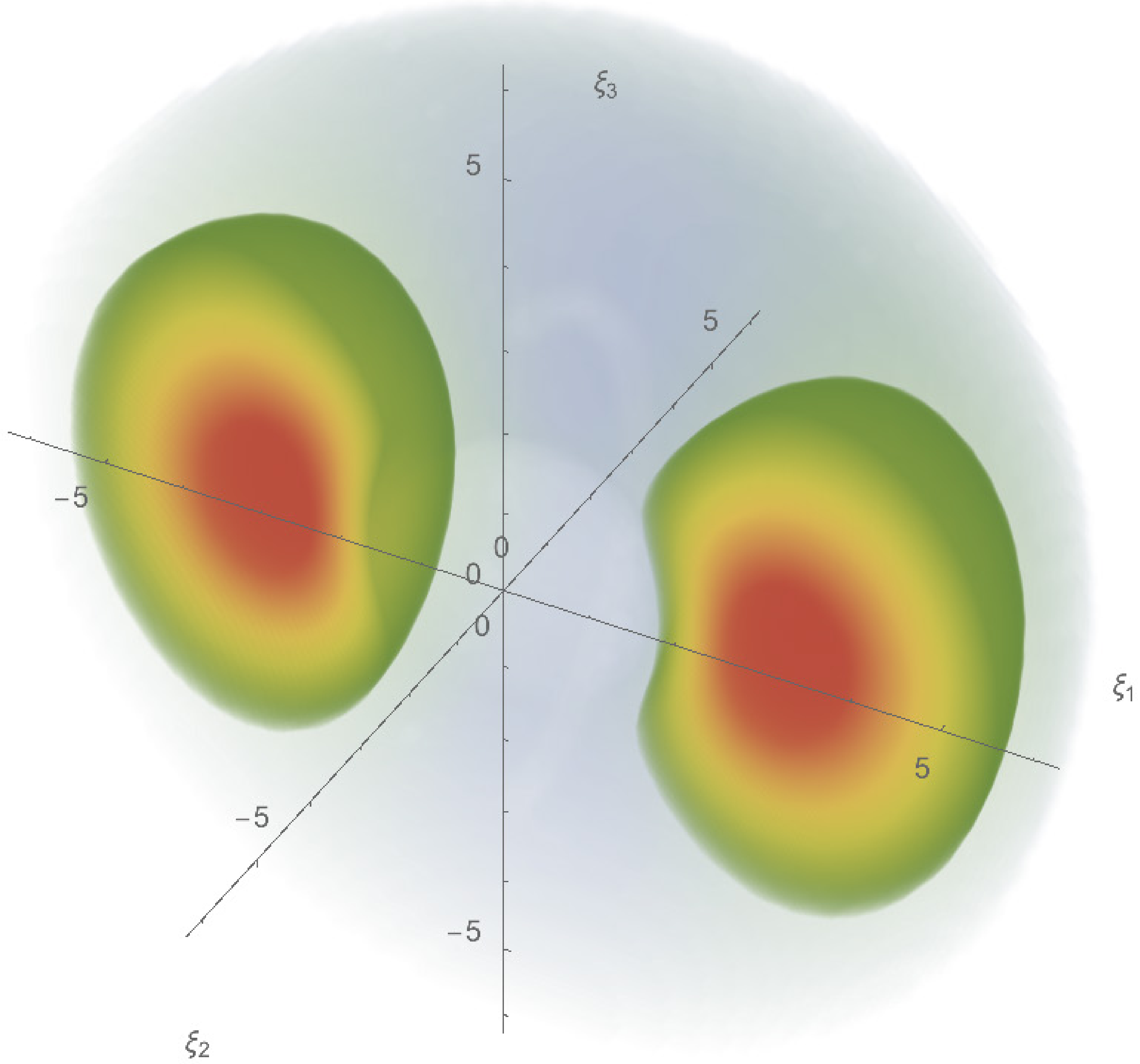
$$\hat{\gamma}(\bar{\xi}) = \sum_{lm} \beta_{lm}^{jt}\, y_{lm}(\bar{\xi})$$

# "Polar" wavelets in $\mathbb{R}^3$

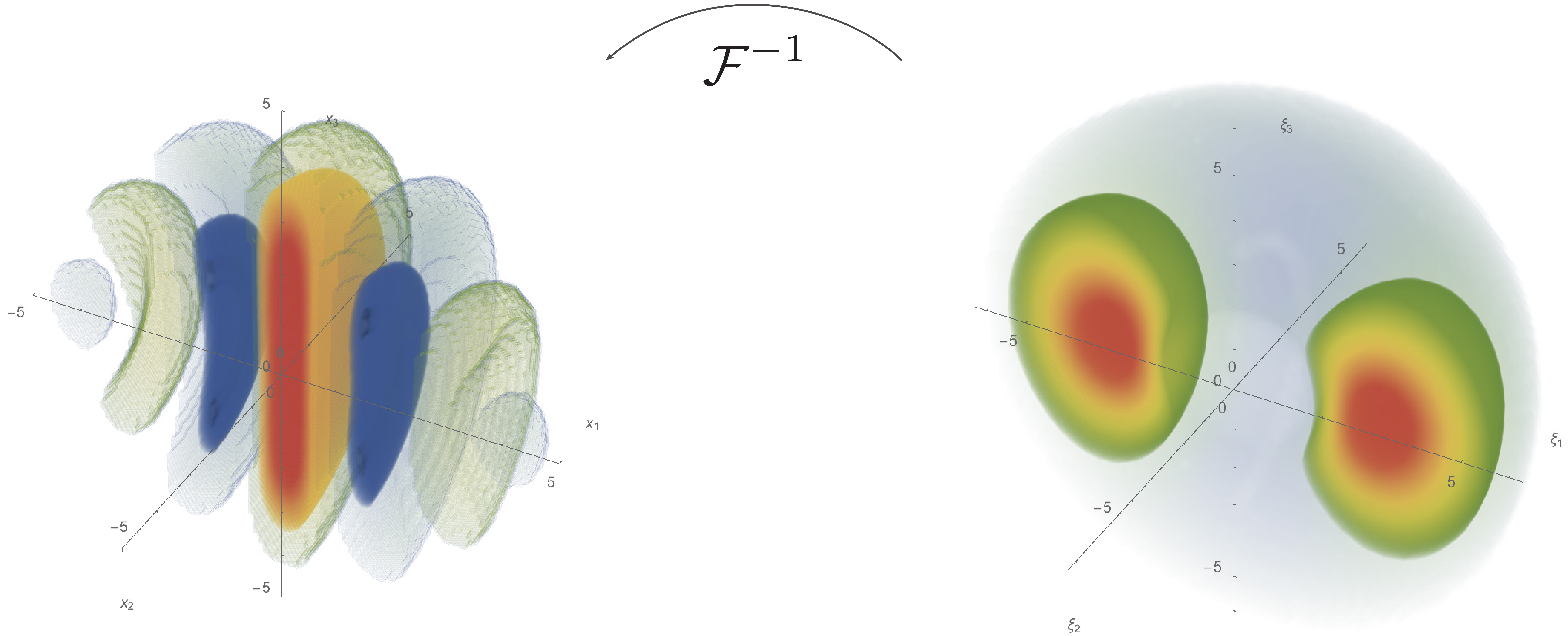
$$
\begin{aligned}
\hat{\psi}(\xi) &= \hat{h}(|\xi|)\,\hat{\gamma}(\bar{\xi}) \\
&= \hat{h}(|\xi|)\left(\sum_{l,m}\beta_{lm}\,y_{lm}(\bar{\xi})\right)
\end{aligned}
$$

$$\Big\downarrow\ \mathcal{F}^{-1}$$

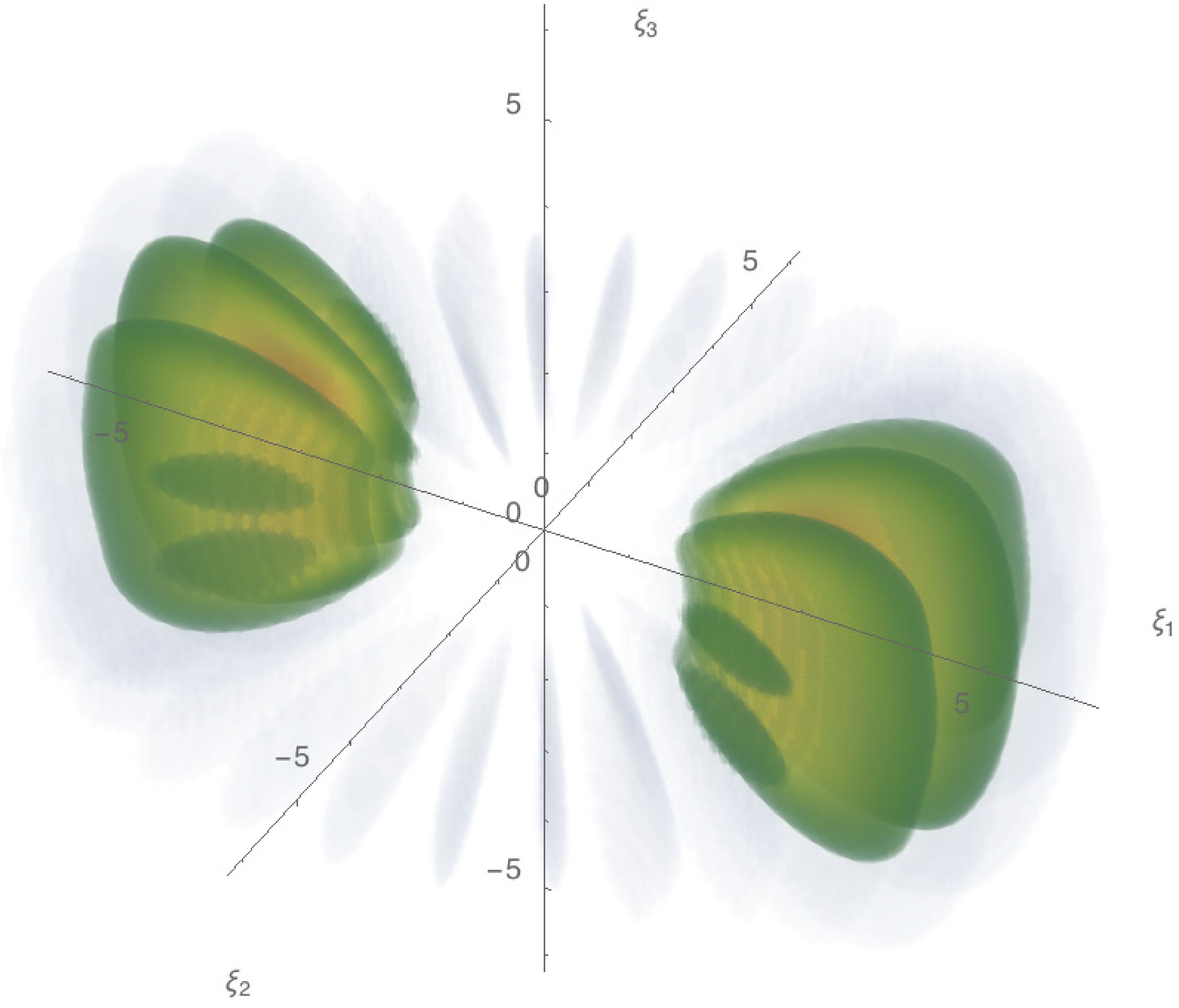
$$\psi_{j,t}(x) = \sum_{l,m} i^l\,\beta_{lm}^{j,t}\,y_{lm}(\bar{x})\,h_l(|\xi|)$$



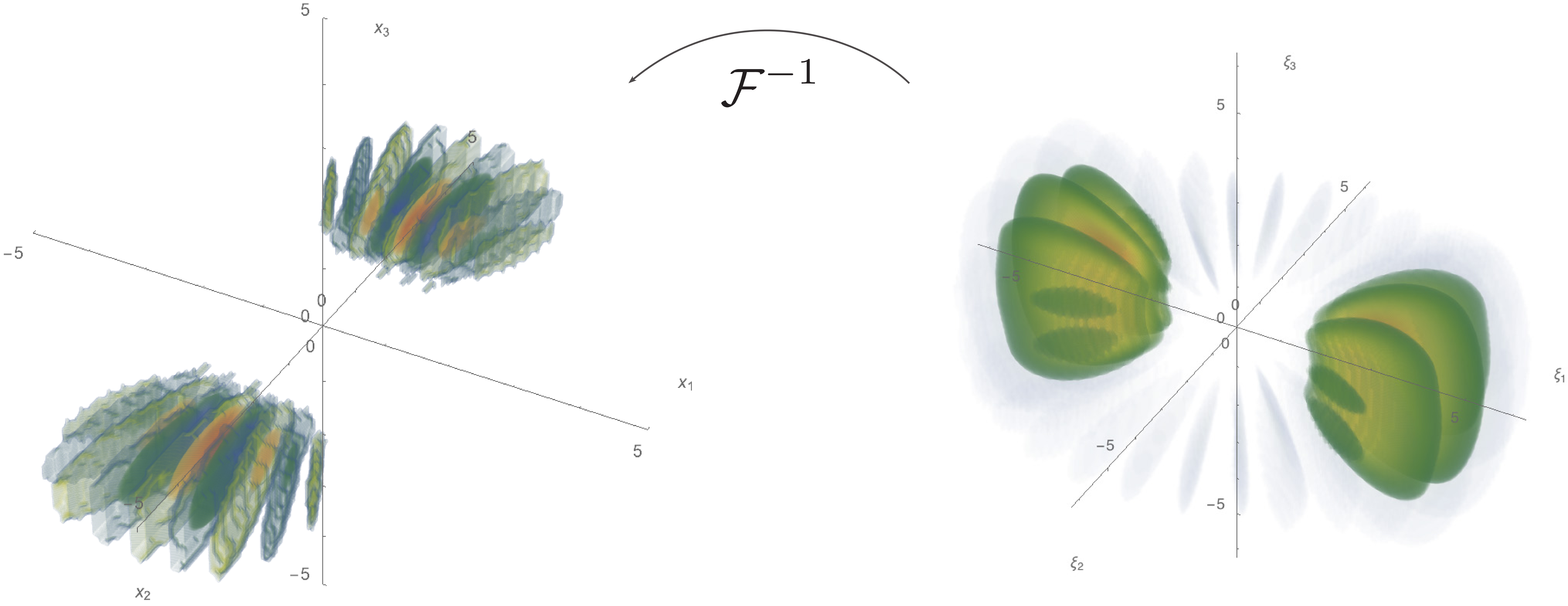
# "Polar" wavelets in $\mathbb{R}^3$


$\mathcal{F}^{-1}$

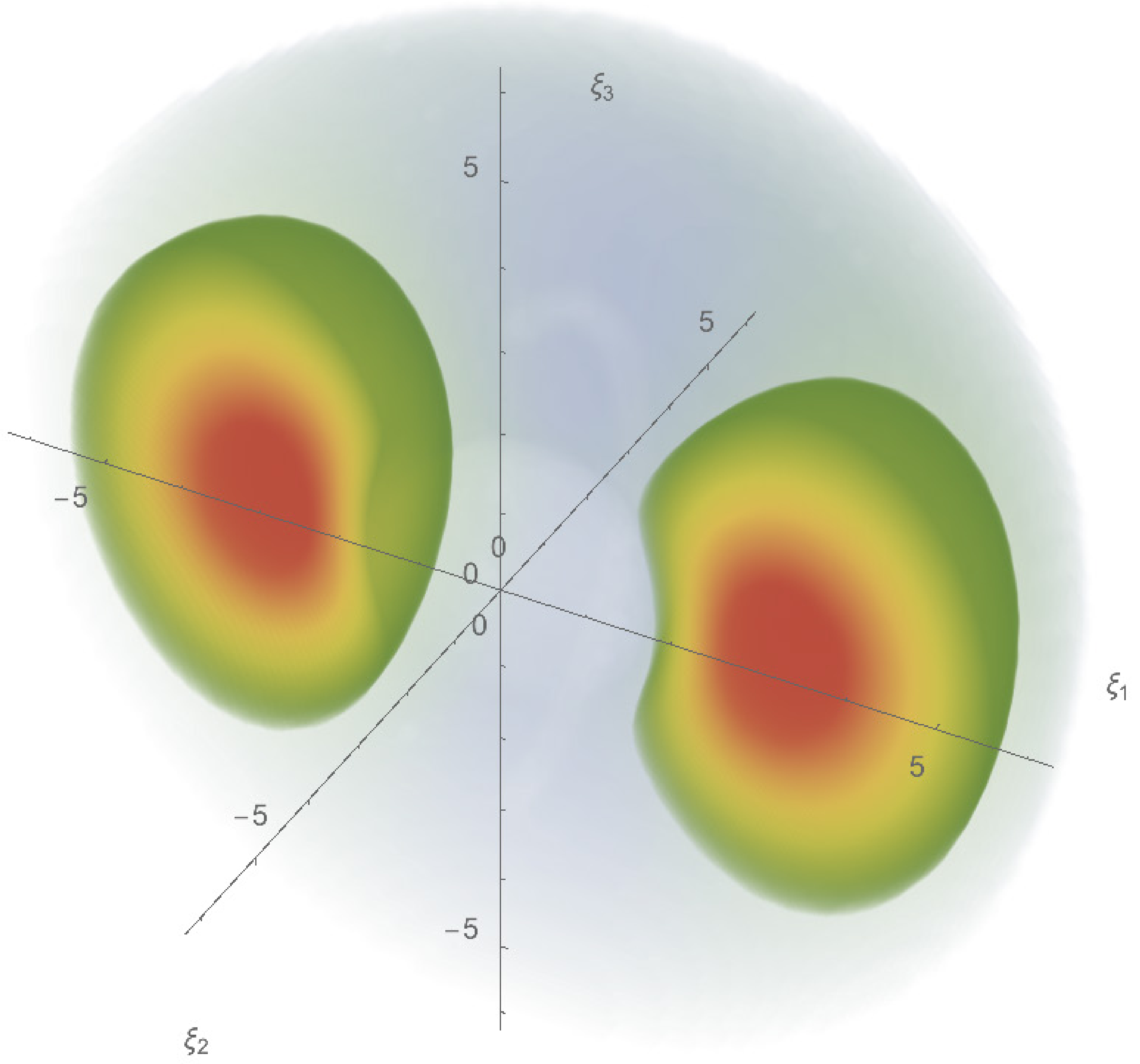
# “Polar” wavelets in $\mathbb{R}^3$



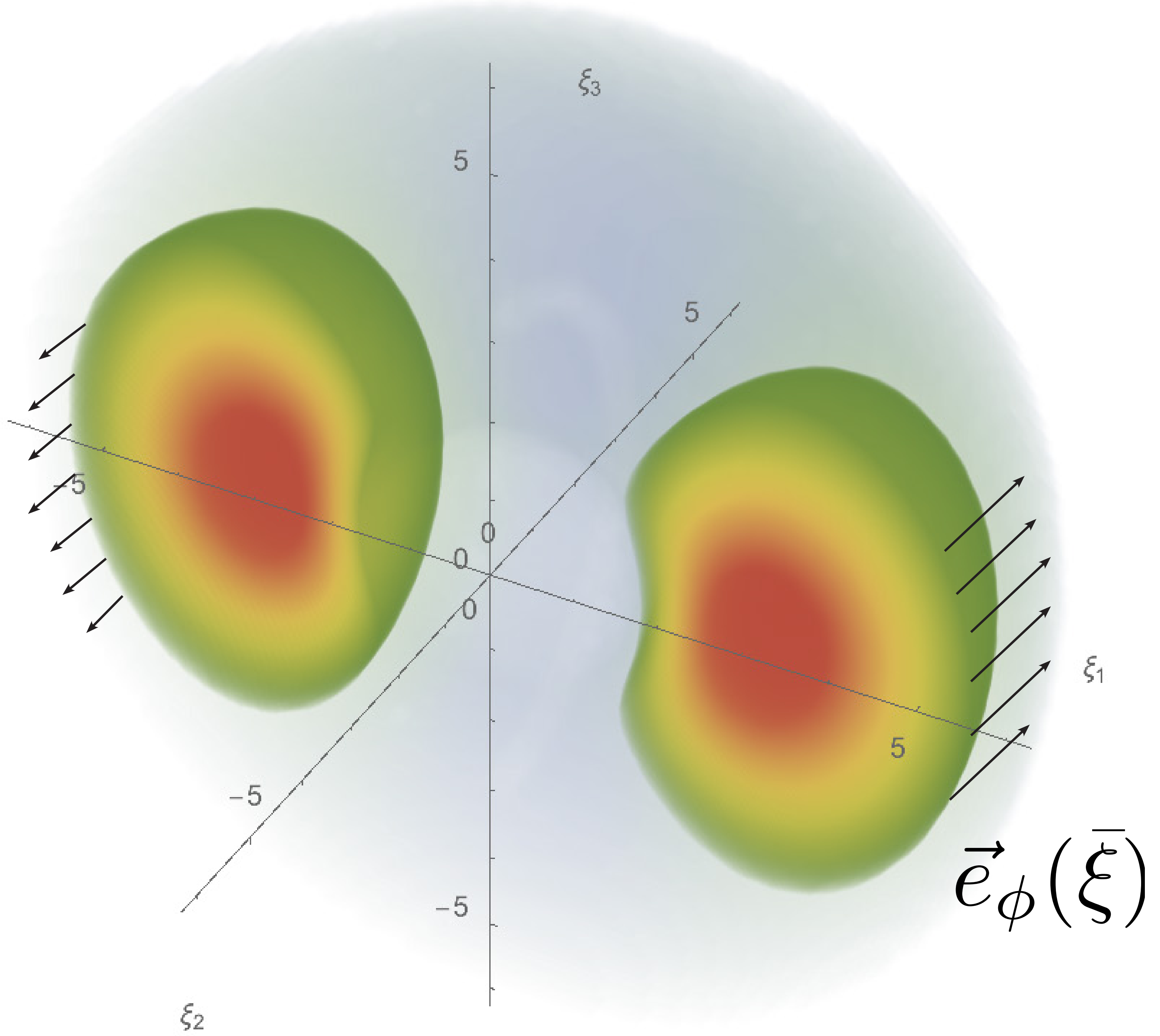
# “Polar” wavelets in $\mathbb{R}^3$



# Divergence free polar wavelets in $\mathbb{R}^3$

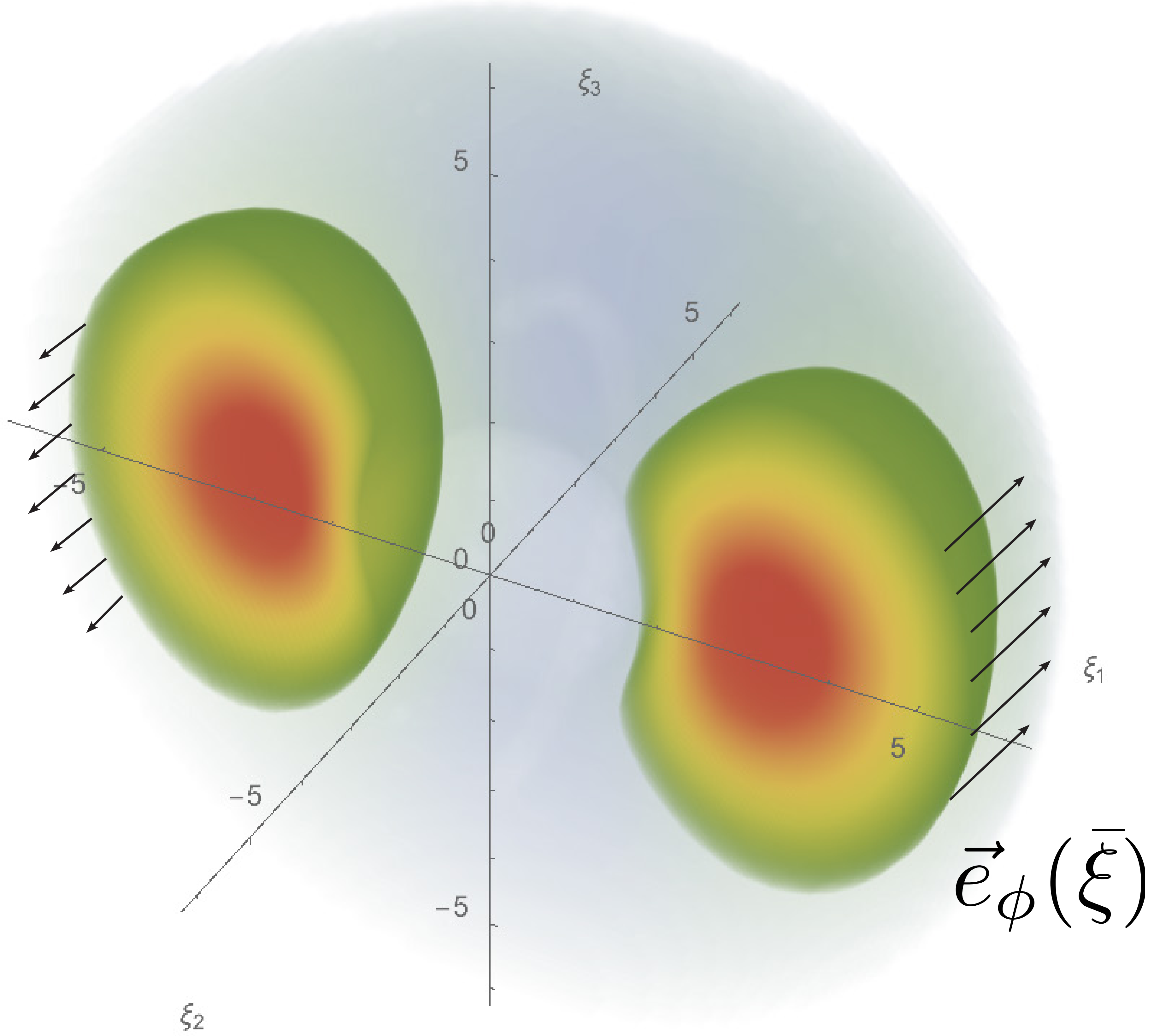


# Divergence free polar wavelets in $\mathbb{R}^3$



# Divergence free polar wavelets in $\mathbb{R}^3$

$$\hat{\psi}(\xi) = \hat{h}(|\xi|)\,\hat{\gamma}(\bar{\xi})\,\vec{e}_{\{\theta,\phi\}}(\bar{\xi})$$



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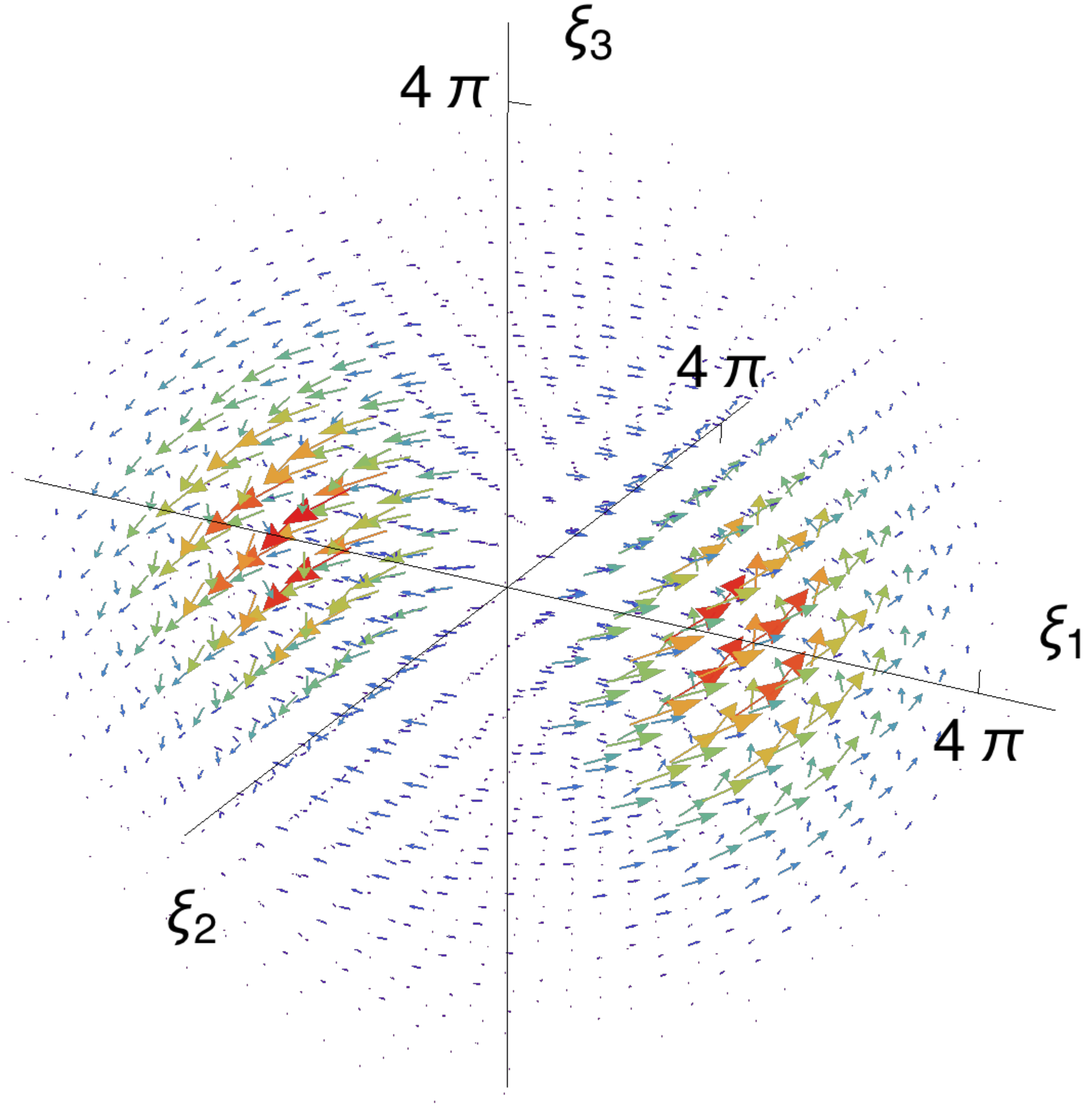
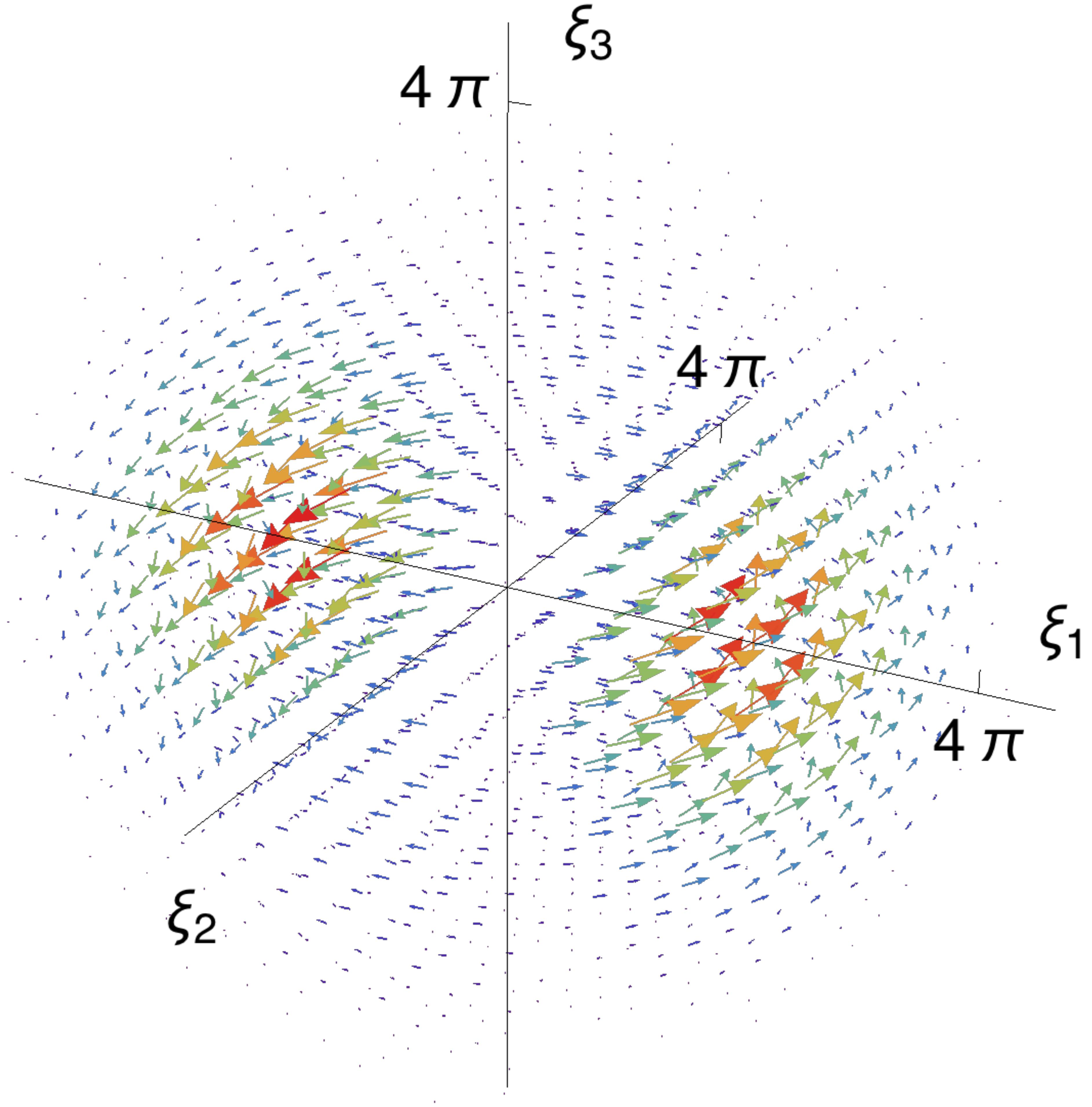


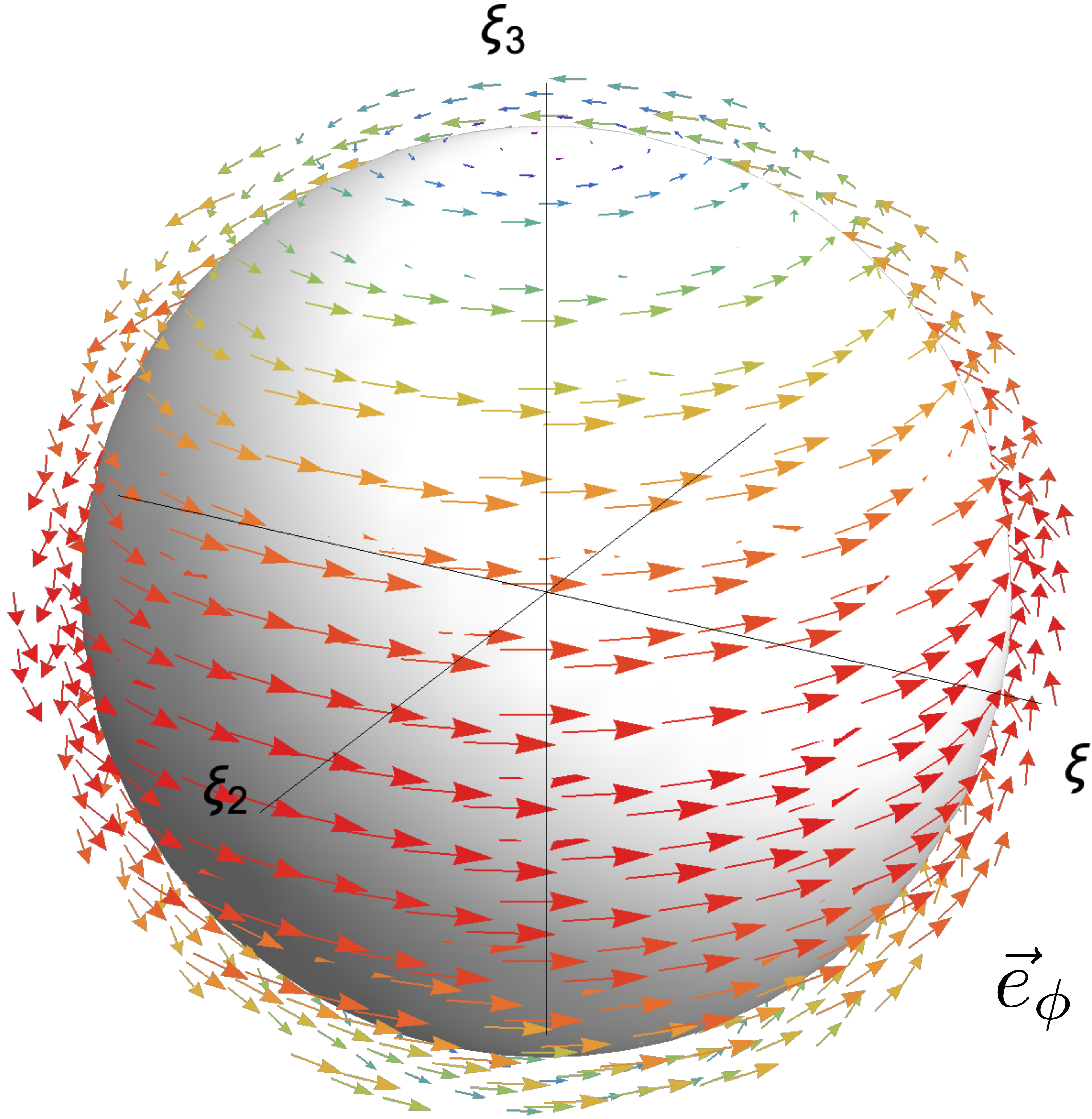
© Christian Lessig, 2019

# Divergence free polar wavelets in $\mathbb{R}^3$

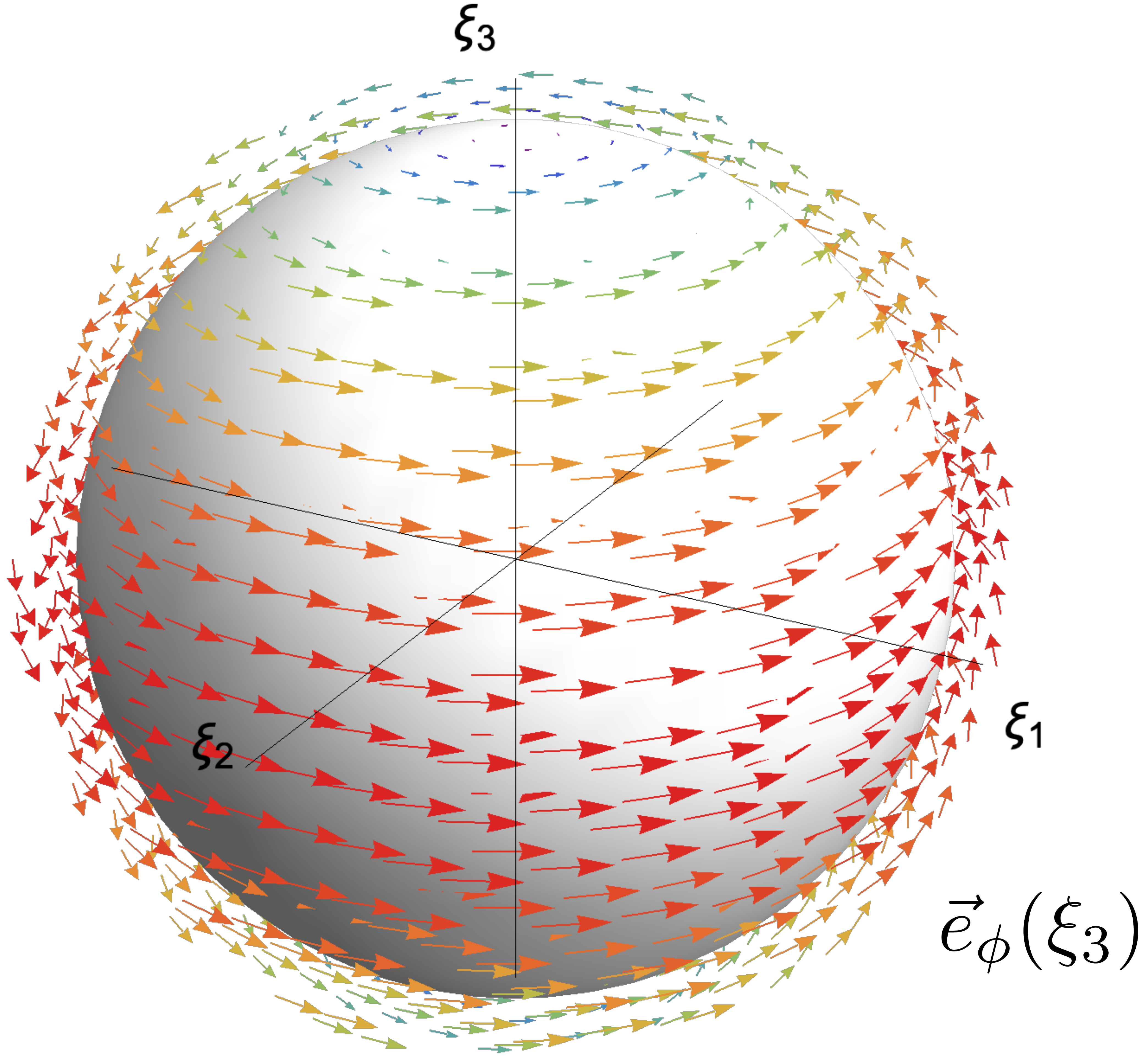
$$\hat{\psi}(\xi) = \hat{h}(|\xi|)\,\hat{\gamma}(\bar{\xi})\,\underbrace{\vec{e}_{\{\theta,\phi\}}(\bar{\xi})}_{\vec{e}_\theta(\bar{\xi}) \text{ is singular at pole}}$$



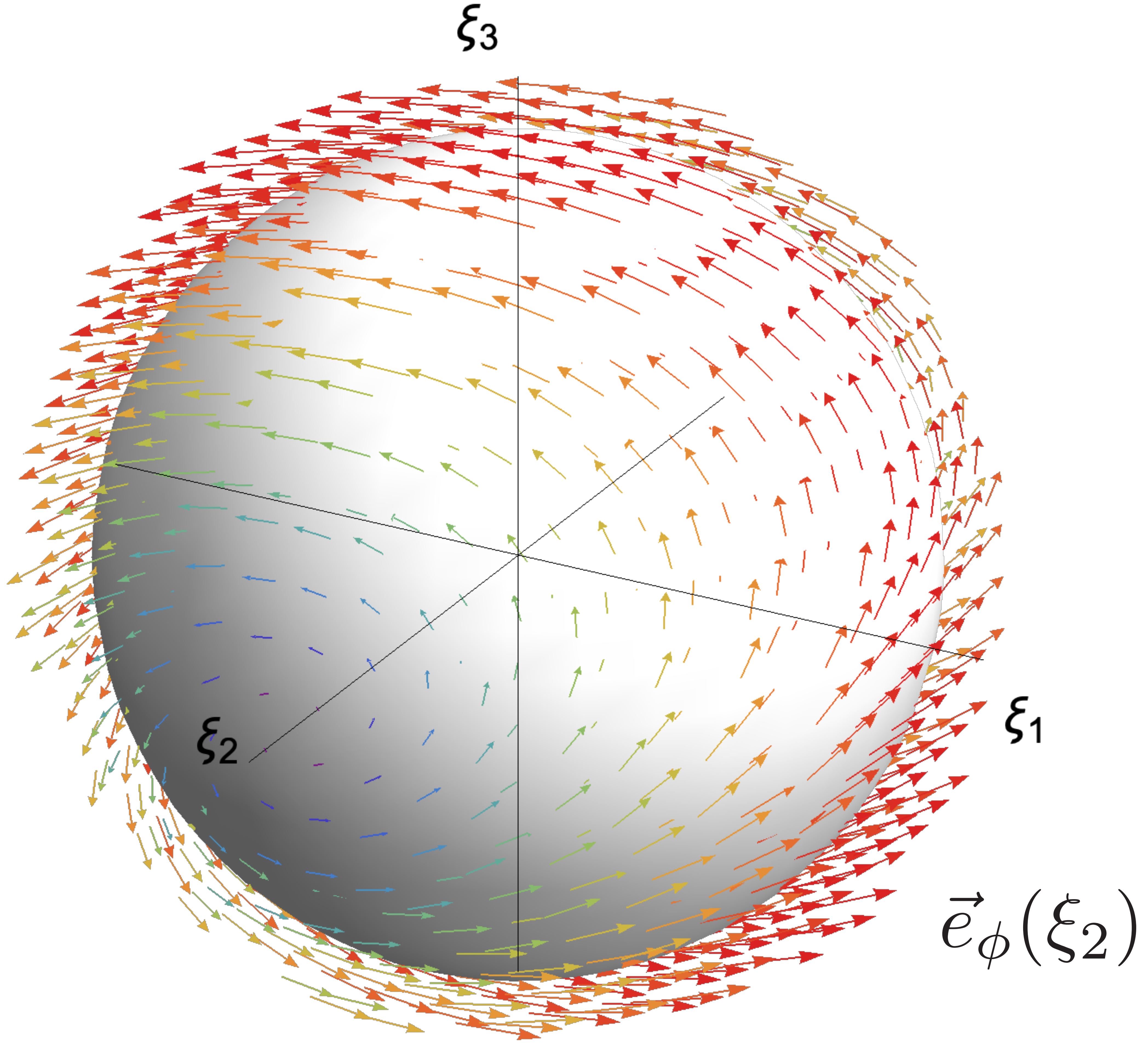
# Divergence free polar wavelets in $\mathbb{R}^3$



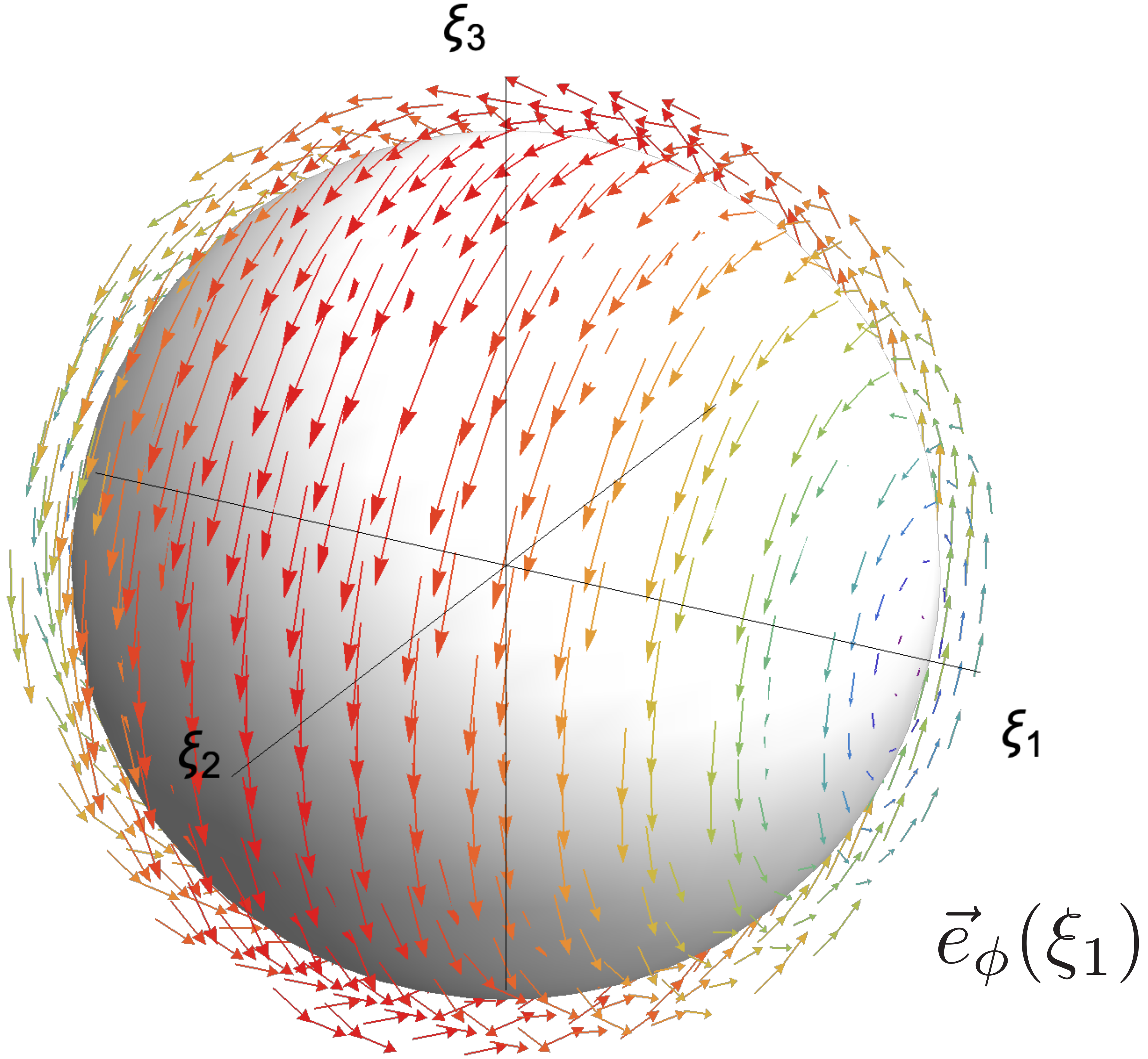
# Divergence free polar wavelets in $\mathbb{R}^3$



# Divergence free polar wavelets in $\mathbb{R}^3$



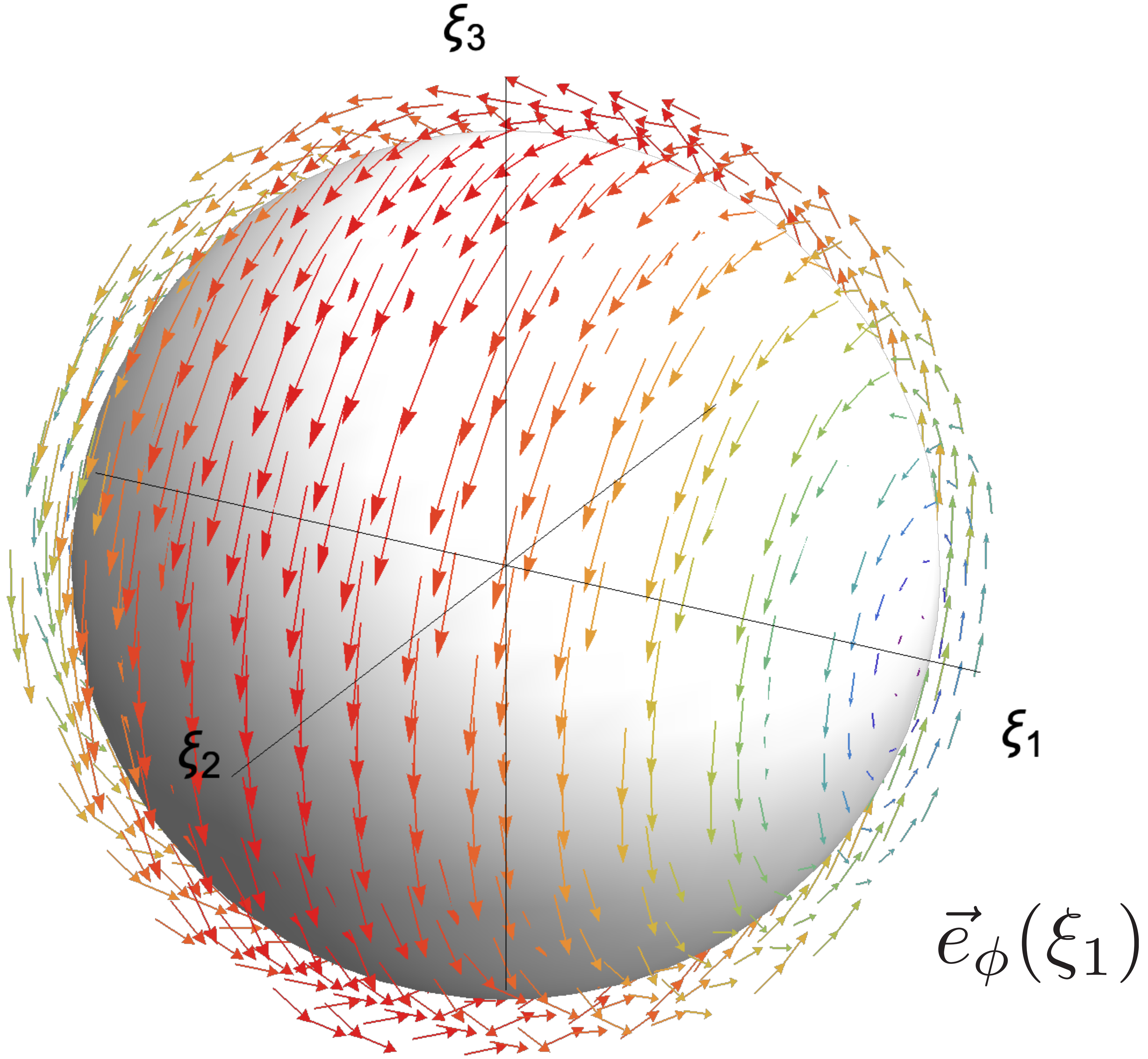
# Divergence free polar wavelets in $\mathbb{R}^3$



# Divergence free polar wavelets in $\mathbb{R}^3$

- Hedgehog frame:

$$H = \left\{ \vec{e}_{\phi}(\xi_1), \vec{e}_{\phi}(\xi_2), \vec{e}_{\phi}(\xi_3) \right\}$$

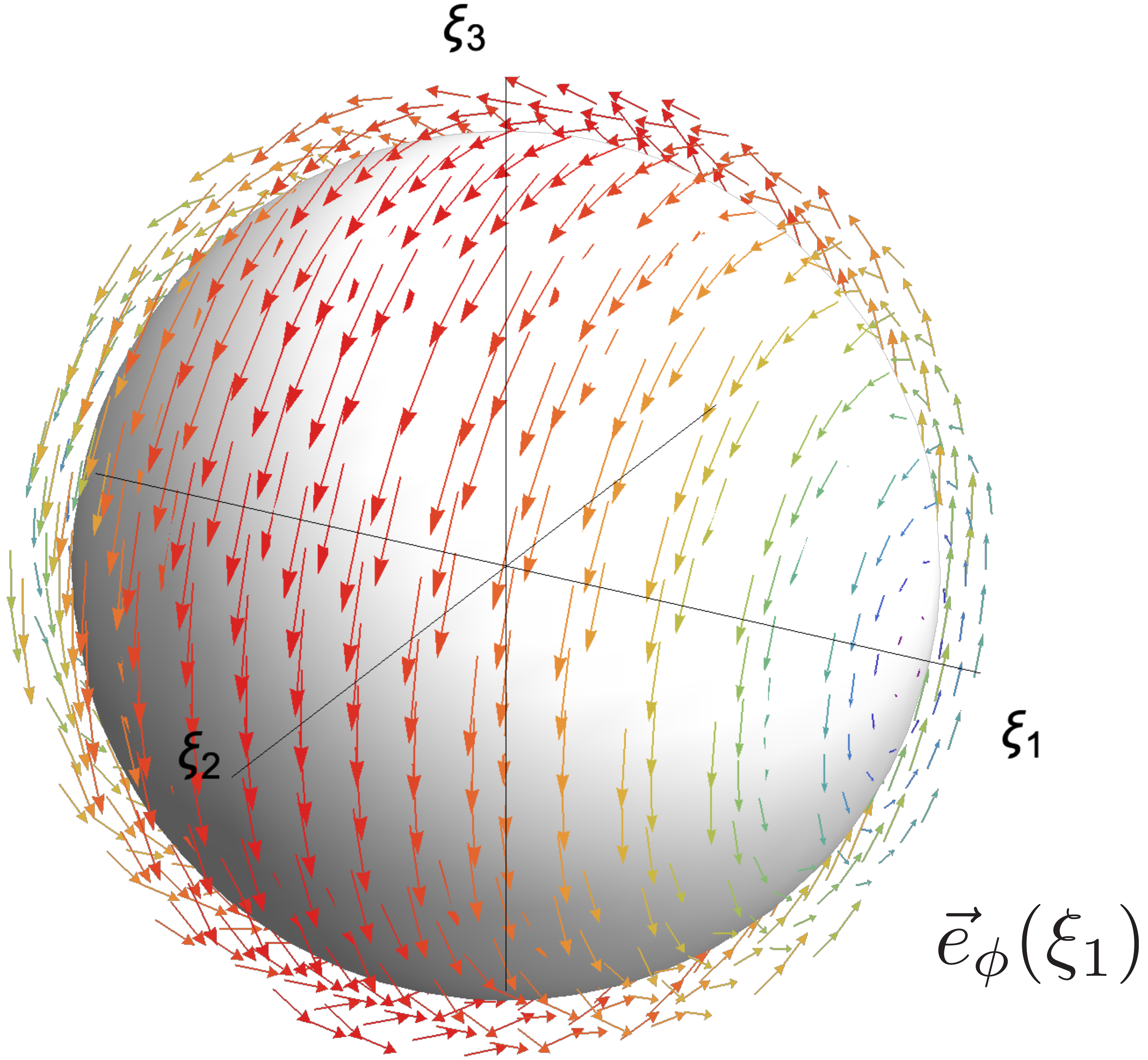


# Divergence free polar wavelets in $\mathbb{R}^3$

- Hedgehog frame:

$$H = \left\{ \vec{e}_\phi(\xi_1), \vec{e}_\phi(\xi_2), \vec{e}_\phi(\xi_3) \right\}$$

  - Tight frame for $TS^2$

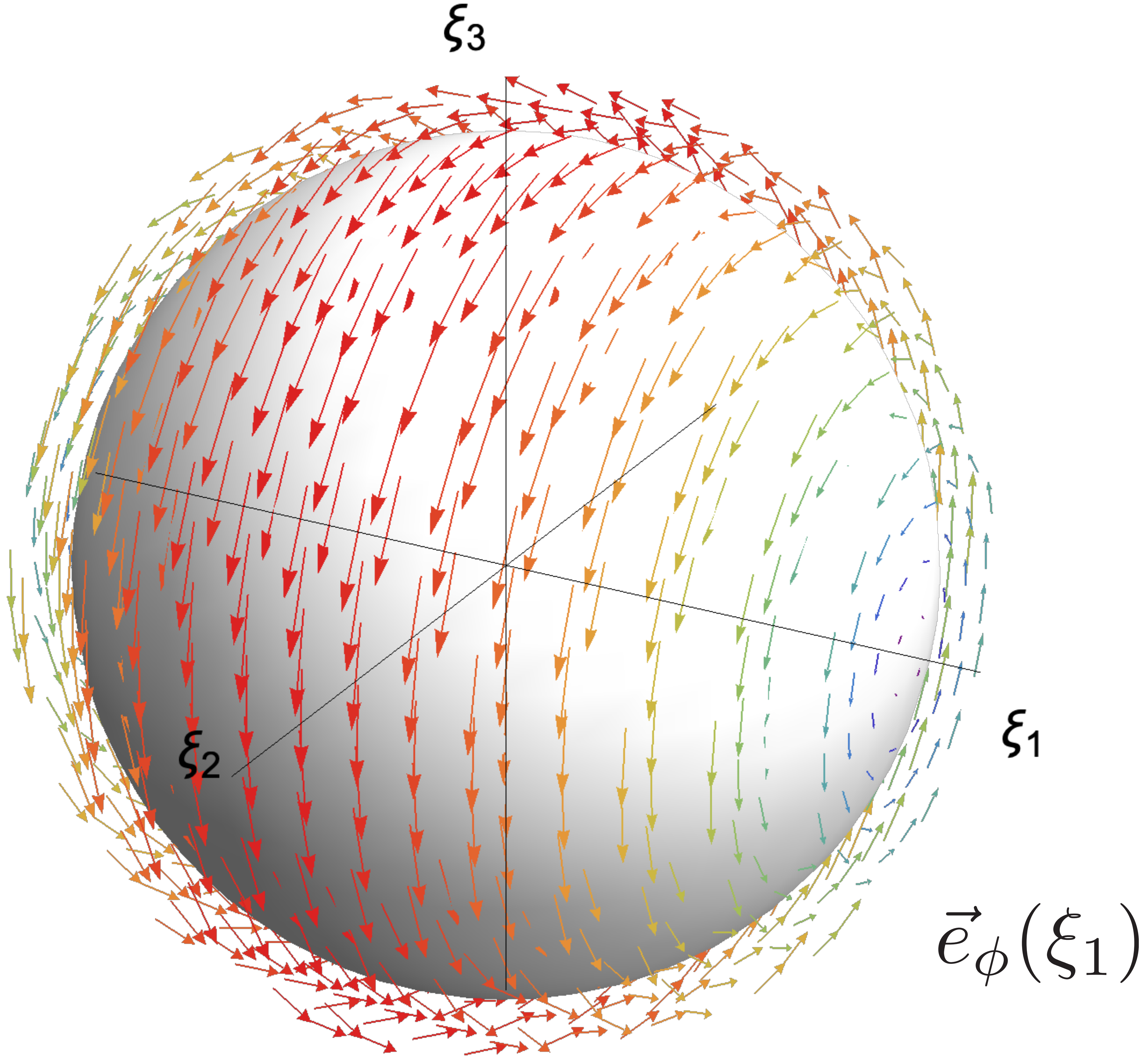


# Divergence free polar wavelets in $\mathbb{R}^3$

- Hedgehog frame:

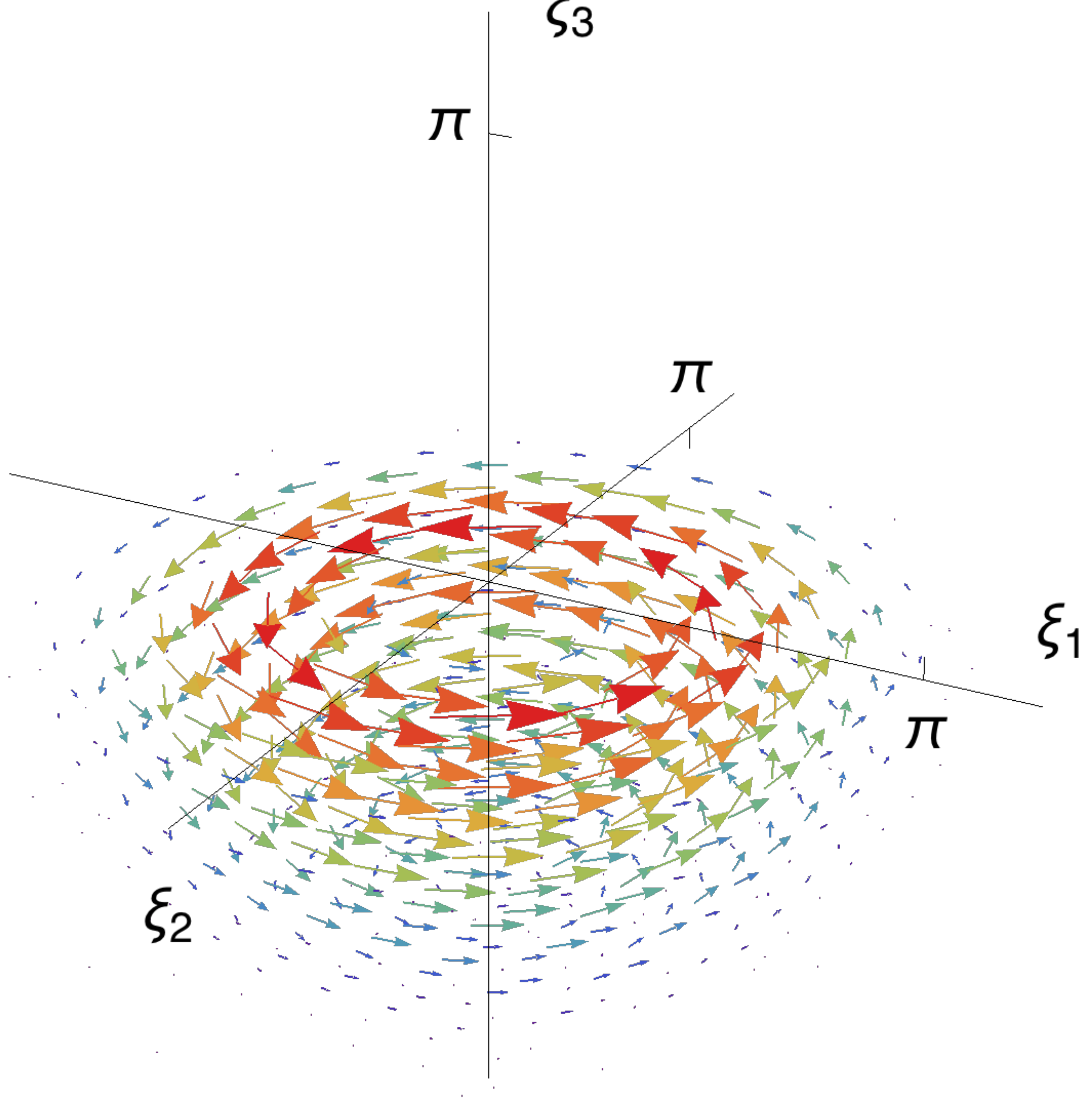
$$H = \left\{ \vec{e}^{\,1}_{\phi}(\bar{\xi}), \vec{e}^{\,2}_{\phi}(\bar{\xi}), \vec{e}^{\,3}_{\phi}(\bar{\xi}) \right\}$$

  – Tight frame for $TS^2$



# Divergence free polar wavelets in $\mathbb{R}^3$

$$\hat{\psi}_{j,t}^{\kappa}(\xi) = \hat{h}_j(|\xi|)\,\hat{\gamma}_{j,t}(\bar{\xi})\,\vec{e}_{\phi}^{\kappa}(\bar{\xi})$$



# Divergence free polar wavelets in $\mathbb{R}^3$

**Proposition 4.** *Let* $w_{j,t}$ *be the* $(L_j+1)^2$*-dimensional vector formed by the rotated angular localization coefficients* $\kappa_{lm}^{j,t} = \sum_{m'=-l}^{l} W(\lambda_t)_{l,m}^{m'} \kappa_{l,m'}$ *for a localization window centered at* $\lambda_t$*, where* $W(\lambda_t)_{l,m}^{m}$ *is the Wigner-D matrix implementing rotation in the spherical harmonics domain, and let* $G^{lm}$ *be the* $(L_j+1)^2 \times (L_j+1)^2$ *dimensional matrix formed by the spherical harmonics product coefficients for fixed* $(l,m)$*. When the Caldèron condition* $\sum_{j \in \mathbb{Z}} \left| \hat{h}(2^{-j}|\xi|) \right|^2 = 1,\ \forall \xi \in \mathbb{R}^3$ *is satisfied and* $\delta_{l,0}\delta_{m,0} = \sum_{t=0}^{M_j} w_{j,t}\, G^{lm}\, w_{j,t}$ *(where* $\delta_{i,j}$ *is the Kronecker delta) then any* $\vec{u}(x) \in L_2^{div}(\mathbb{R}^{3,3})$ *has the representation*

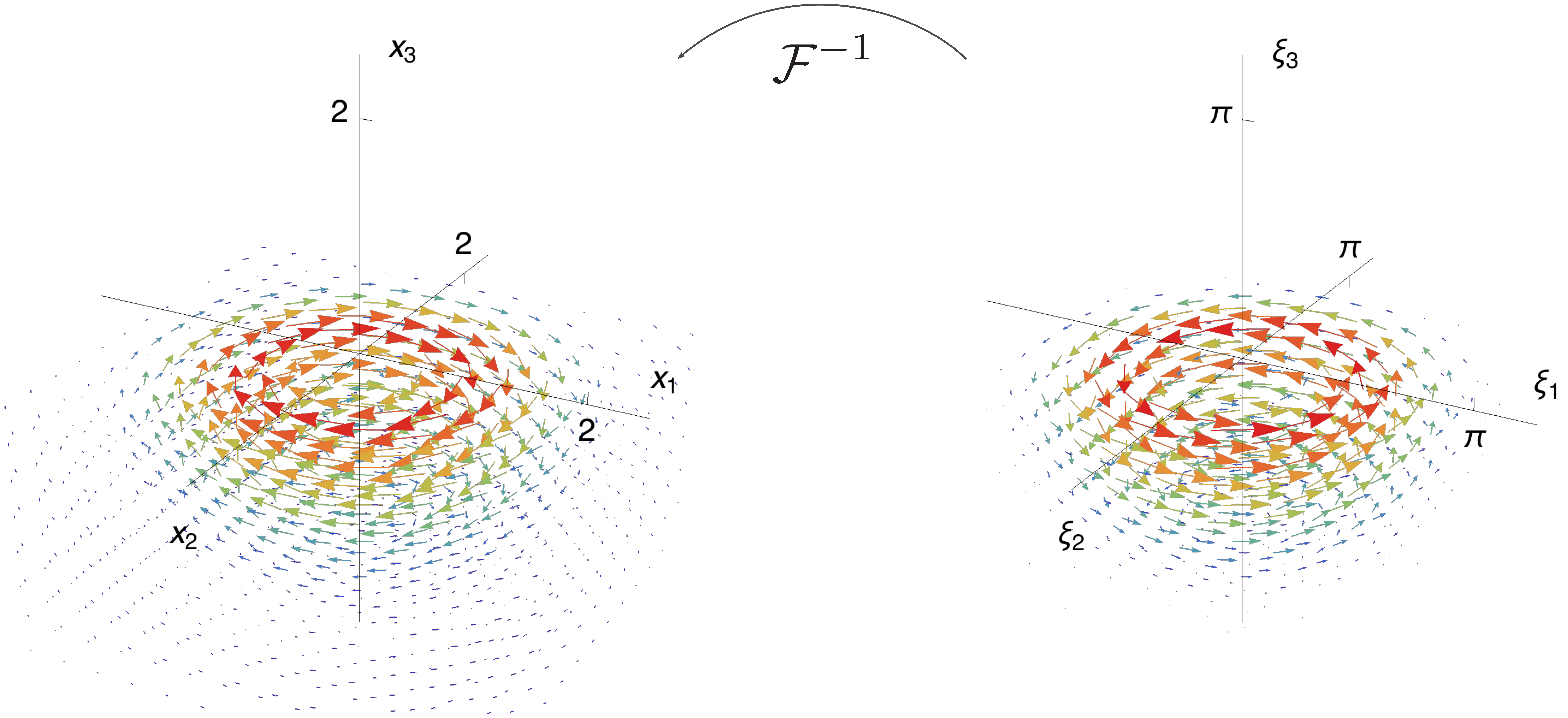
$$\vec{u}(x) = \sum_{a=1}^{3} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^3} \sum_{t=1}^{M_j} \left\langle \vec{u}(y), \vec{\psi}_{j,k,t}^{a}(y) \right\rangle \vec{\psi}_{j,k,t}^{a}(x) \tag{12a}$$

*with frame functions*

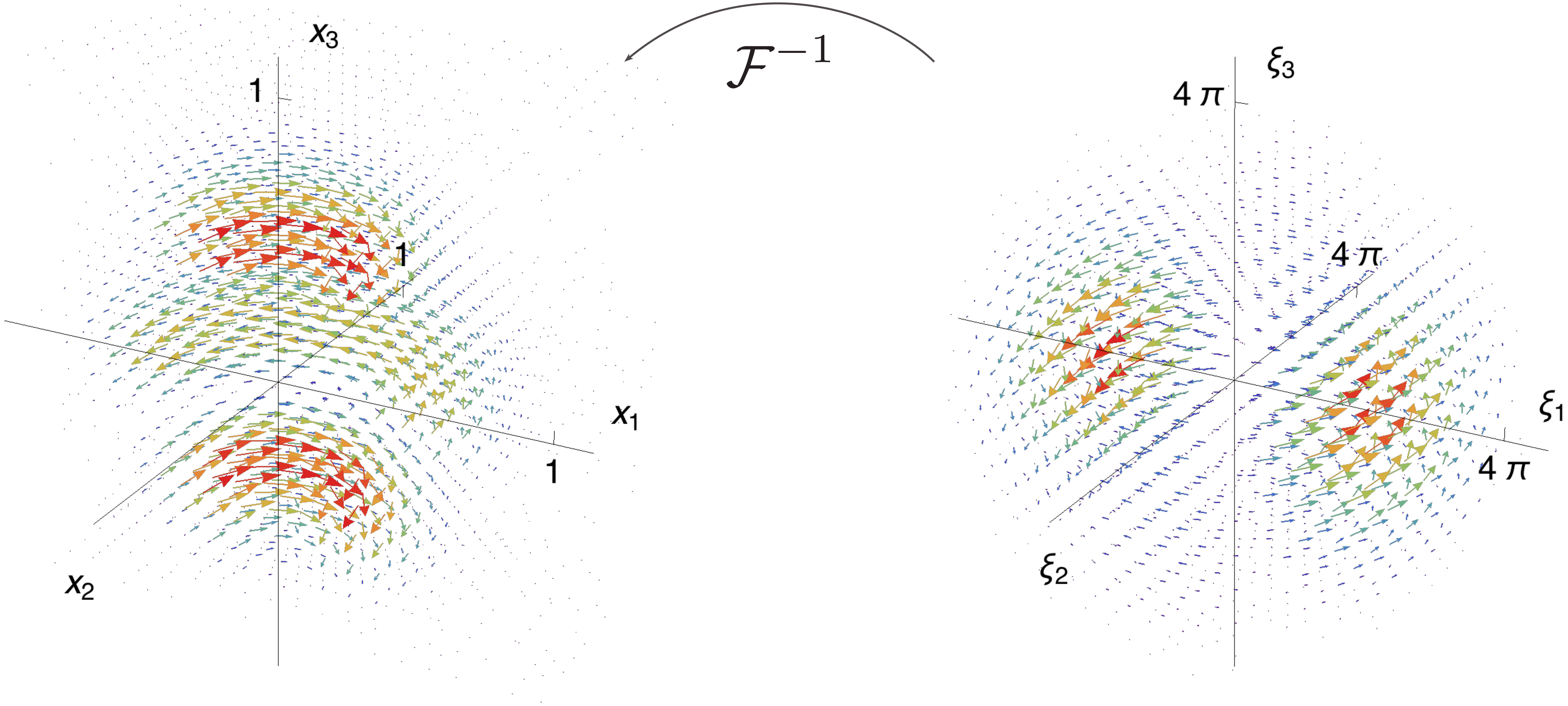
$$\vec{\psi}_{j,k,t}^{a}(x) = \frac{2^{3j/2}}{(2\pi)^{3/2}}\, \vec{\psi}_a\big(R_{\lambda_t}(2^j x - k)\big), \tag{12b}$$

*for* $\vec{\psi}_a(x)$ *defined in Eq. 11 and* $R_{\lambda_t}$ *the rotation from the North pole to* $\lambda_t$*.*

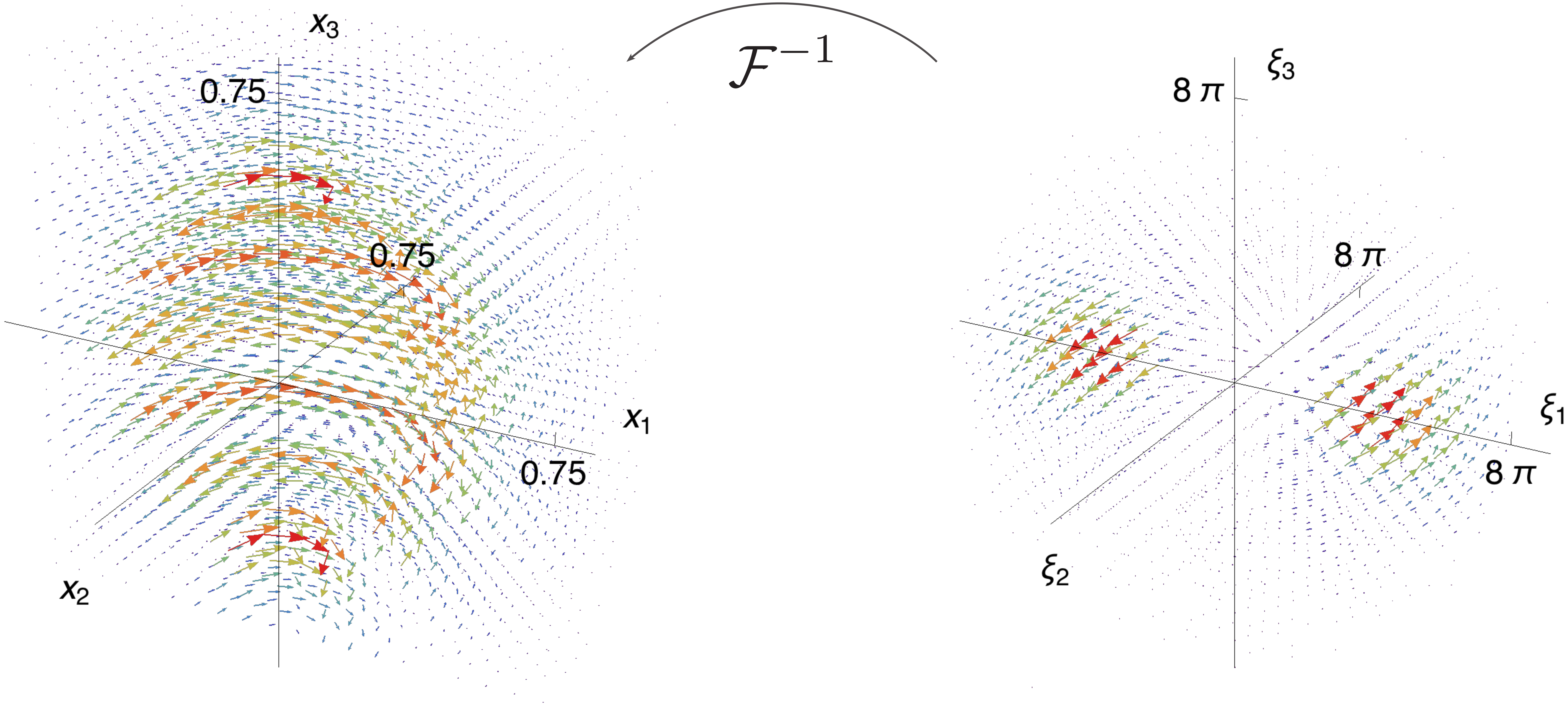
# Divergence free polar wavelets in $\mathbb{R}^3$



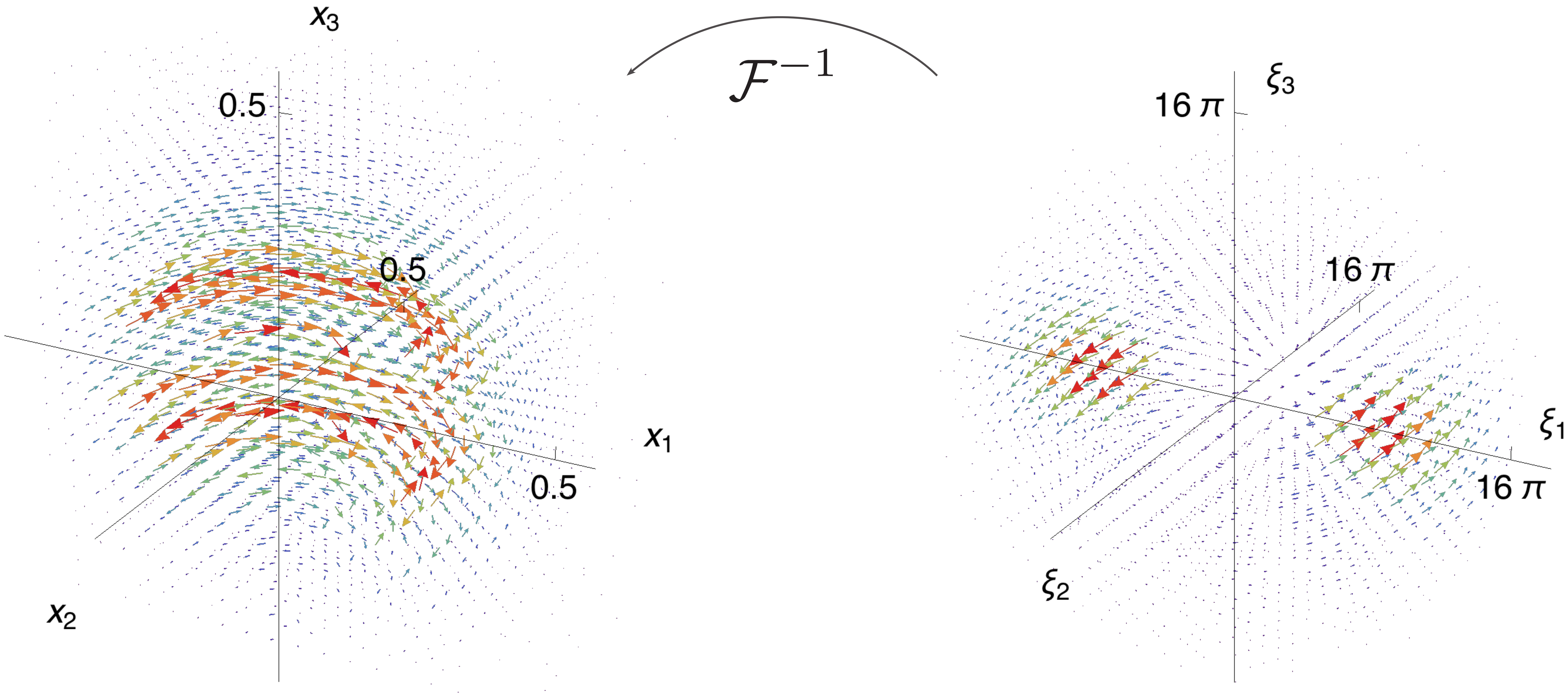
# Divergence free polar wavelets in $\mathbb{R}^3$



# Divergence free polar wavelets in $\mathbb{R}^3$

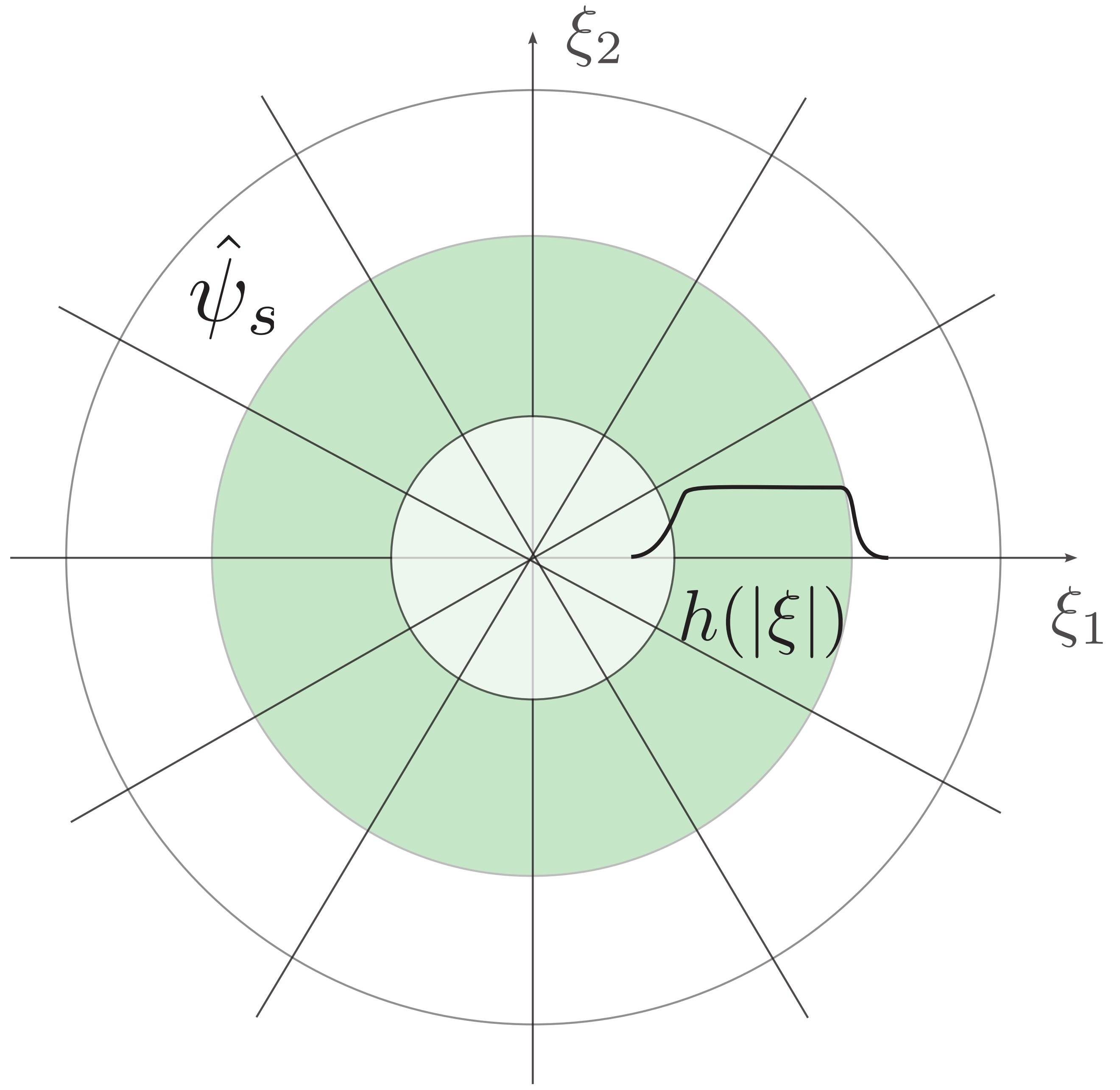


# Divergence free polar wavelets in $\mathbb{R}^3$

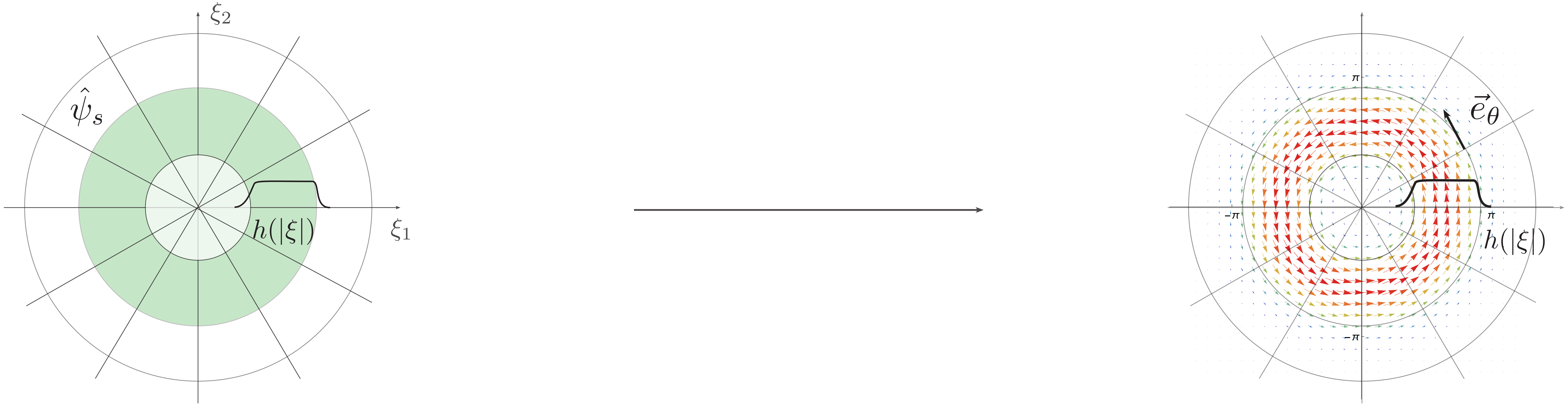


# III. The bigger picture

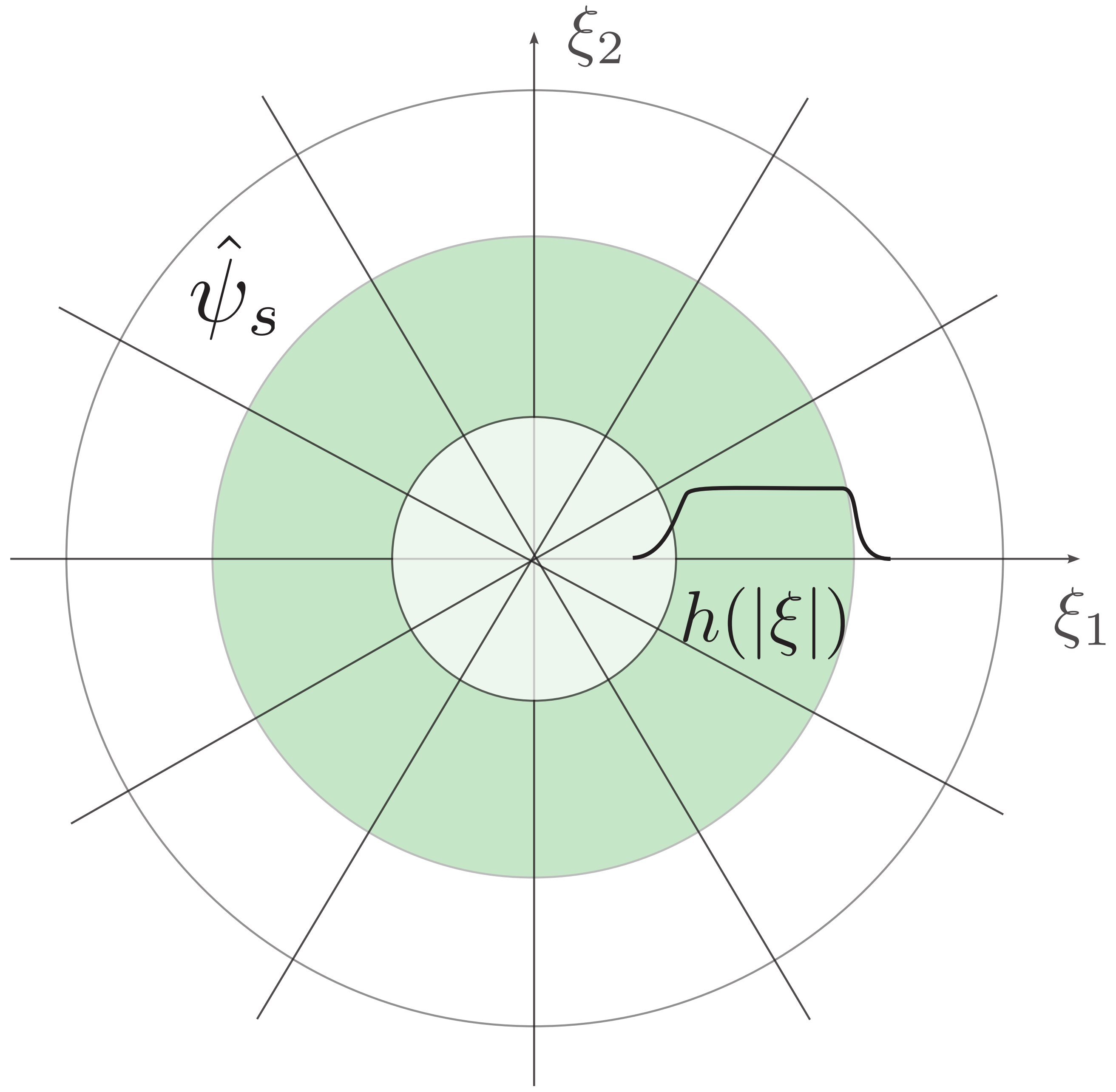
# Polar wavelets in the plane



# Polar wavelets in the plane



# Polar wavelets in the plane



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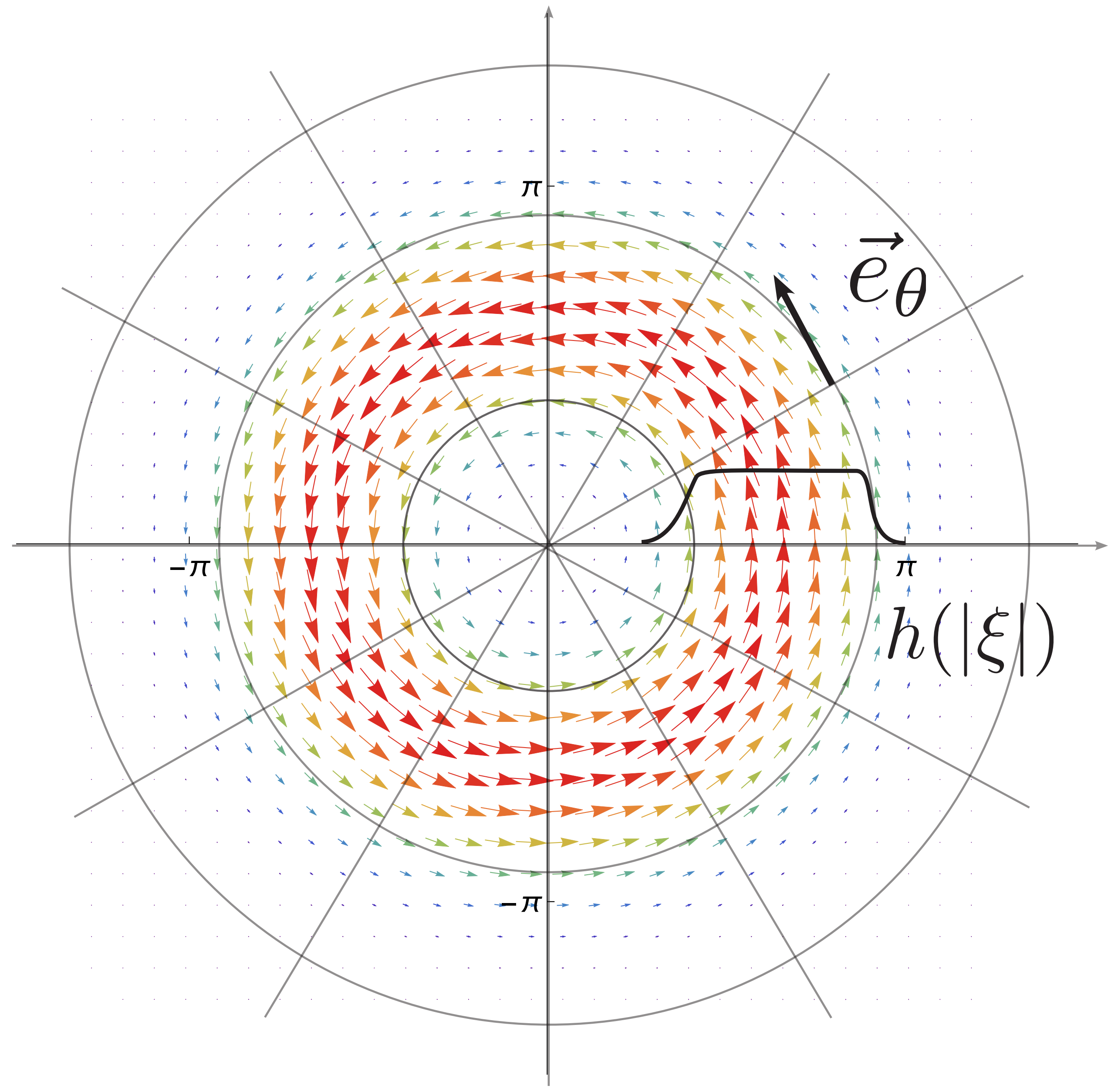
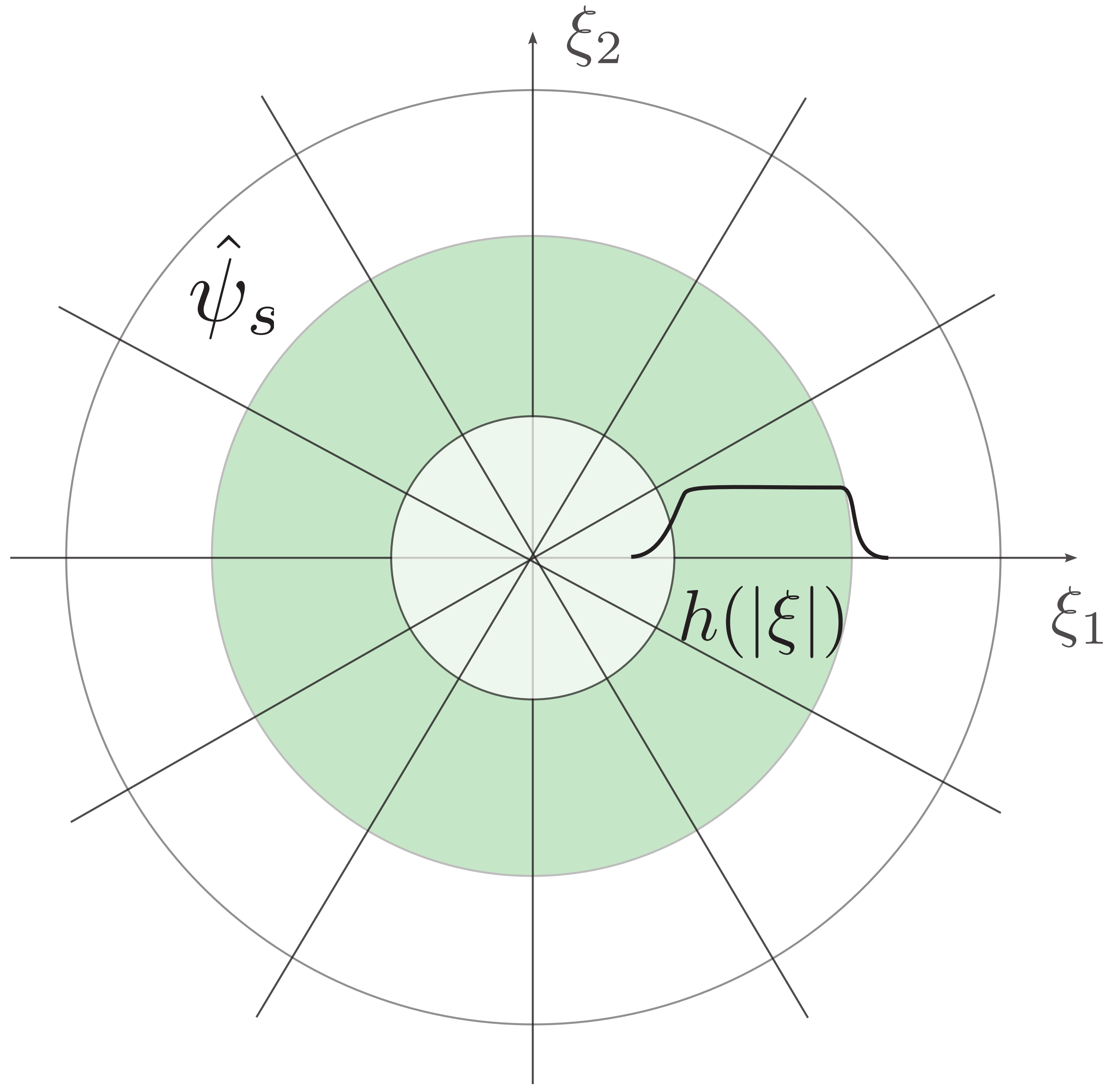
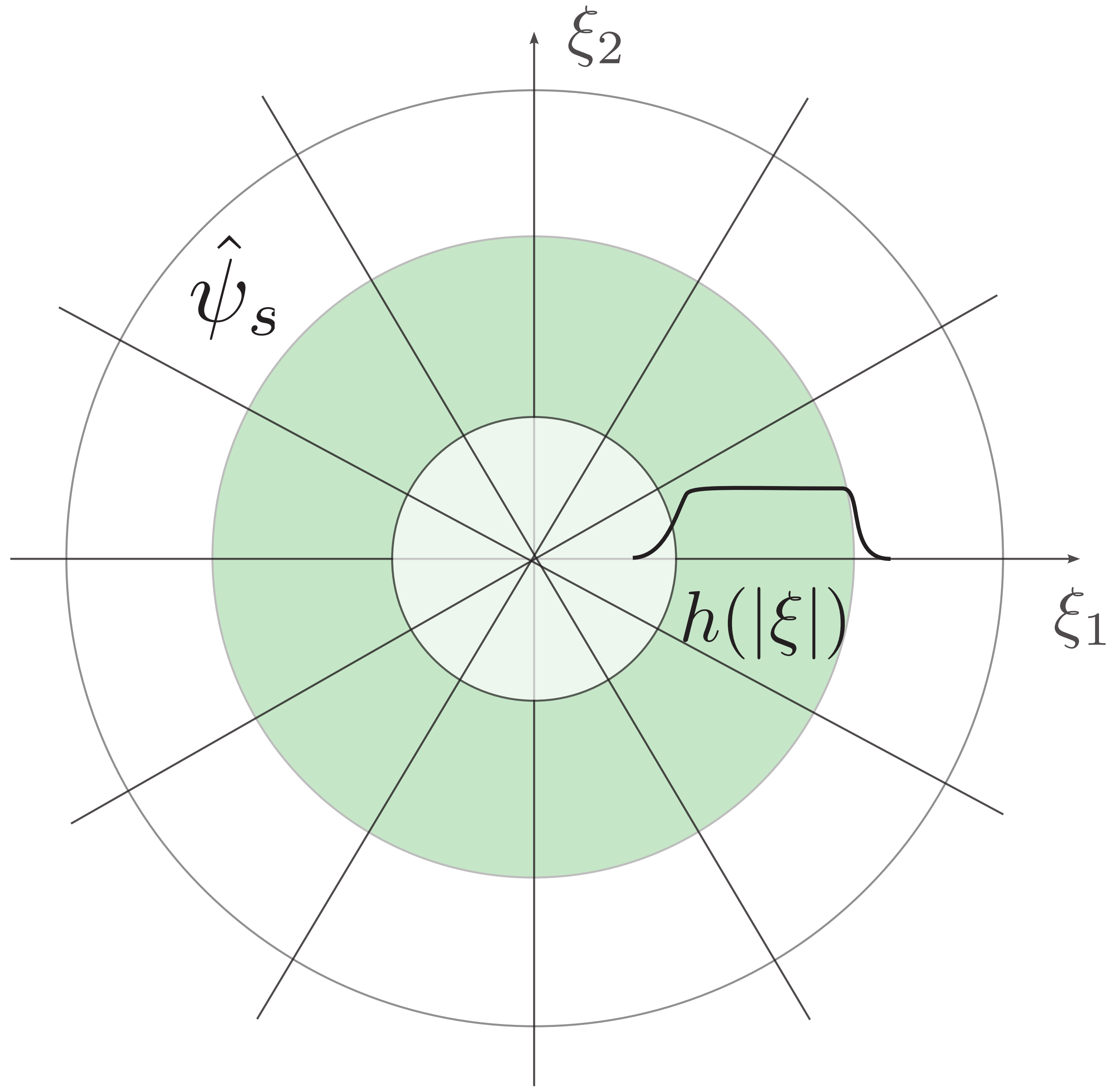


# Polar wavelets in the plane

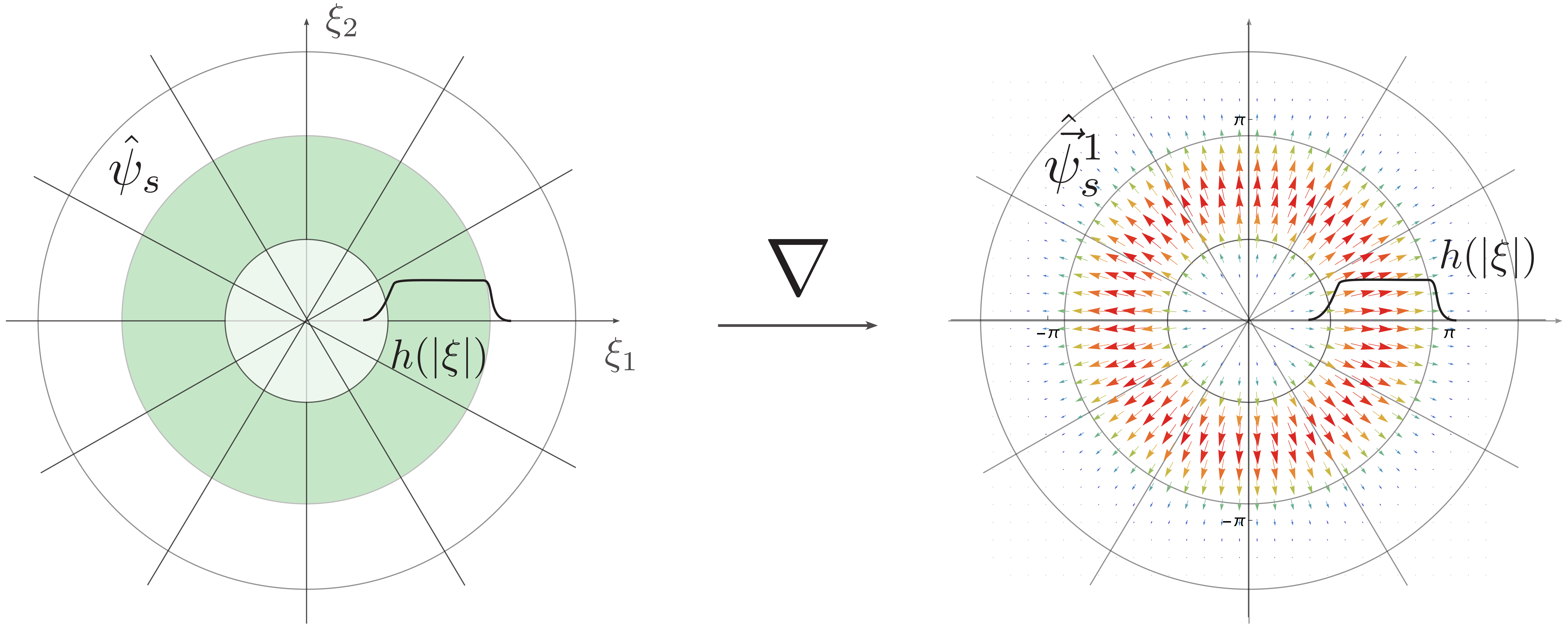


# Polar wavelets in the plane



$$\xrightarrow{\nabla}$$

# Polar wavelets in the plane



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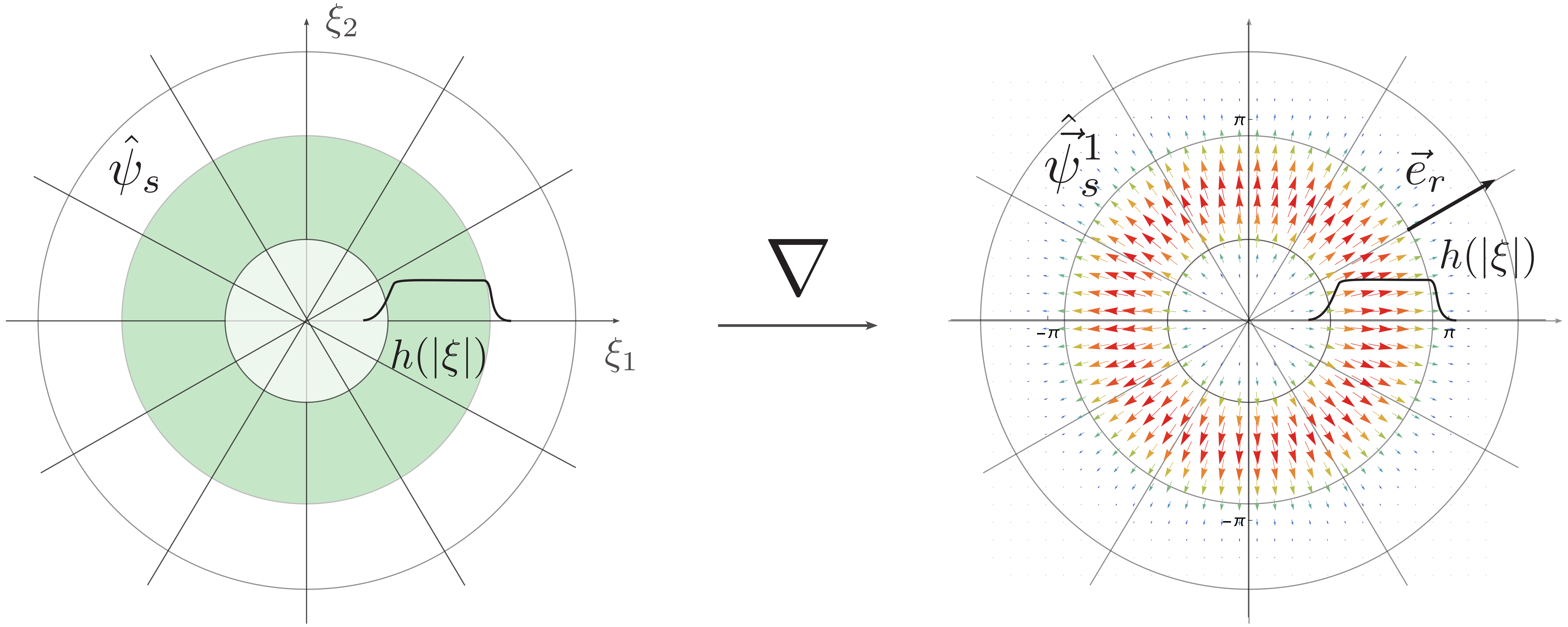

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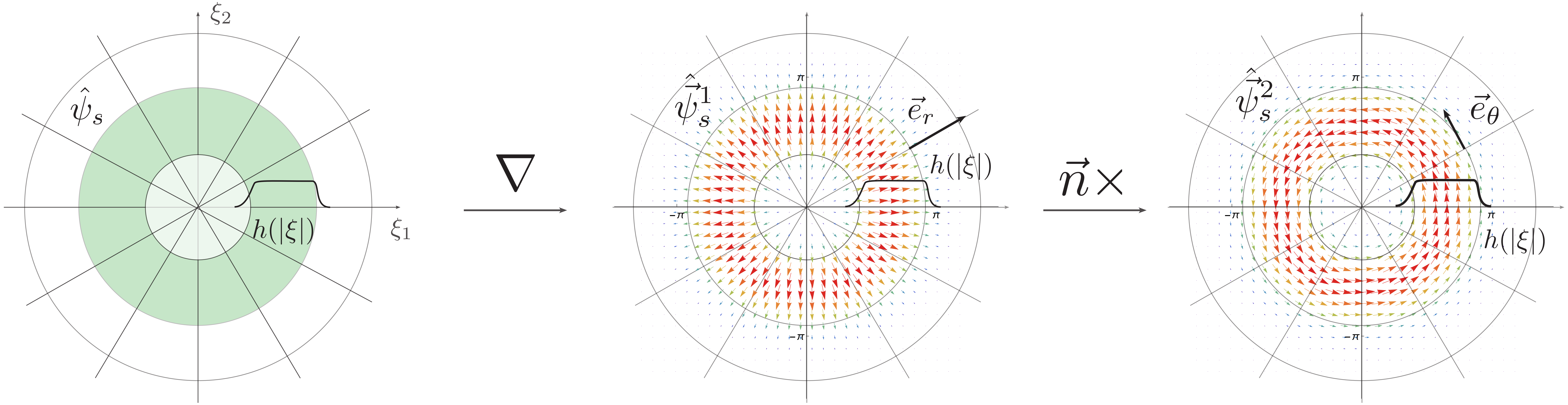

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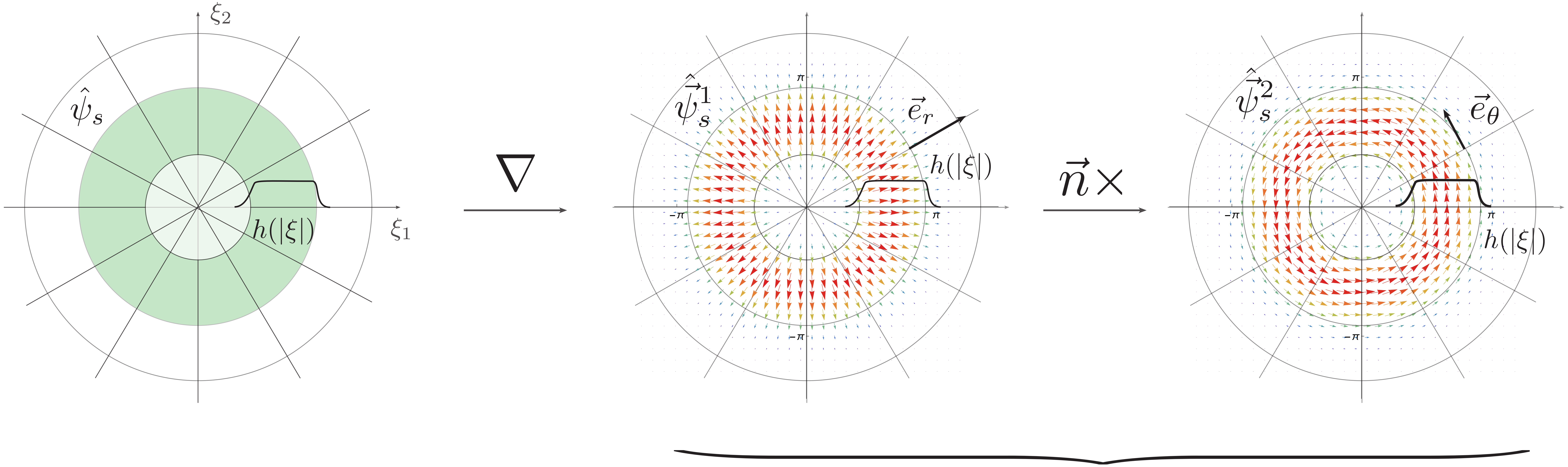


Helmholtz decomposition of vector field

# Polar wavelets in the plane

Helmholtz decomposition of vector field

# Polar wavelets in the plane

Helmholtz decomposition of vector field

# Polar wavelets in the plane

$\Omega_{\mathrm{d}}^2(\mathbb{R}^2)$

Helmholtz decomposition of
vector field

# Polar wavelets in the plane

$$\Omega^1_\delta(\mathbb{R}^2) \xrightarrow{\mathrm{d}} \Omega^2_{\mathrm{d}}(\mathbb{R}^2)$$

Helmholtz decomposition of vector field

# Polar wavelets in the plane

$$\begin{tikzcd}
\Omega^1_{\mathrm{d}}(\mathbb{R}^2) \arrow[r, leftrightarrow, "\star"] & \Omega^1_{\delta}(\mathbb{R}^2) \arrow[r, "\mathrm{d}"] & \Omega^2_{\mathrm{d}}(\mathbb{R}^2)
\end{tikzcd}$$

Hodge-Helmholtz decomposition

# Polar wavelets in the plane

$$\begin{tikzcd}\Omega^0_\delta(\mathbb{R}^2) \arrow[r, "\mathrm{d}"] & \Omega^1_{\mathrm{d}}(\mathbb{R}^2) \arrow[r, leftrightarrow, "\star"] & \Omega^1_\delta(\mathbb{R}^2) \arrow[r, "\mathrm{d}"] & \Omega^2_{\mathrm{d}}(\mathbb{R}^2)\end{tikzcd}$$

Hodge-Helmholtz decomposition

# $\Psi$ec: Local spectral exterior calculus[5]

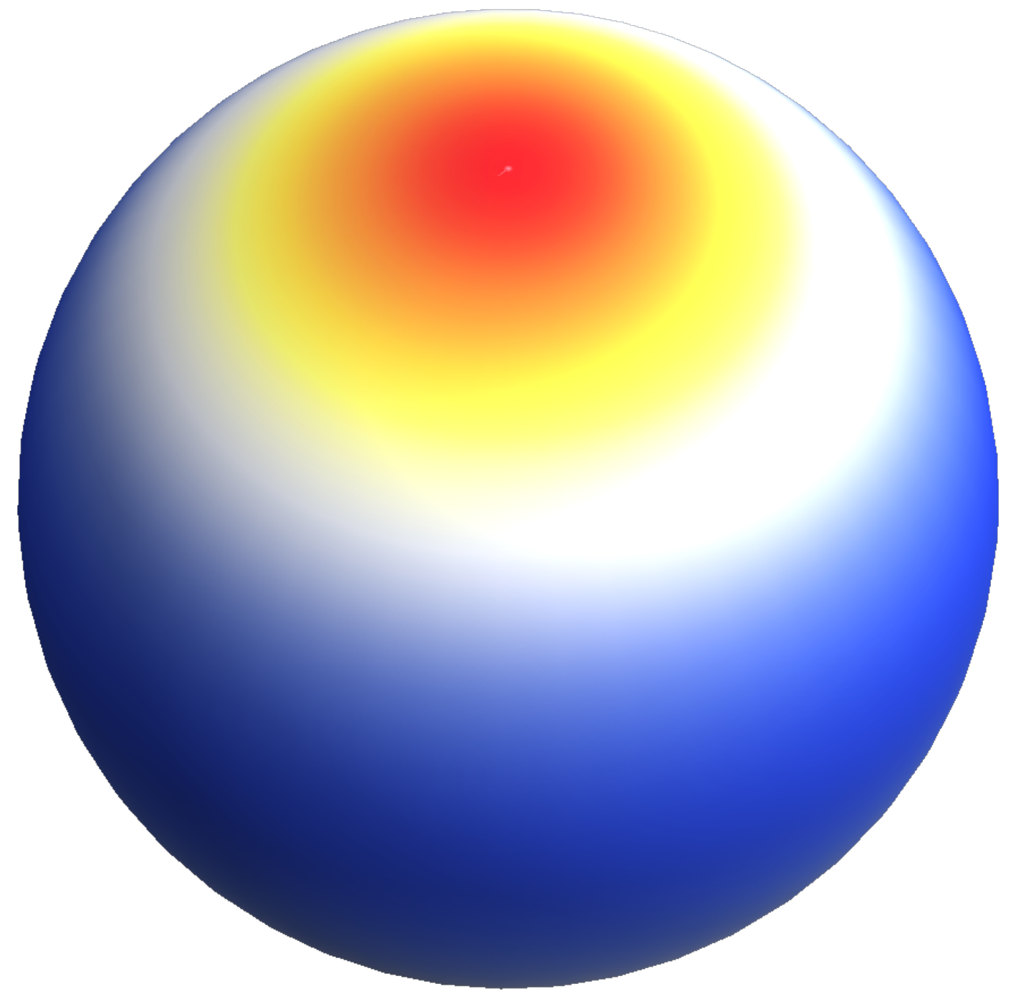
$$\begin{tikzcd}
\Omega_{\delta}^{0}(\mathbb{R}^2) \arrow[r, "\mathrm{d}"] & \Omega_{\mathrm{d}}^{1}(\mathbb{R}^2) \arrow[r, leftrightarrow, "\star"] & \Omega_{\delta}^{1}(\mathbb{R}^2) \arrow[r, "\mathrm{d}"] & \Omega_{\mathrm{d}}^{2}(\mathbb{R}^2)
\end{tikzcd}$$

---

[5] C. Lessig, "PsiEC: A Local Spherical Exterior Calculus," Submitt. to Appl. Comput. Harmon. Anal., https://arxiv.org/abs/1811.12269, 2018.

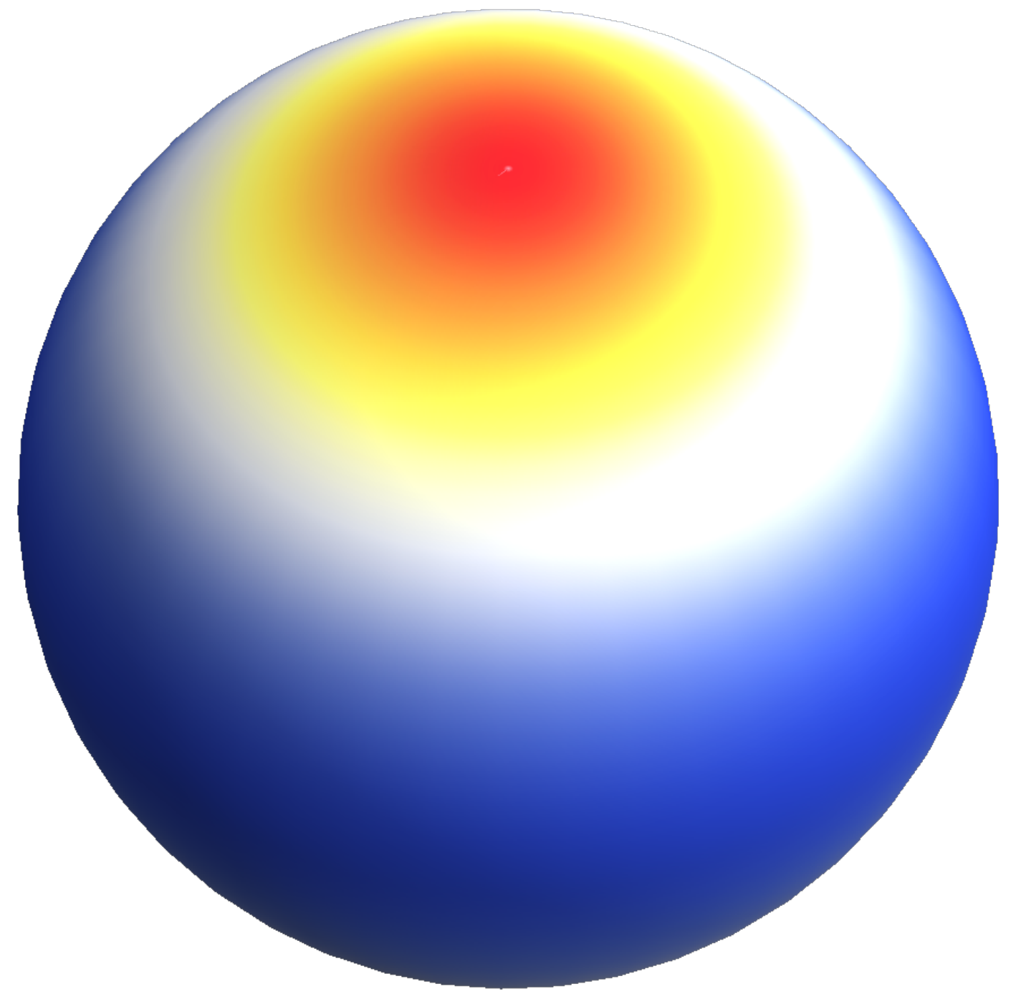
# $\Psi$ec: Local spectral exterior calculus

$\Omega^0_\delta(\mathbb{S}^2)$



# $\Psi\mathrm{ec}$: Local spectral exterior calculus

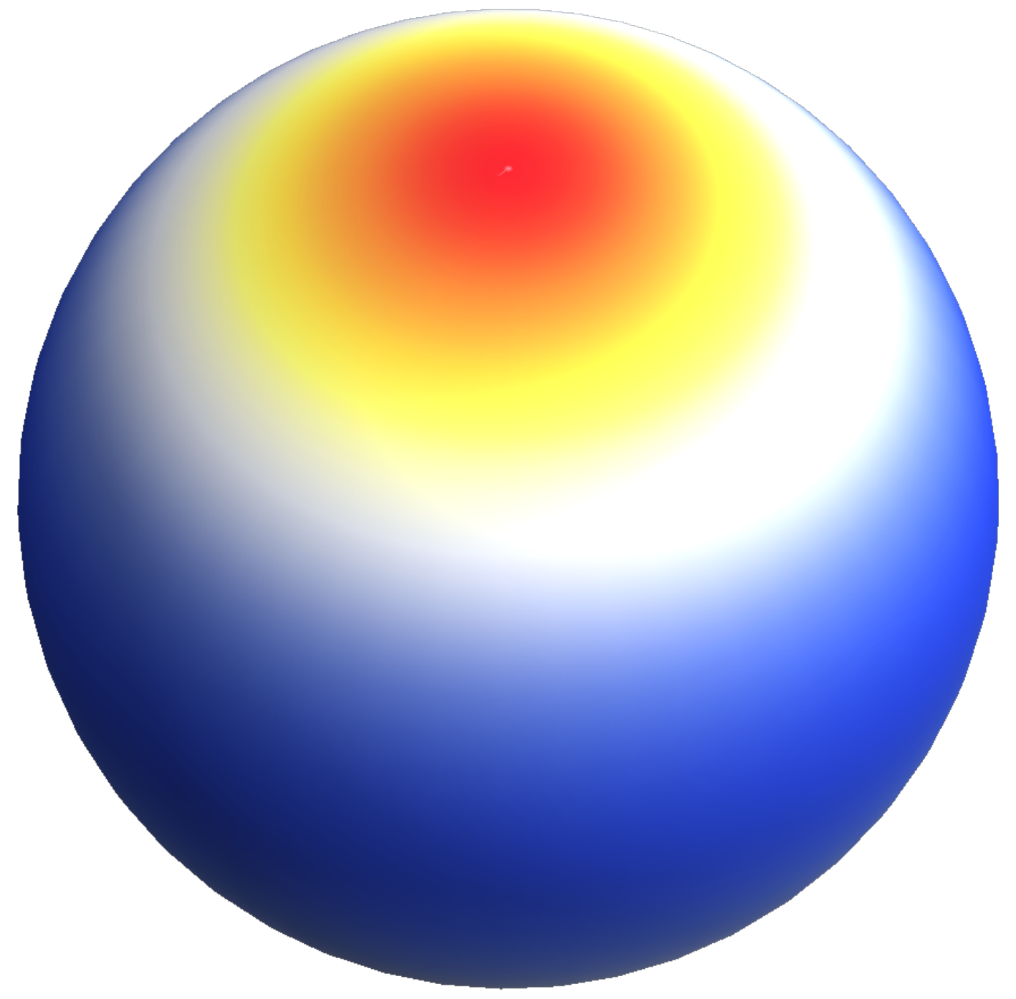
$\Omega^0_\delta(\mathbb{S}^2)$



$\hat{\gamma}_{j,t}(\bar{\xi})$

# $\Psi\mathrm{ec}$: Local spectral exterior calculus

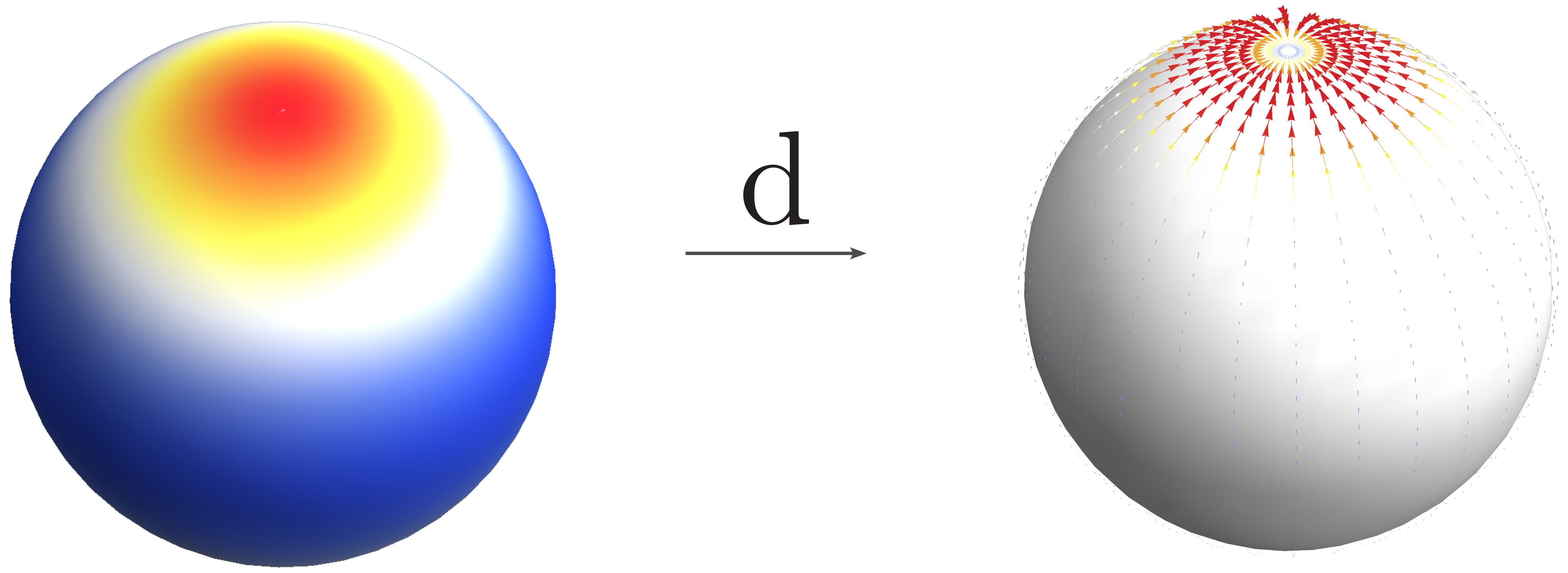
$$\Omega^0_\delta(\mathbb{S}^2)$$



$$\hat{\gamma}_{j,t}(\bar{\xi}\,) = \sum_{l,m} \beta_{lm}\, y_{lm}(\bar{\xi})$$

# $\Psi\mathrm{ec}$: Local spectral exterior calculus

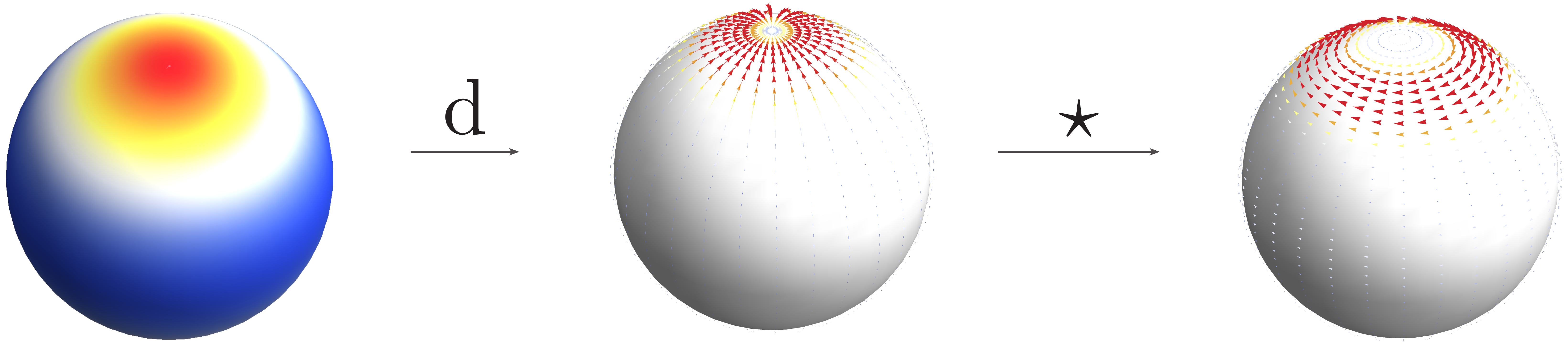
$\Omega^0_\delta(\mathbb{S}^2)$ $\Omega^1_\mathrm{d}(\mathbb{S}^2)$



$$\hat{\gamma}_{j,t}(\bar{\xi}\,) = \sum_{l,m} \beta_{lm}\, y_{lm}(\bar{\xi})$$

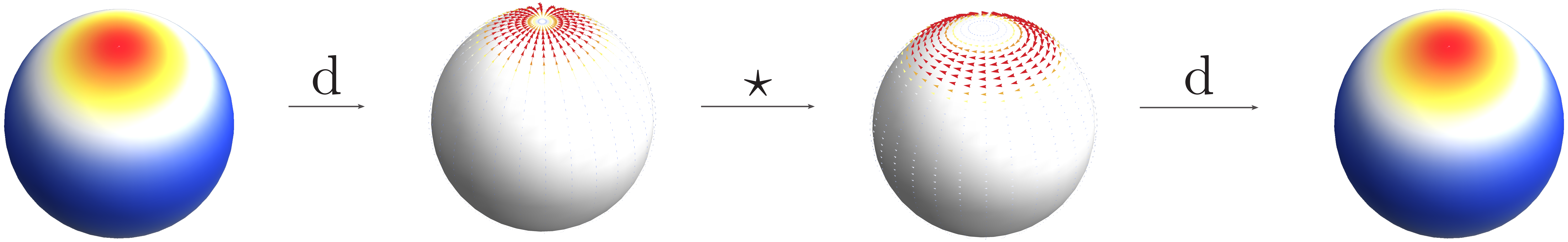
# $\Psi$ec: Local spectral exterior calculus

$$\begin{tikzcd} \Omega^0_\delta(\mathbb{S}^2) \arrow[r, "\mathrm{d}"] & \Omega^1_{\mathrm{d}}(\mathbb{S}^2) \arrow[r, "\star"] & \Omega^1_\delta(\mathbb{S}^2) \end{tikzcd}$$



$$\hat{\gamma}_{j,t}(\bar{\xi}\,) = \sum_{l,m} \beta_{lm}\, y_{lm}(\bar{\xi})$$

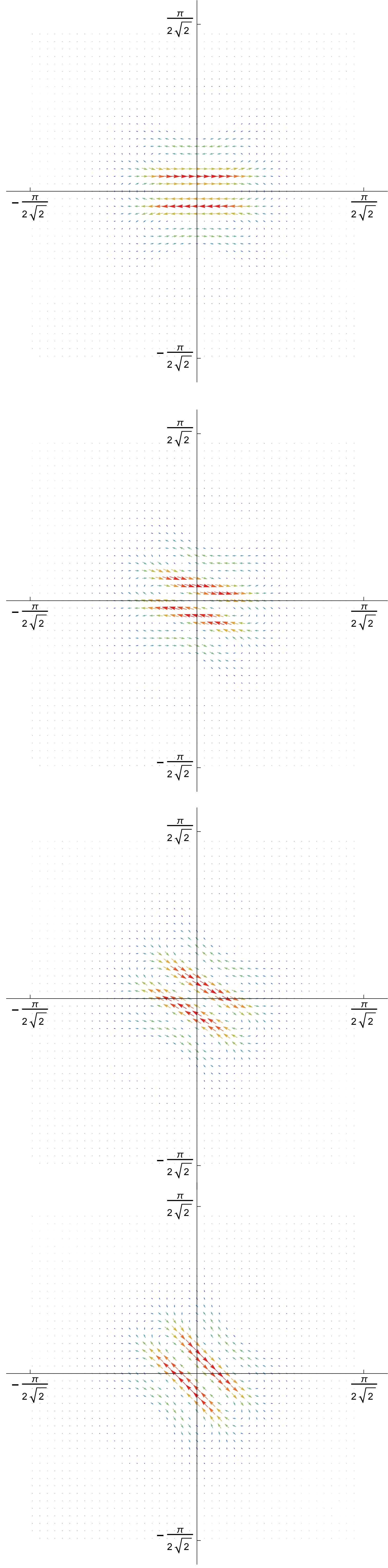
# $\Psi$ec: Local spectral exterior calculus



$$\begin{tikzcd} \Omega^0_\delta(\mathbb{S}^2) \arrow[r, "\mathrm{d}"] & \Omega^1_\mathrm{d}(\mathbb{S}^2) \arrow[r, "\star"] & \Omega^1_\delta(\mathbb{S}^2) \arrow[r, "\mathrm{d}"] & \Omega^2_\mathrm{d}(\mathbb{S}^2) \end{tikzcd}$$

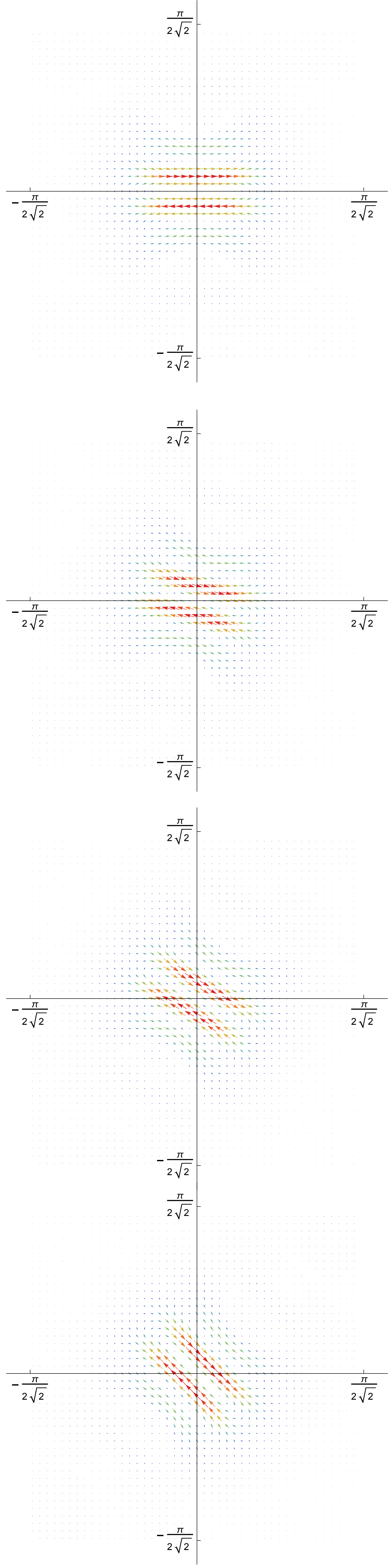
$$\hat{\gamma}_{j,t}(\bar{\xi}) = \sum_{l,m} \beta_{lm}\, y_{lm}(\bar{\xi})$$

# Summary

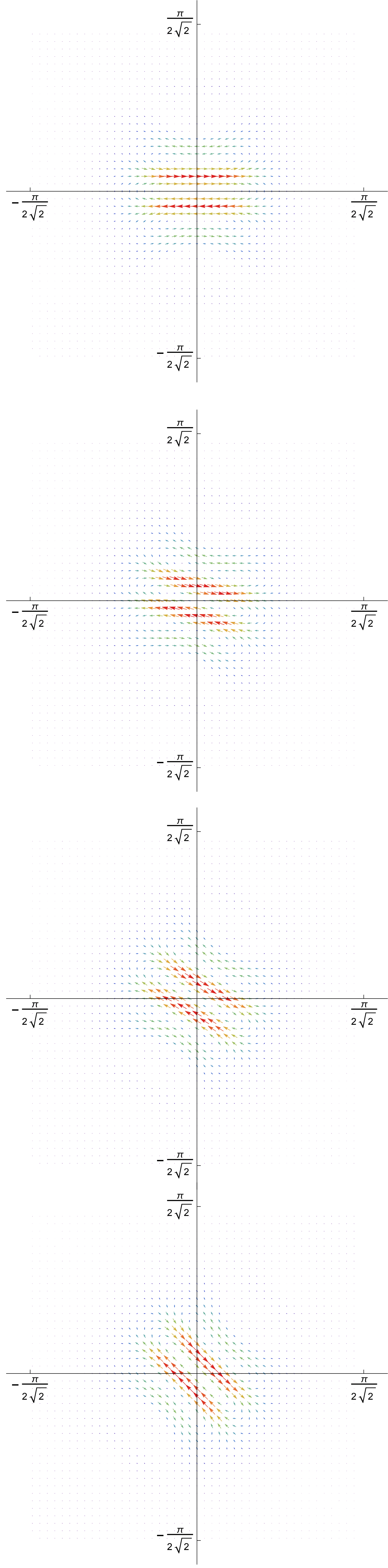


# Summary

- Tight frames of divergence free wavelets
  - Flexible angular localization
  - Closed form expressions in spatial and frequencey domain
  - Quasi optimal approximation properties in 2D
  - Intuitive correspondence to natural flow phenomena

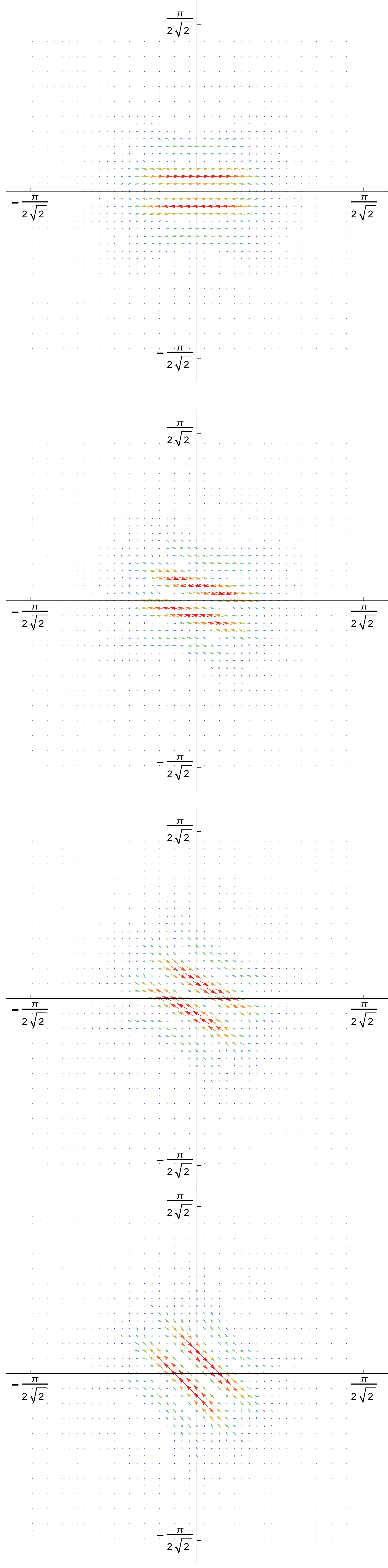


# Open Questions



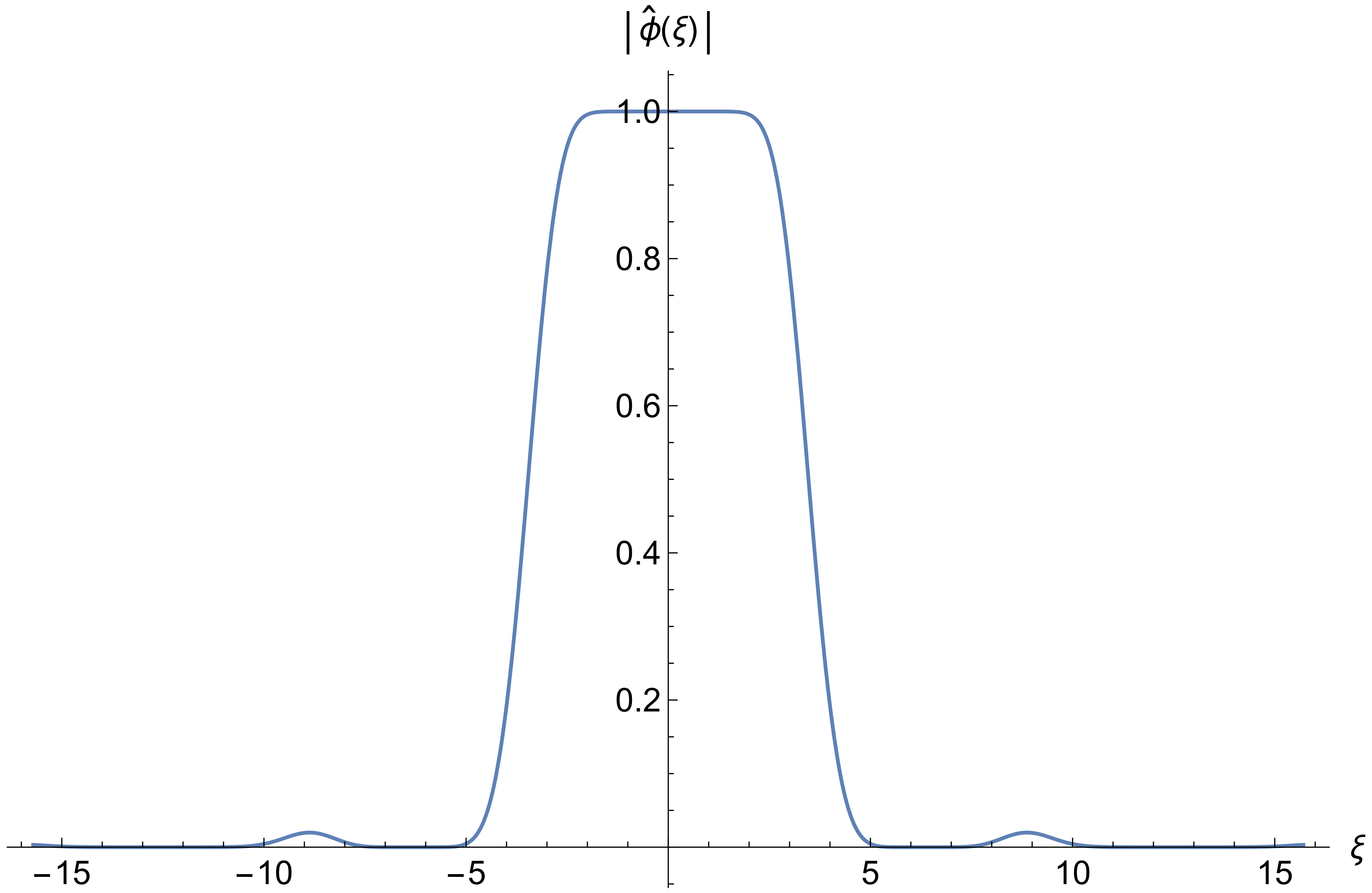
# Open Questions

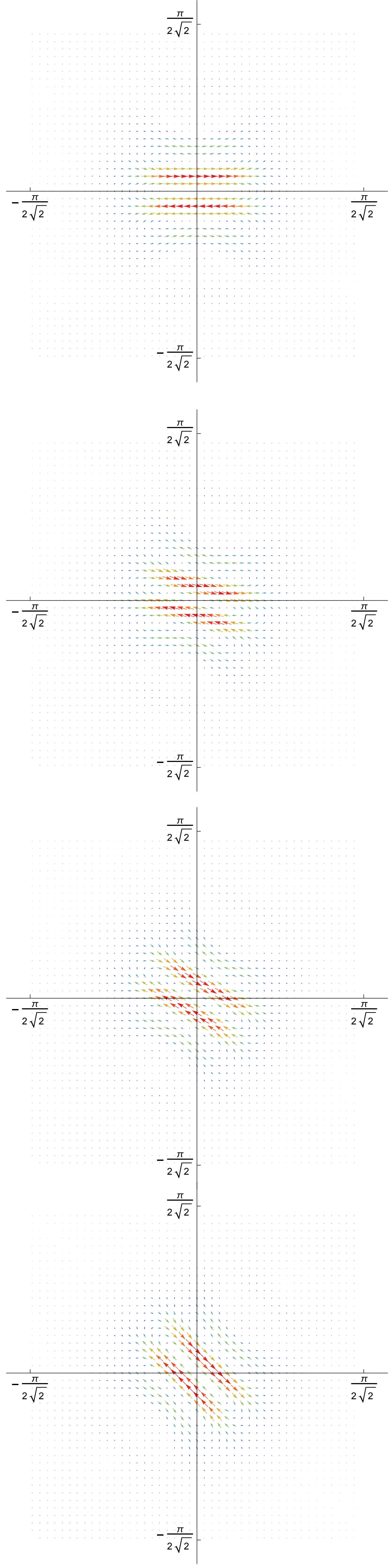
- Compactly supported polar wavelets



# Open Questions

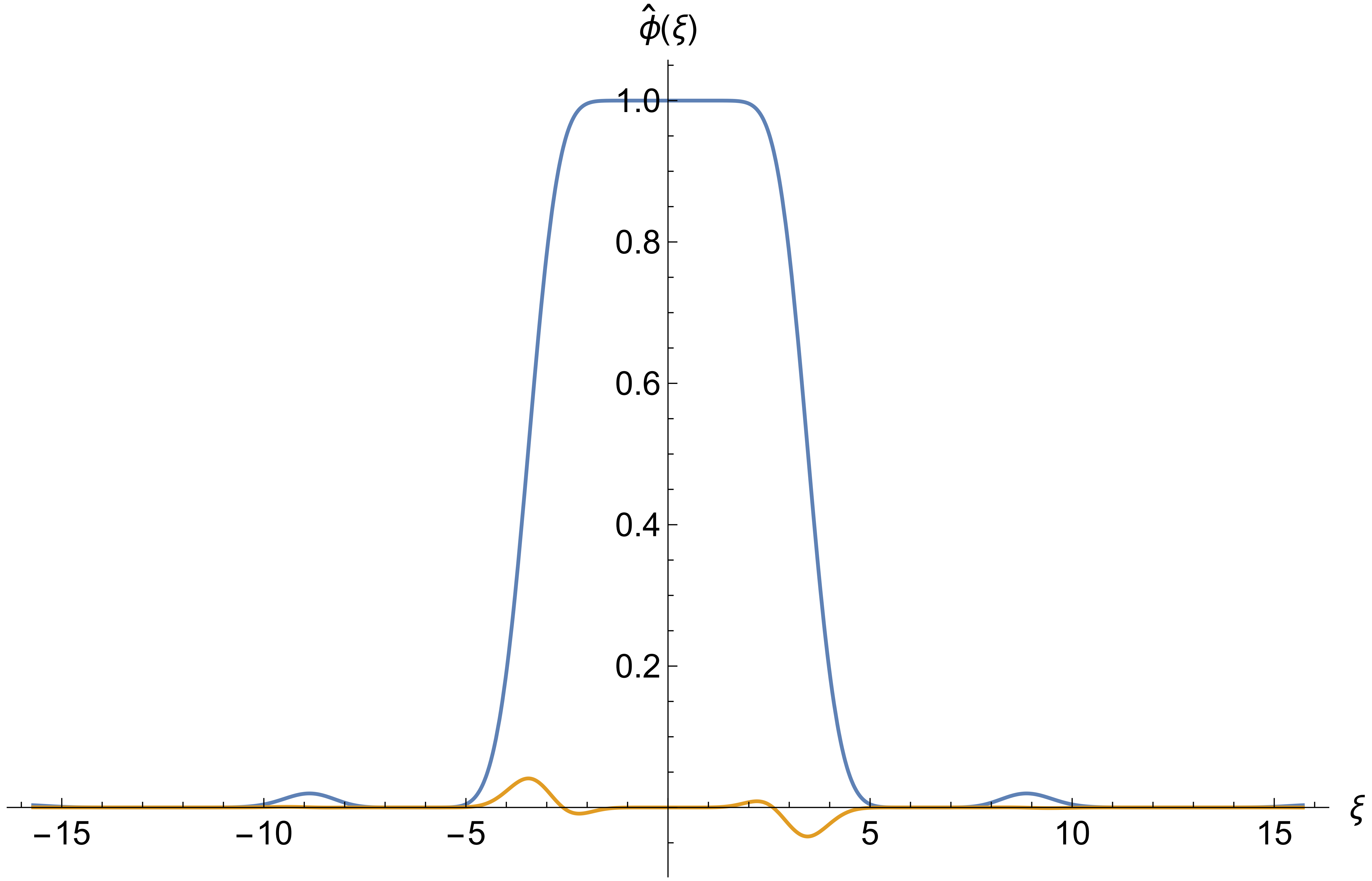
- Compactly supported polar wavelets

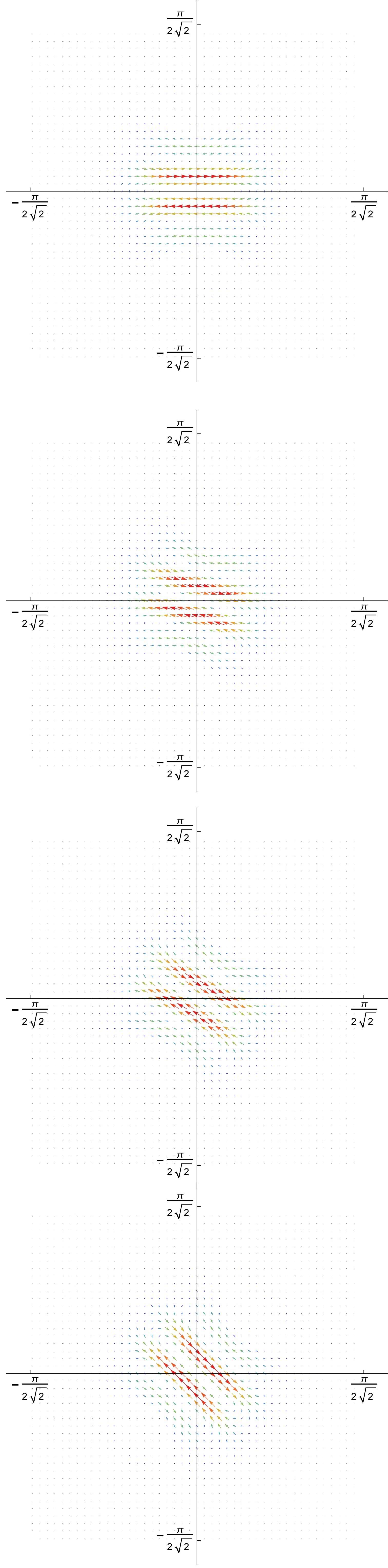




# Open Questions

- Compactly supported polar wavelets





# Open Questions

- Compactly supported polar wavelets
- Boundary layer theory and curvelet-like anisotropic wavelets
- Approximation properties and implementation of 3D div-free wavelets