# Divergence Free Polar Wavelets

Christian Lessig
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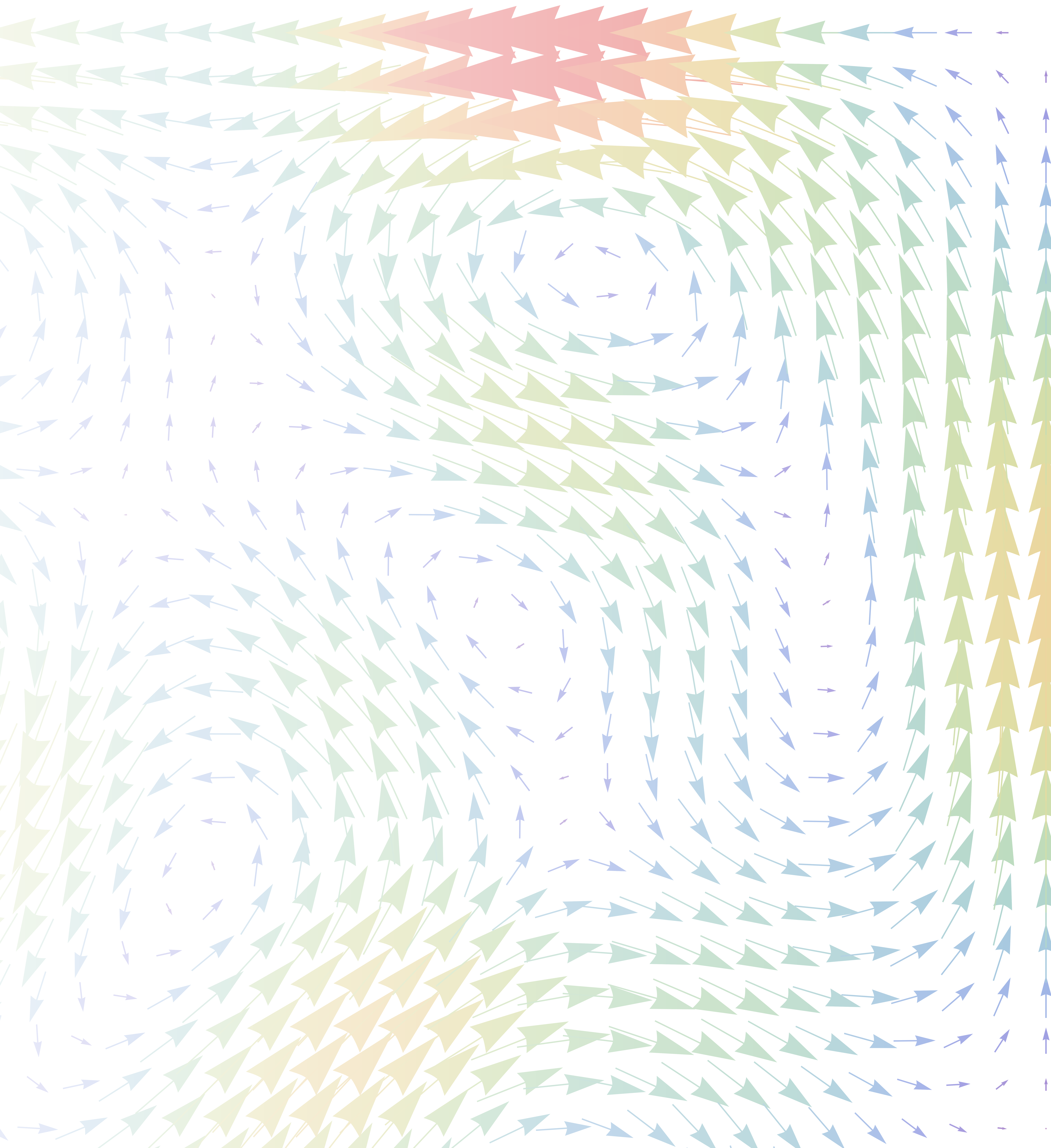
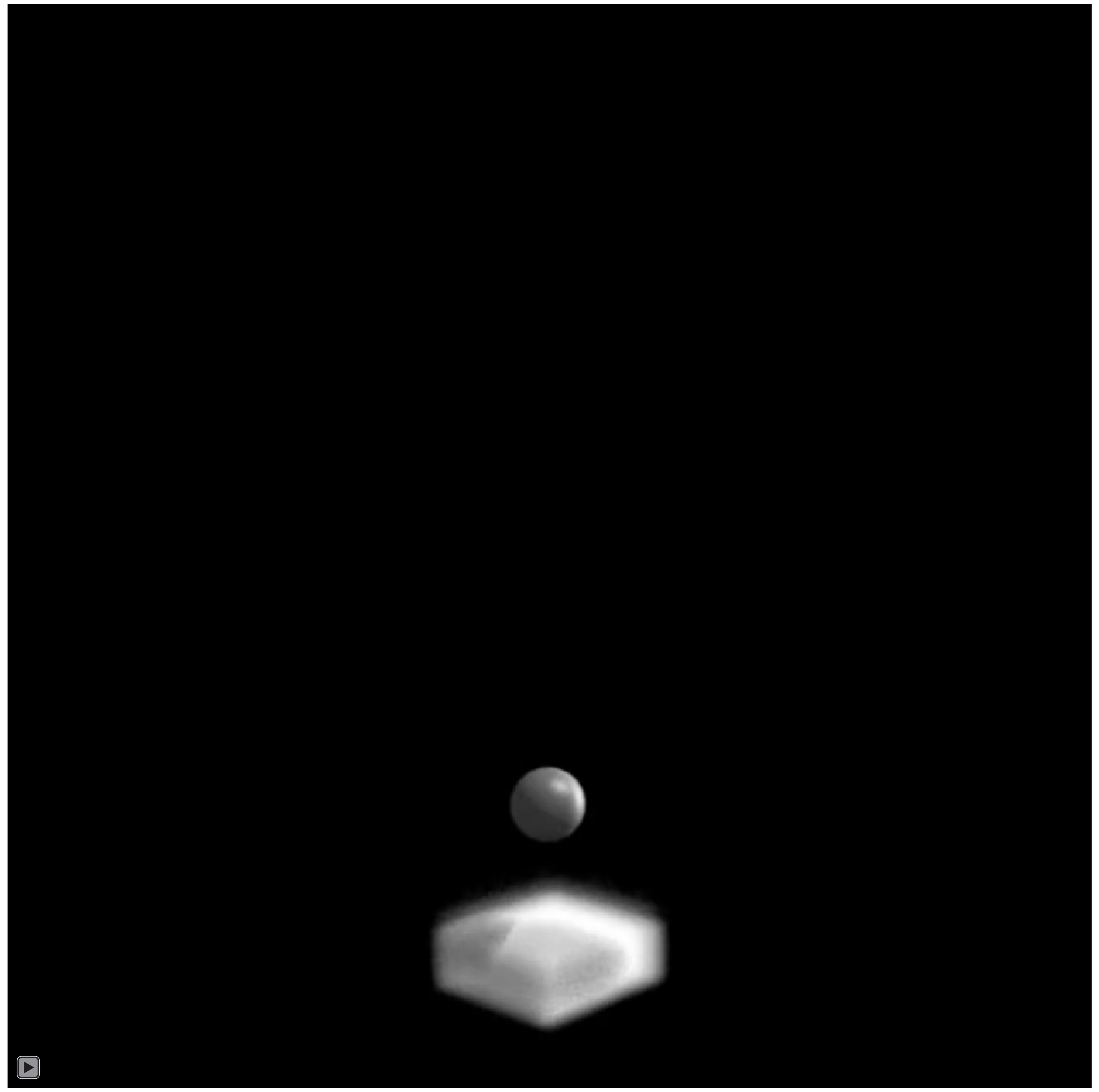


# Motivation



B. Liu, G. Mason, J. Hodgson, Y. Tong, and M. Desbrun, “Model-reduced variational fluid simulation,” ACM Trans. Graph., vol. 34, no. 6, pp. 1–12, Oct. 2015.

# Fluid simulation

- Incompressible Navier-Stokes equations:

$$\frac{\partial \vec{u}}{\partial t} + \mathcal{L}_{\vec{u}} \vec{u} = -\nabla p + \nu \Delta \vec{u}$$

$$\nabla \cdot \vec{u} = 0$$

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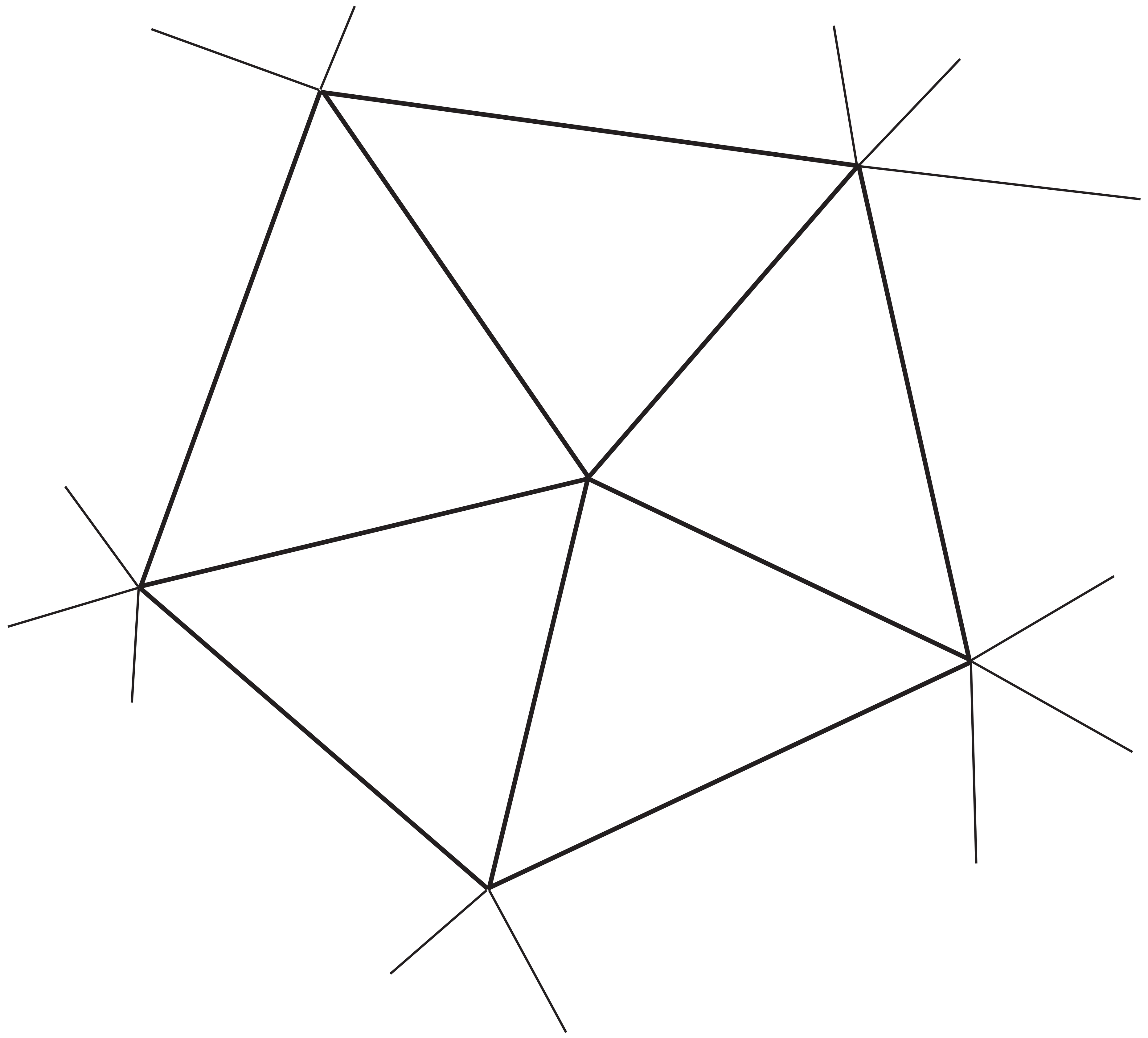
# Divergence freedom

- Finite element approach:



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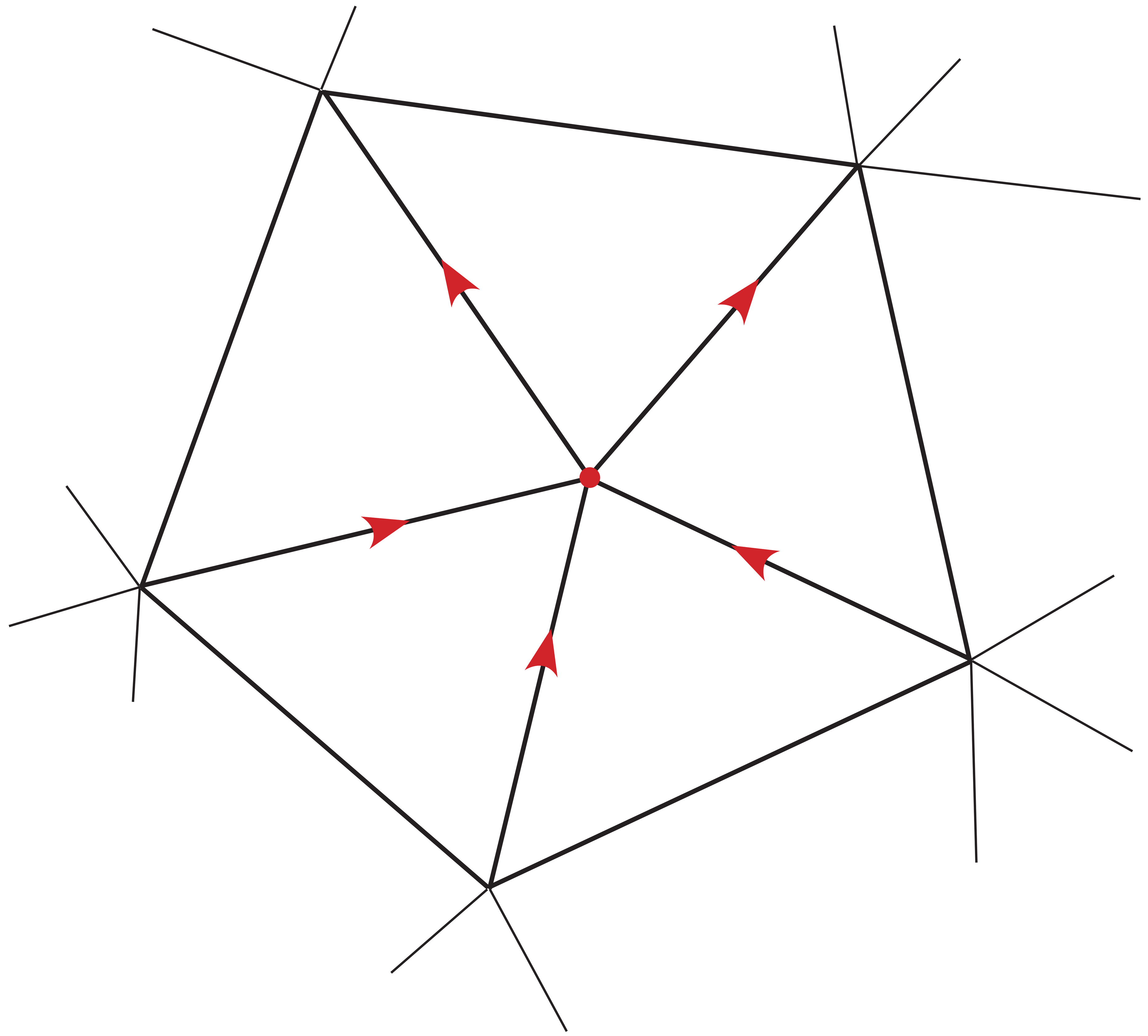

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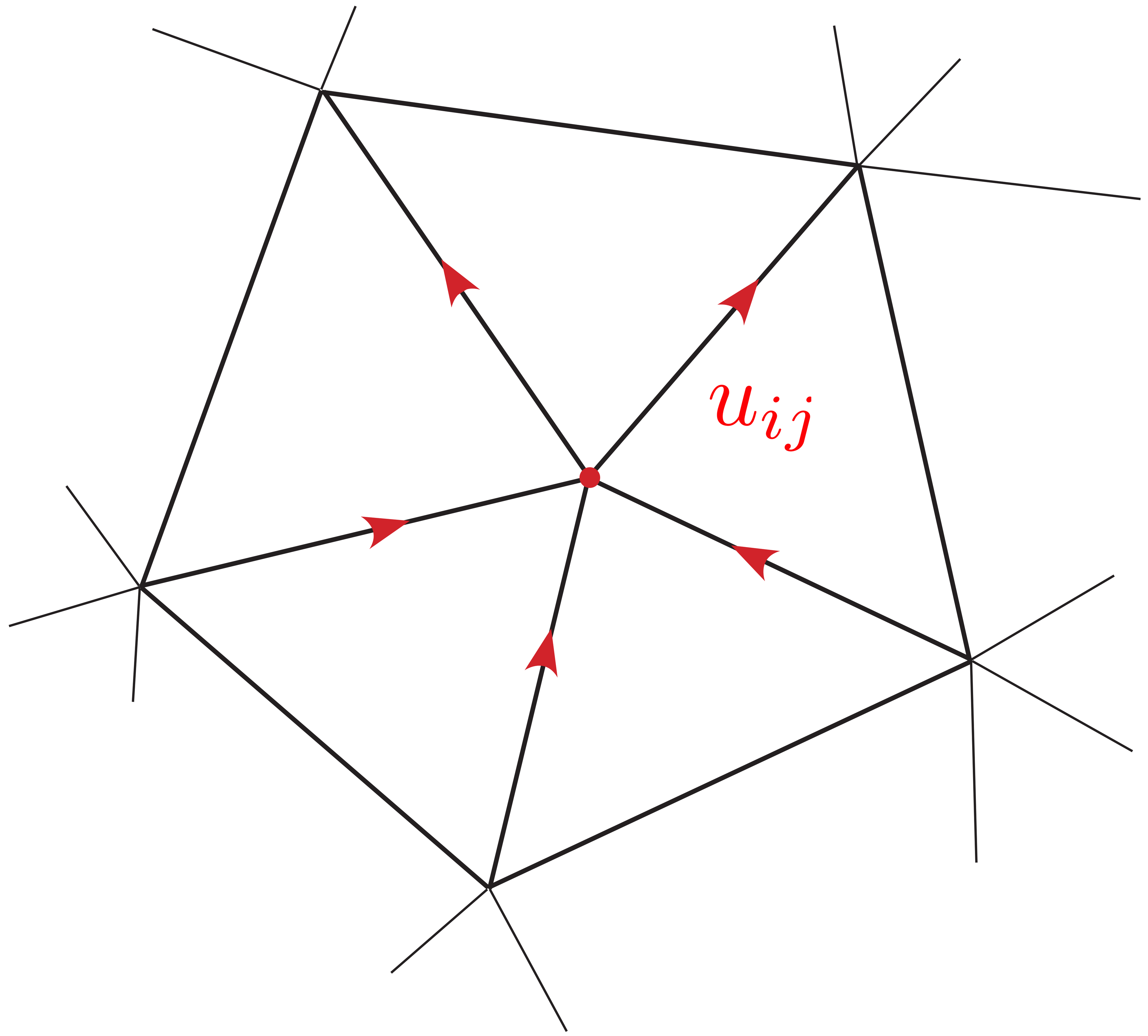

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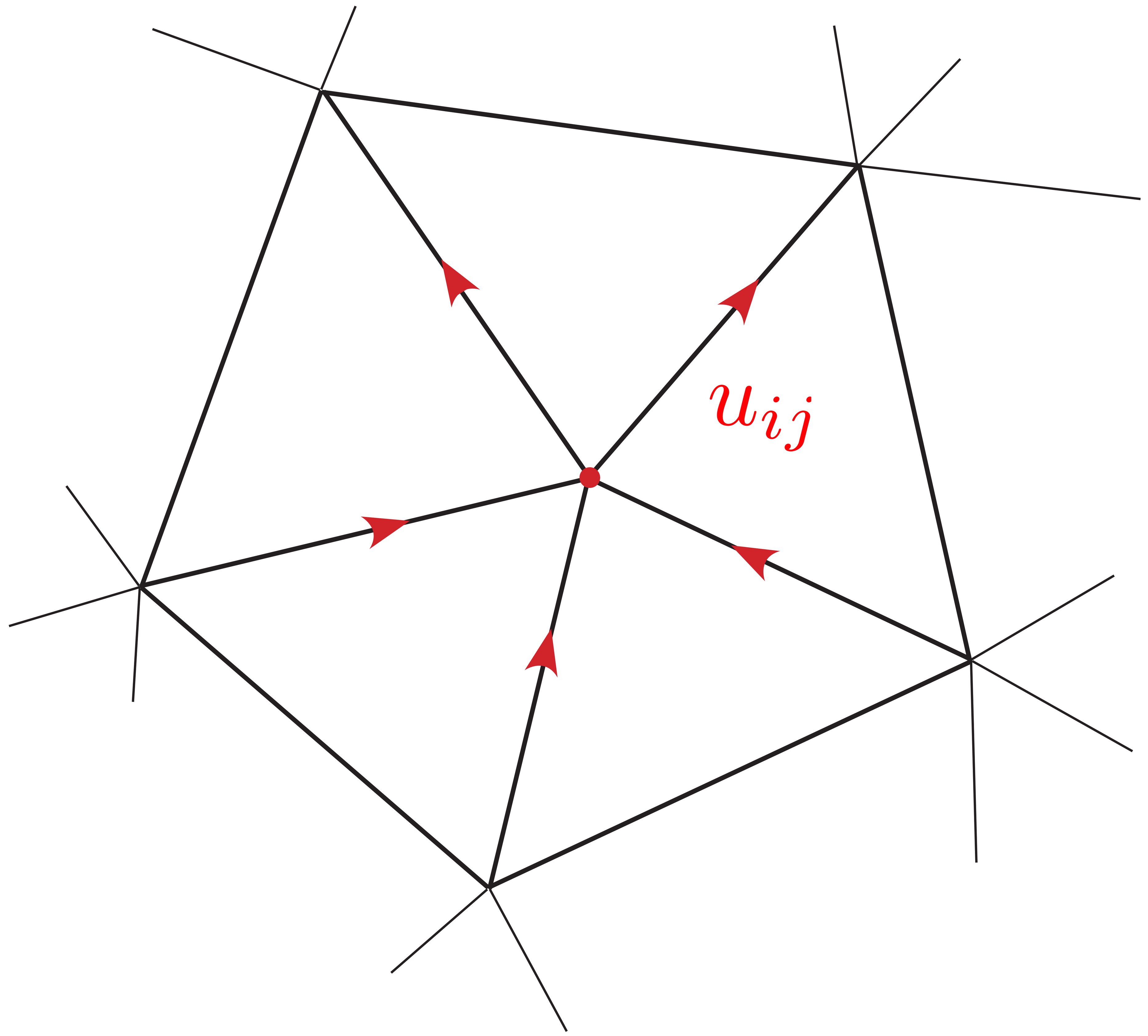
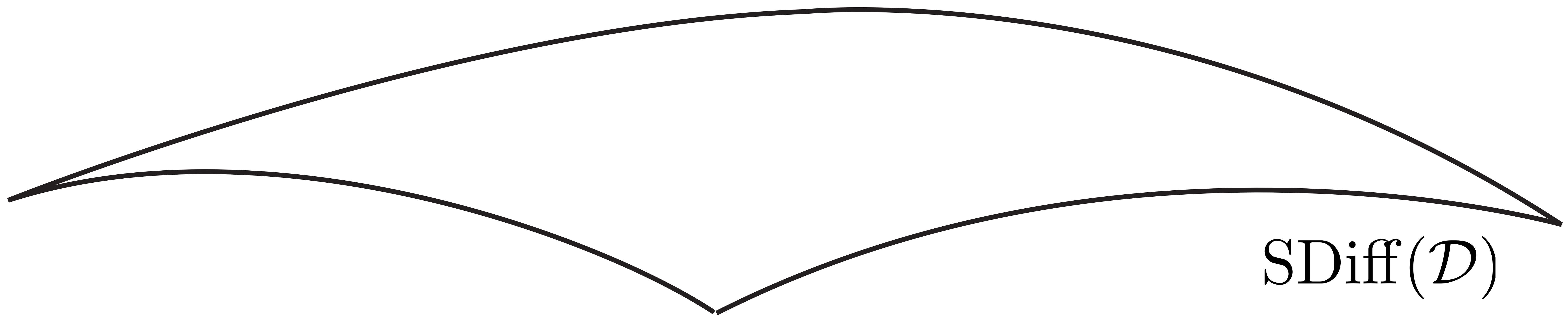



$$\sum_j u_{ij} = 0 \ , \ \forall i$$

# Divergence freedom

- Time integration:



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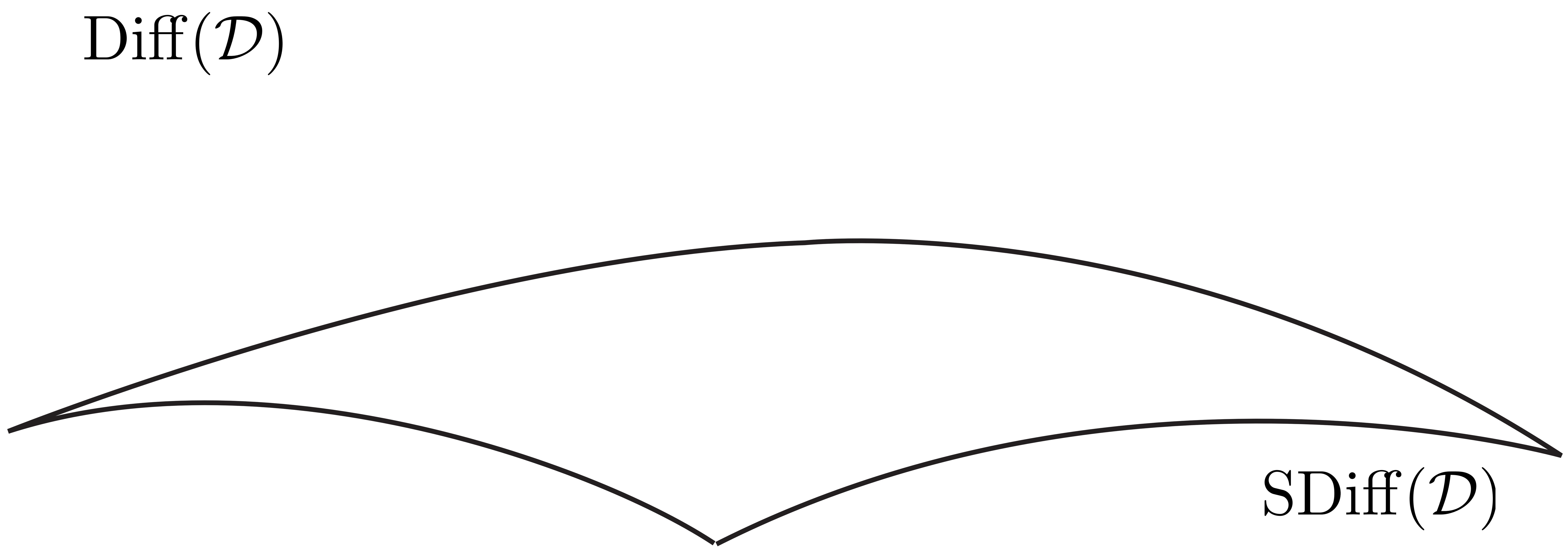

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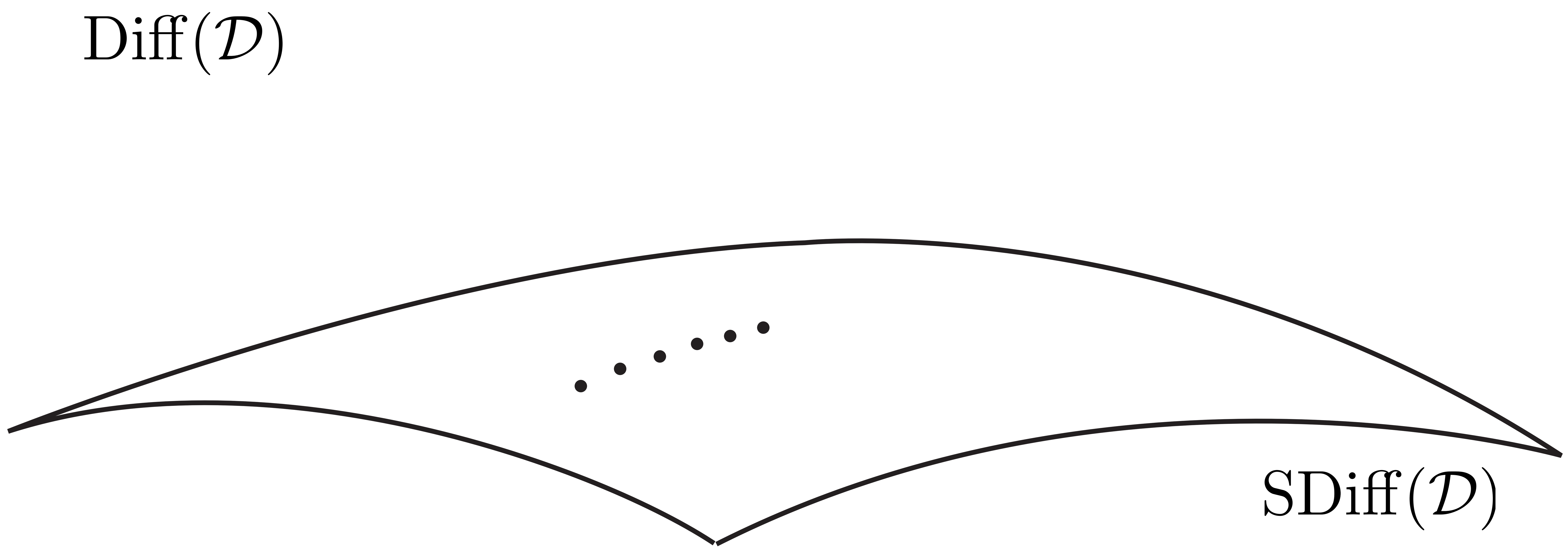

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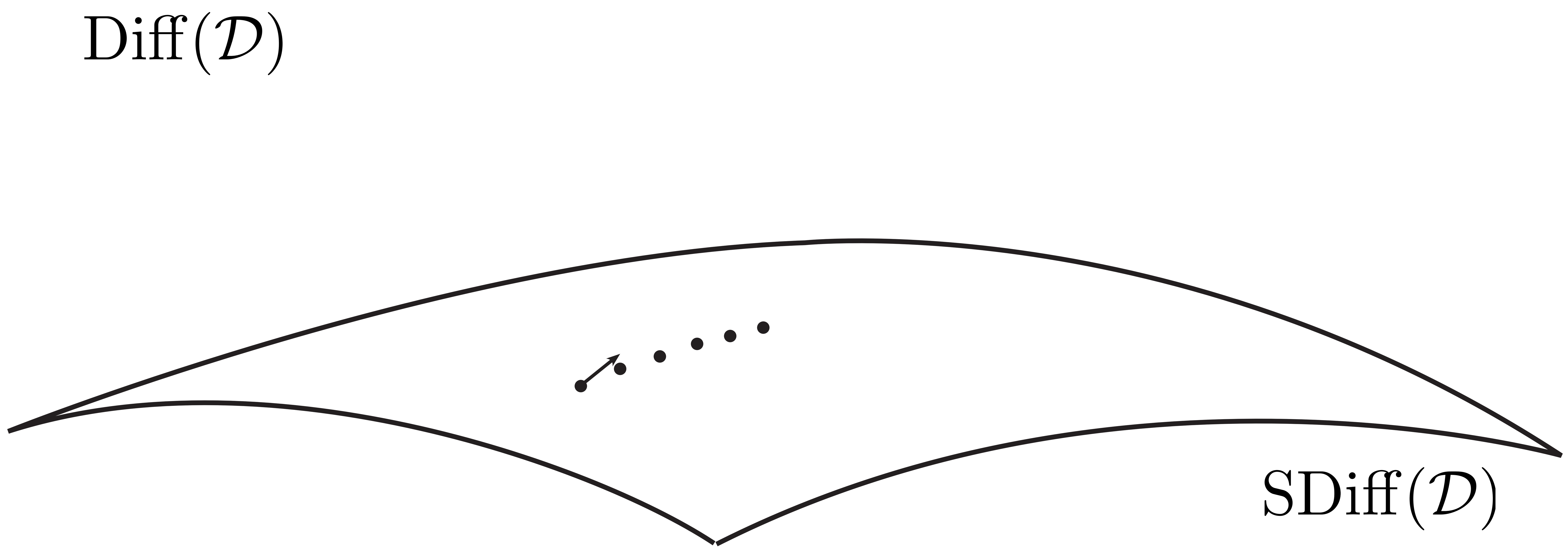

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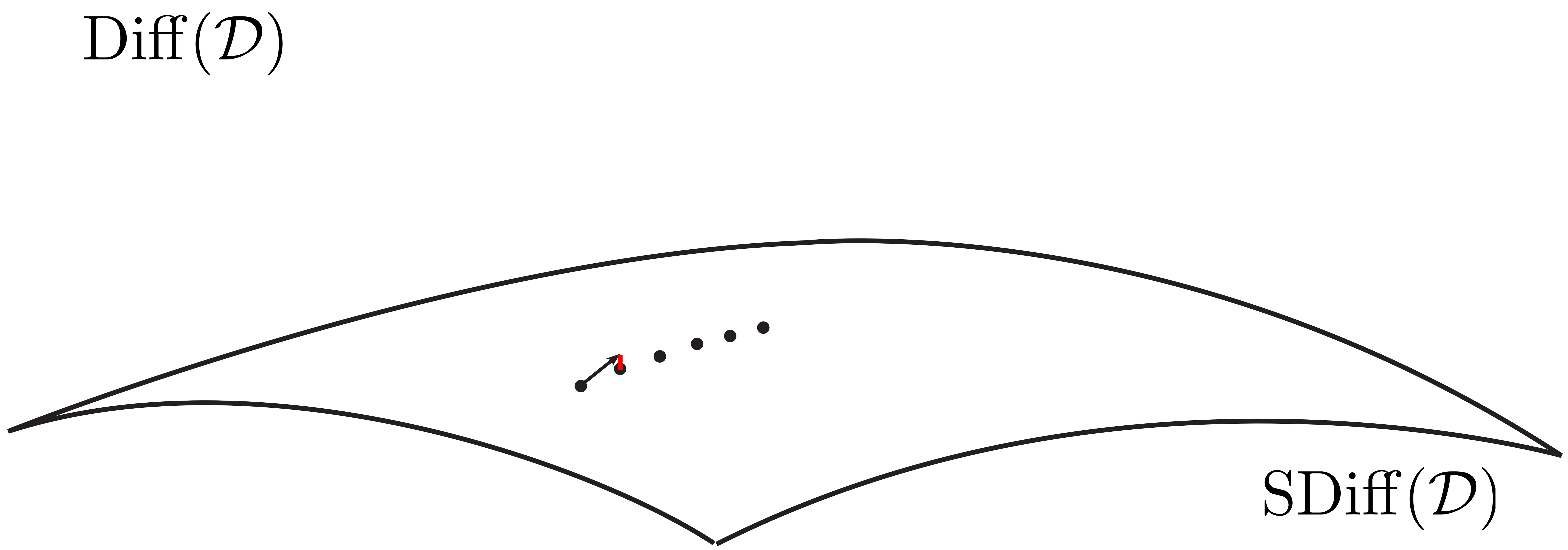

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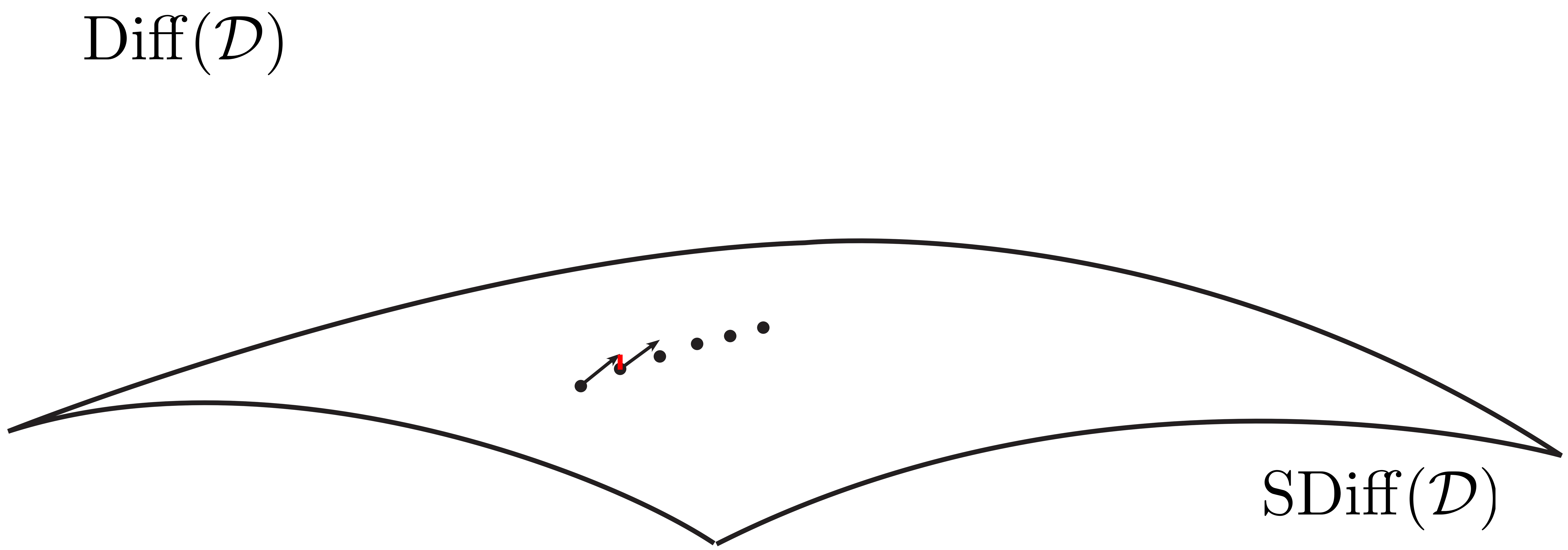

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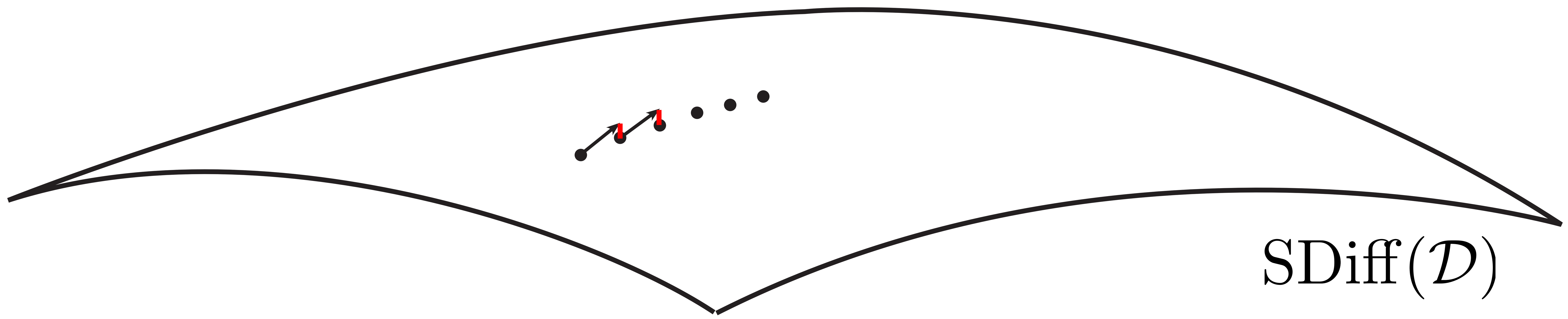
$\mathrm{Diff}(\mathcal{D})$



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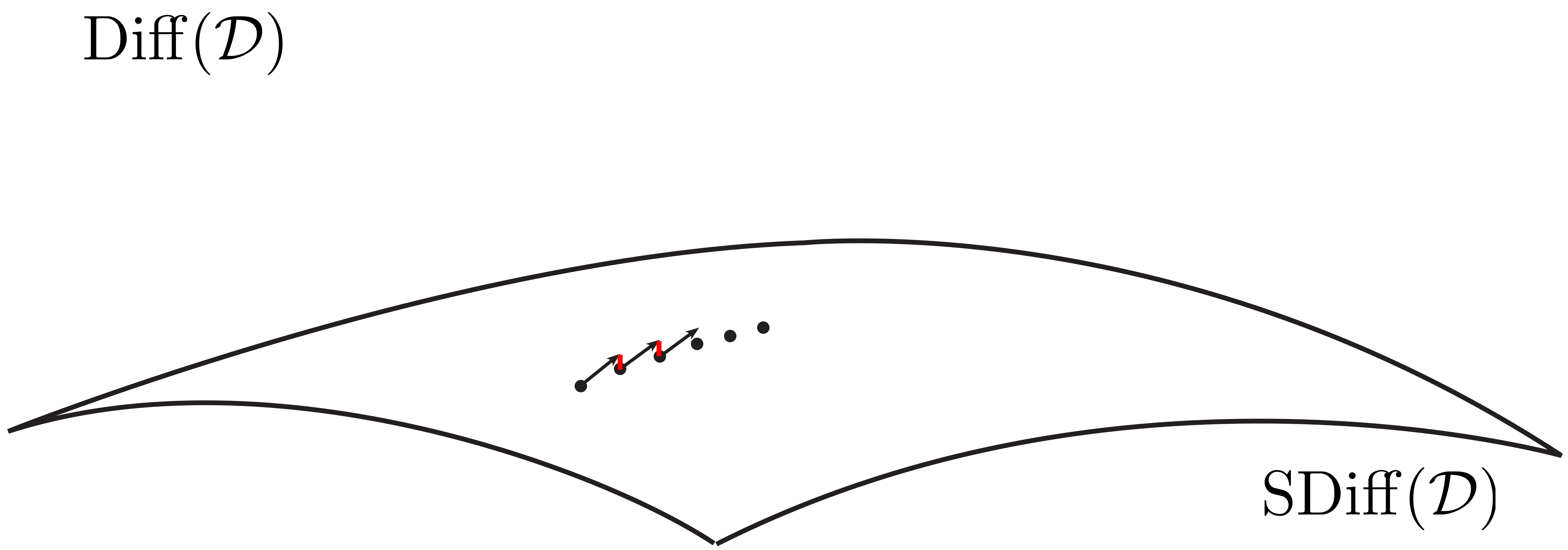

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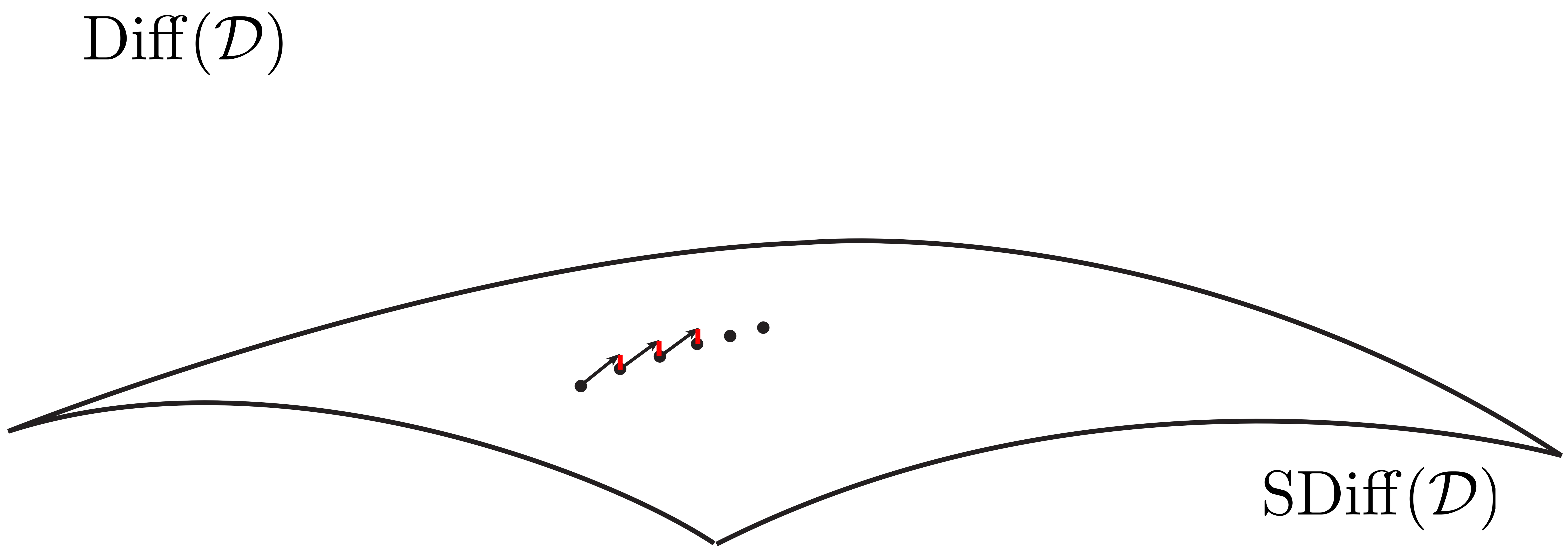

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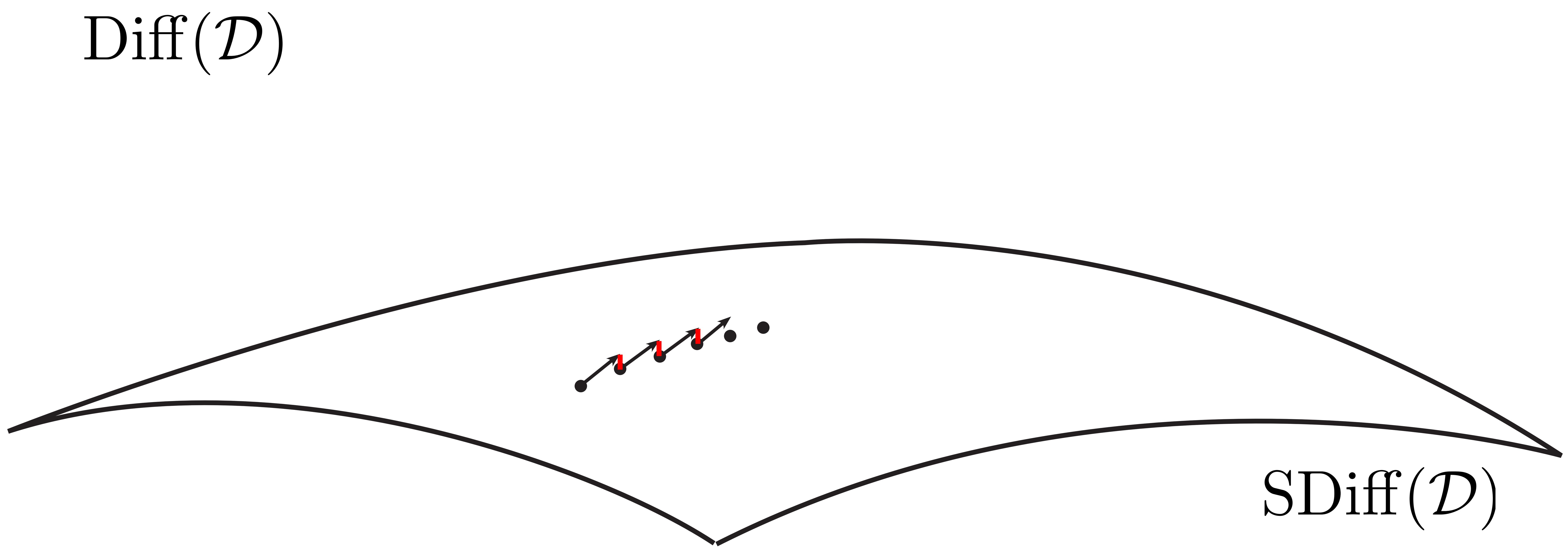

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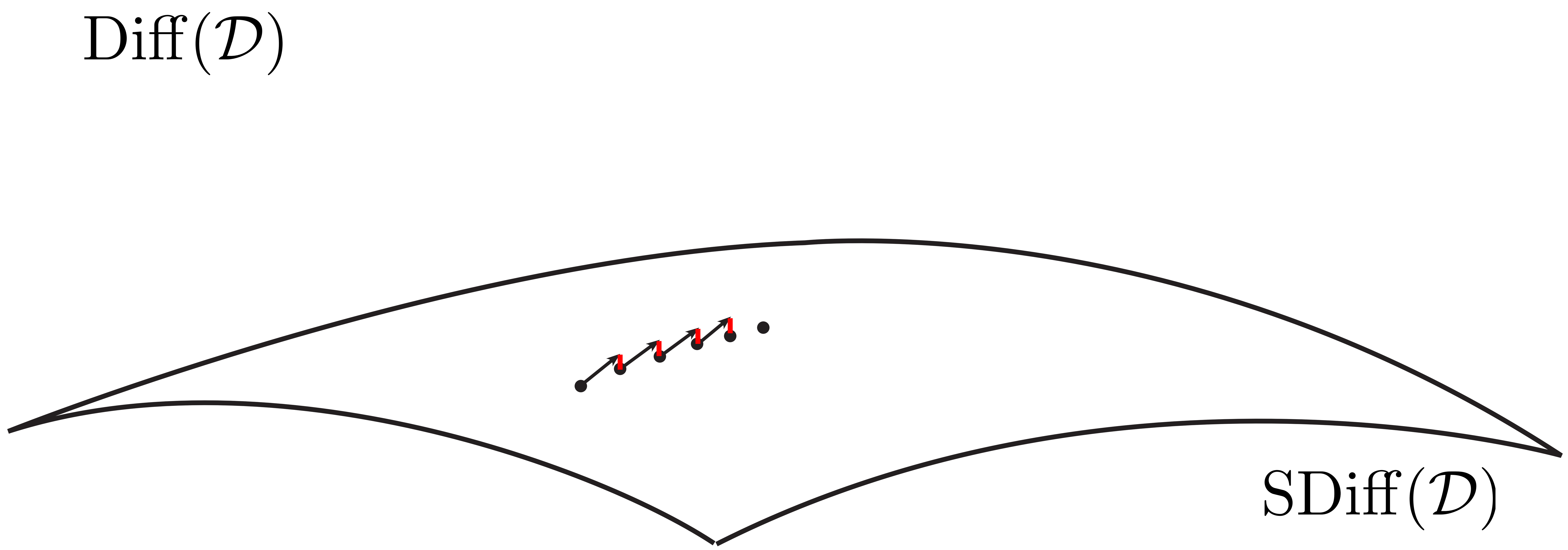

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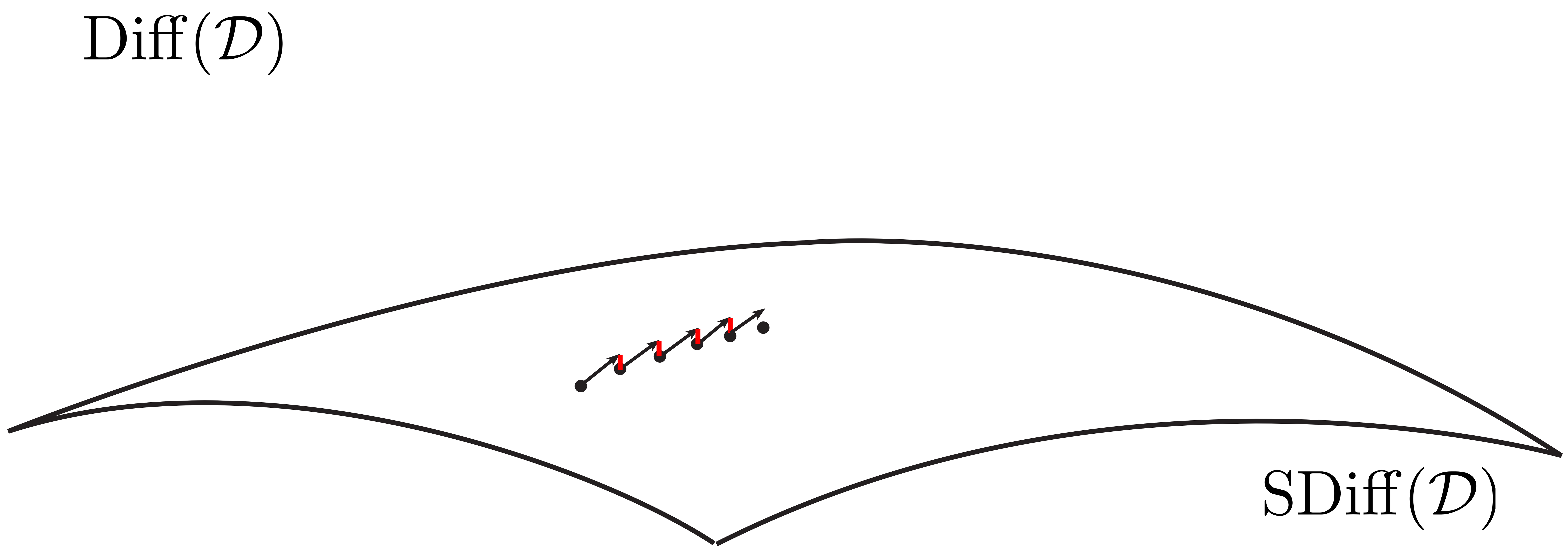

# Fluid simulation

- Incompressible Navier-Stokes equations:

$$\frac{\partial \vec{u}}{\partial t} + \mathcal{L}_{\vec{u}}\vec{u} = -\nabla p + \nu \Delta \vec{u}$$

$$\boxed{\nabla \cdot \vec{u} = 0}$$

needs to be re-enforced
after every time step

# Fluid simulation

- Incompressible Navier-Stokes equations in vorticity form:

$$\frac{\partial \omega}{\partial t} + \mathcal{L}_{\vec{u}}\, \omega = \nu \Delta \omega$$

$$\nabla \times \vec{u} = \omega$$

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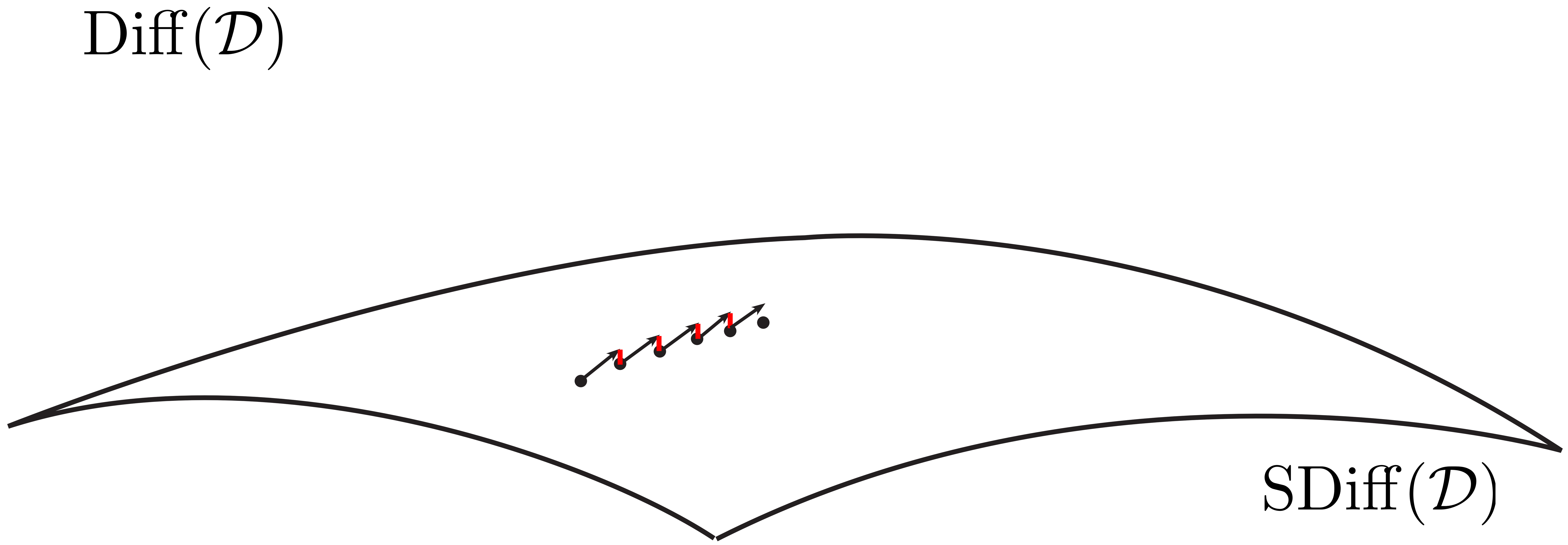
Enforces much of the intrinsic structure of imcompressible fluids

# Divergence freedom

- Time integration:



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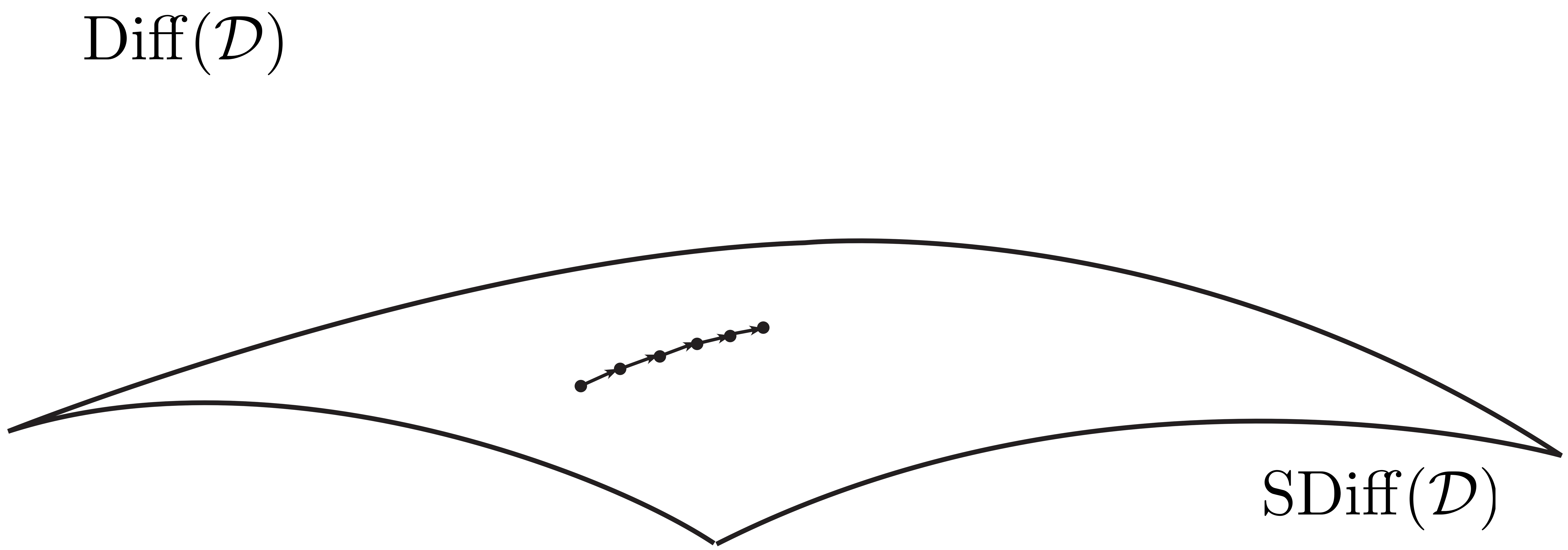

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$$\vec{\Delta}\,\vec{\phi}_i = \lambda_i\,\vec{\phi}_i$$

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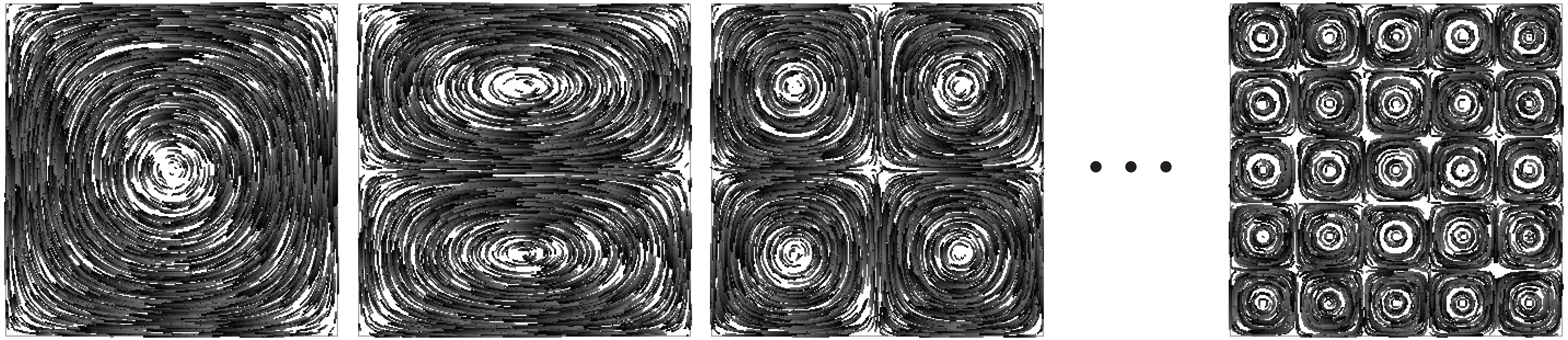
  – Analytic form for simple geometries (square, disk, …)
  – Numerical computation for meshed domains

# Fluid simulation

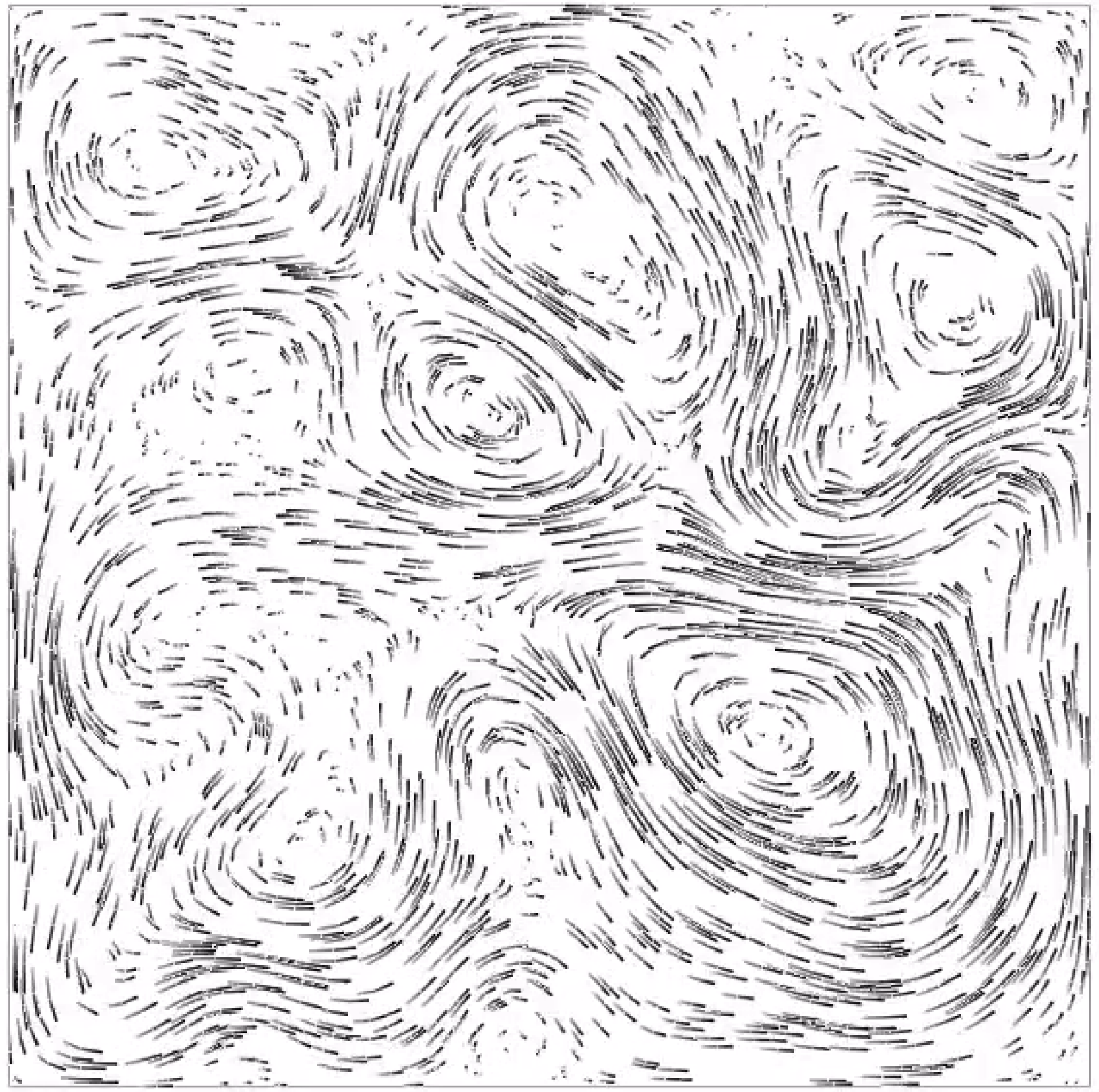
- Divergence free basis functions for unit square:



from T. de Witt, C. Lessig, and E. Fiume, "Fluid Simulation Using Laplacian Eigenfunctions," ACM Trans. Graph., vol. 31, no. 1, pp. 1–11, Jan. 2012.

# Fluid simulation

◦ Fluid simulation using Laplacian eigenfunctions:



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- Fluid simulation using Laplacian eigenfunctions:
  - + Analytical divergence freedom
  - + Efficient and simple (no pressure projection)
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- Fluid simulation using Laplacian eigenfunctions:
  - + Analytical divergence freedom
  - + Efficient and simple (no pressure projection)
  - + Plausible flows with few degrees of freedom
  - – Fixed geometry
  - – Global support of basis functions (e.g. difficult to resolve turbulent details)

How to construct a local divergence free representation for velocity?

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(that preserves all the other advantages)

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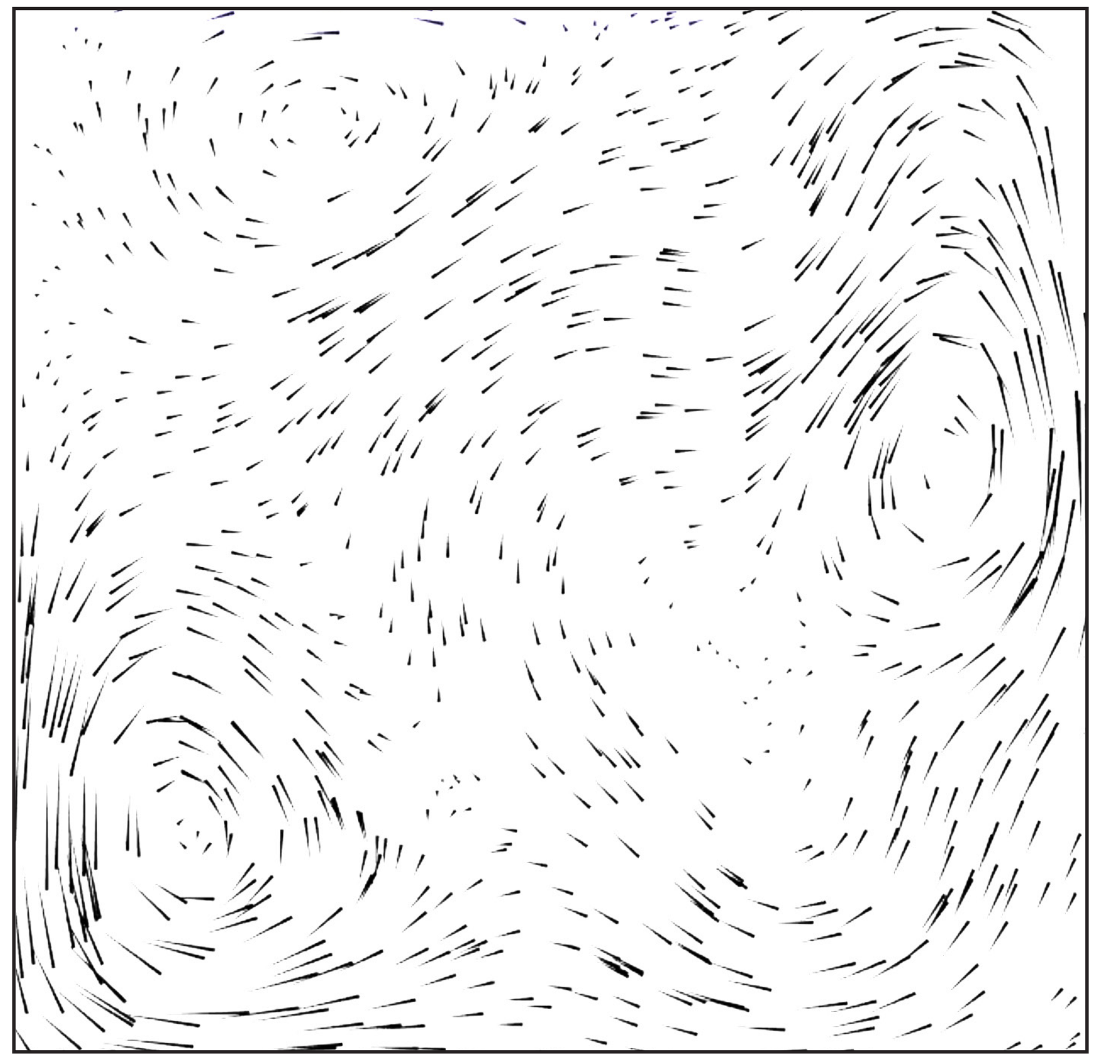
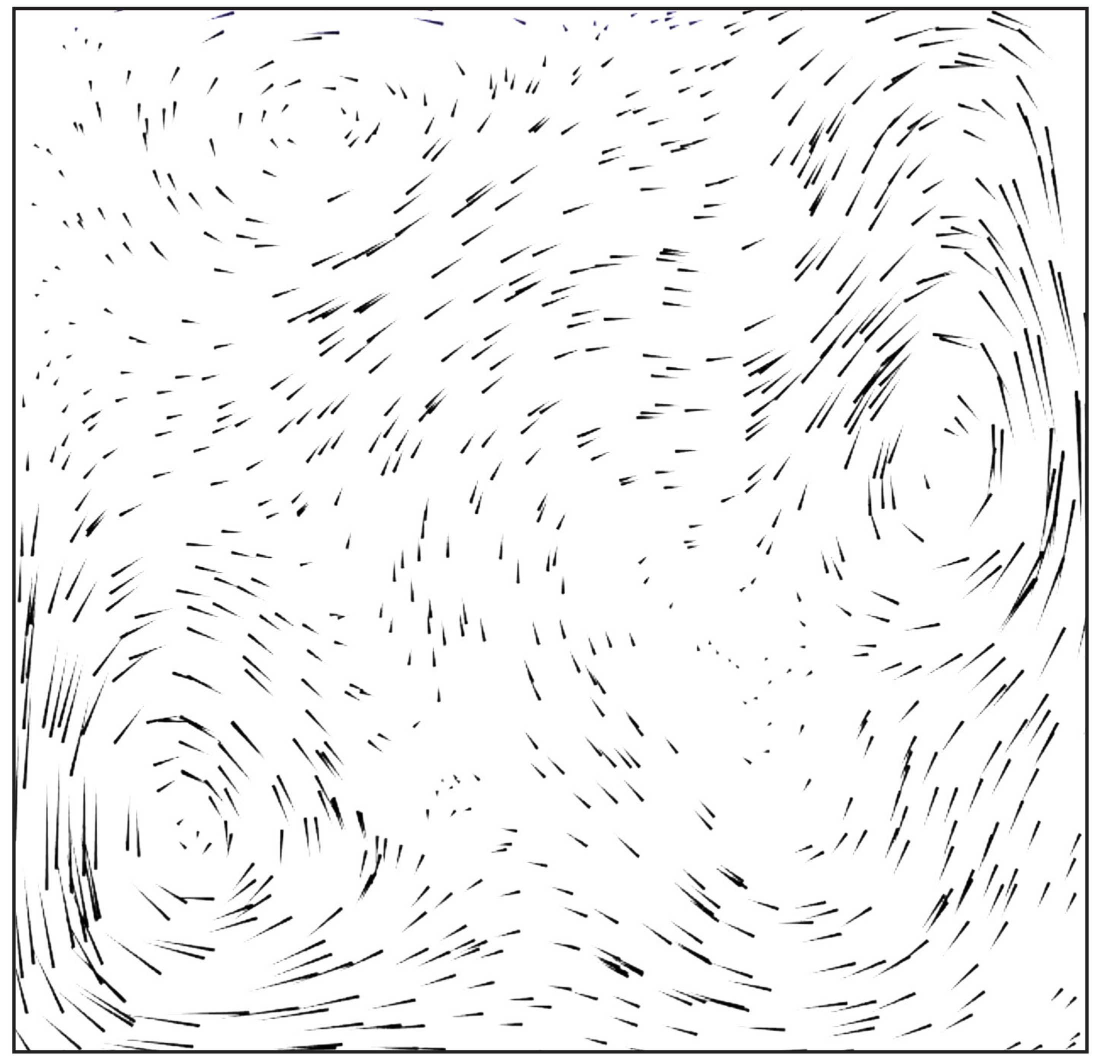


# Divergence freedom

$$\nabla \cdot \vec{u} = 0 \xrightarrow[\text{transform}]{\text{Fourier}} \vec{\xi} \cdot \hat{\vec{u}} = 0$$



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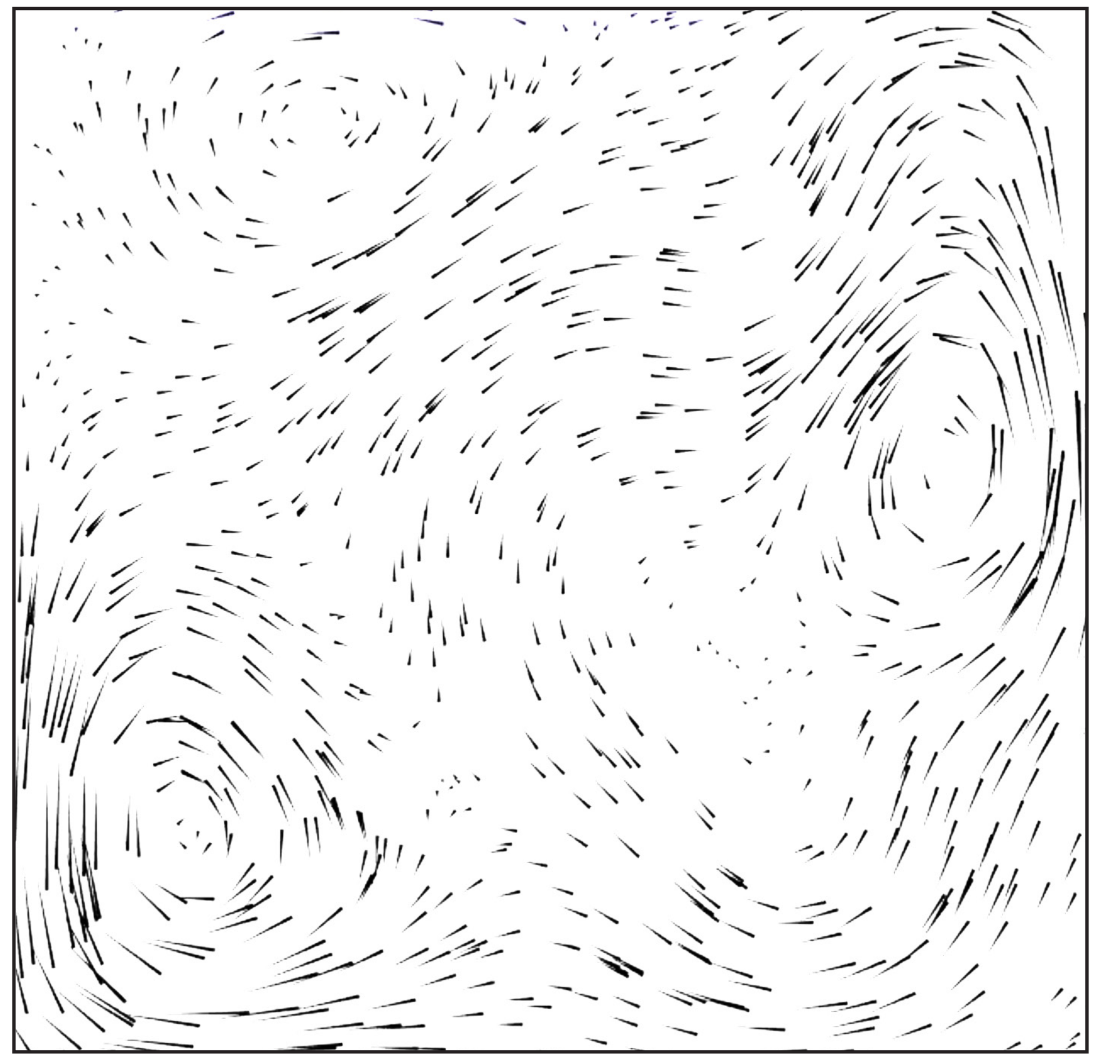

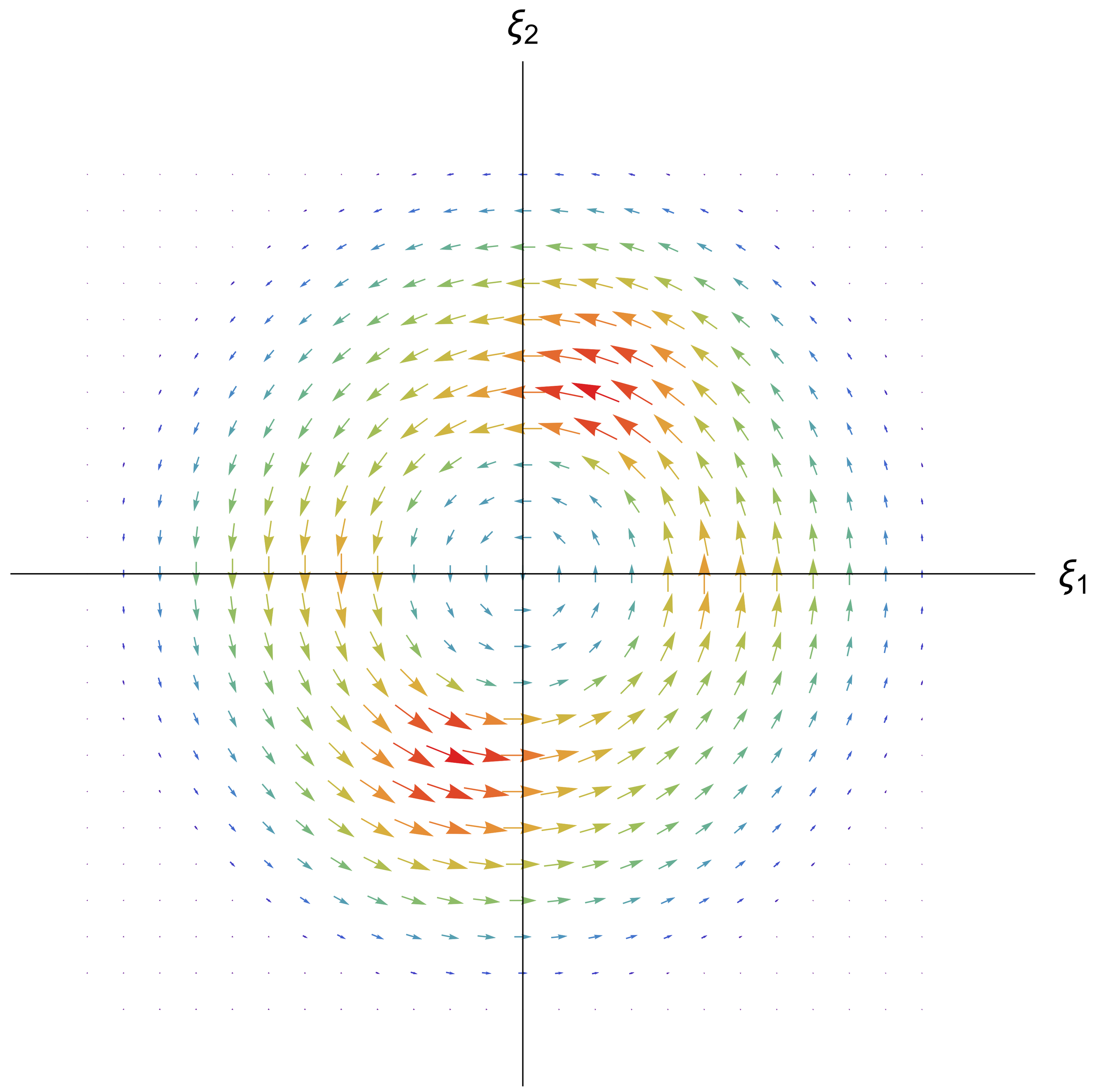

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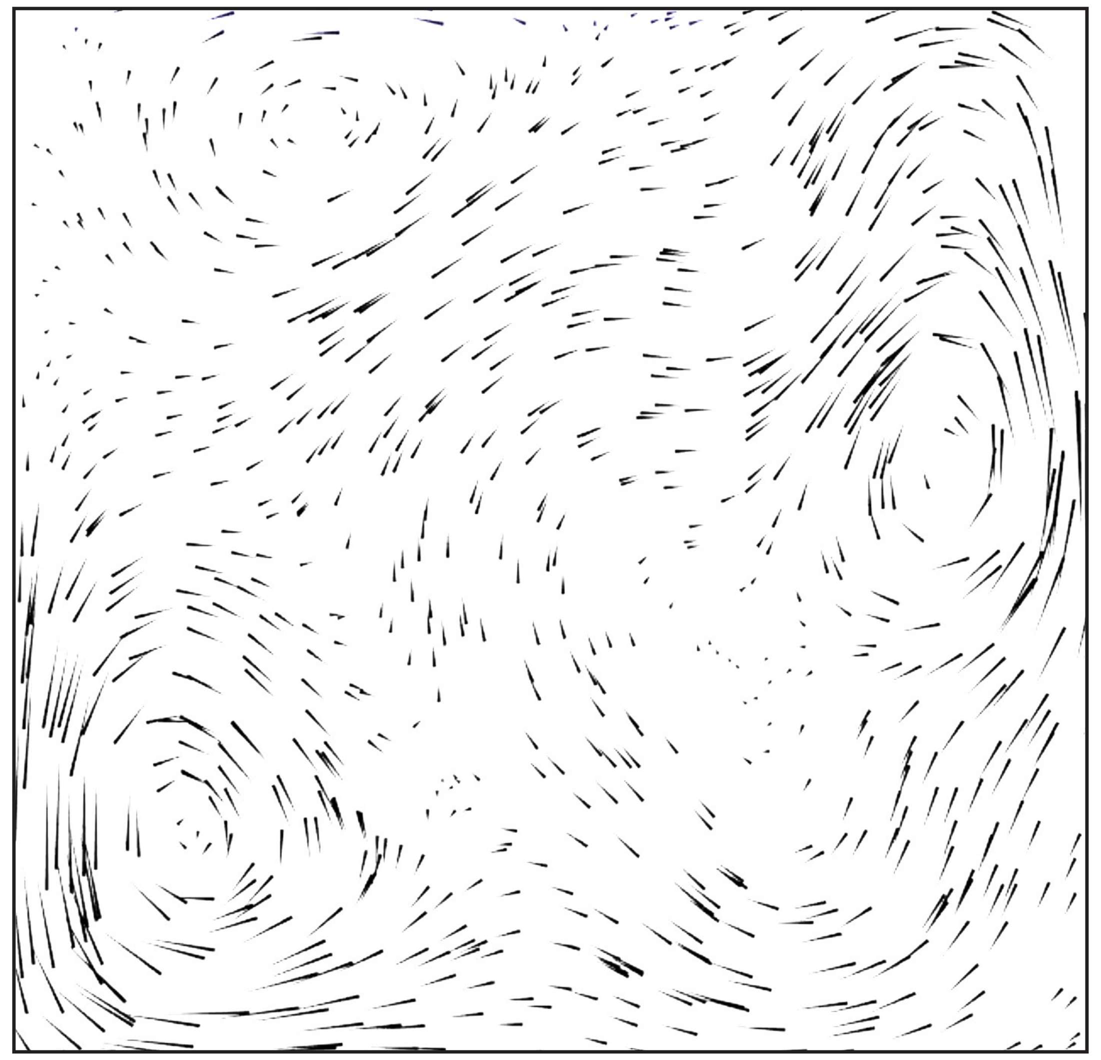

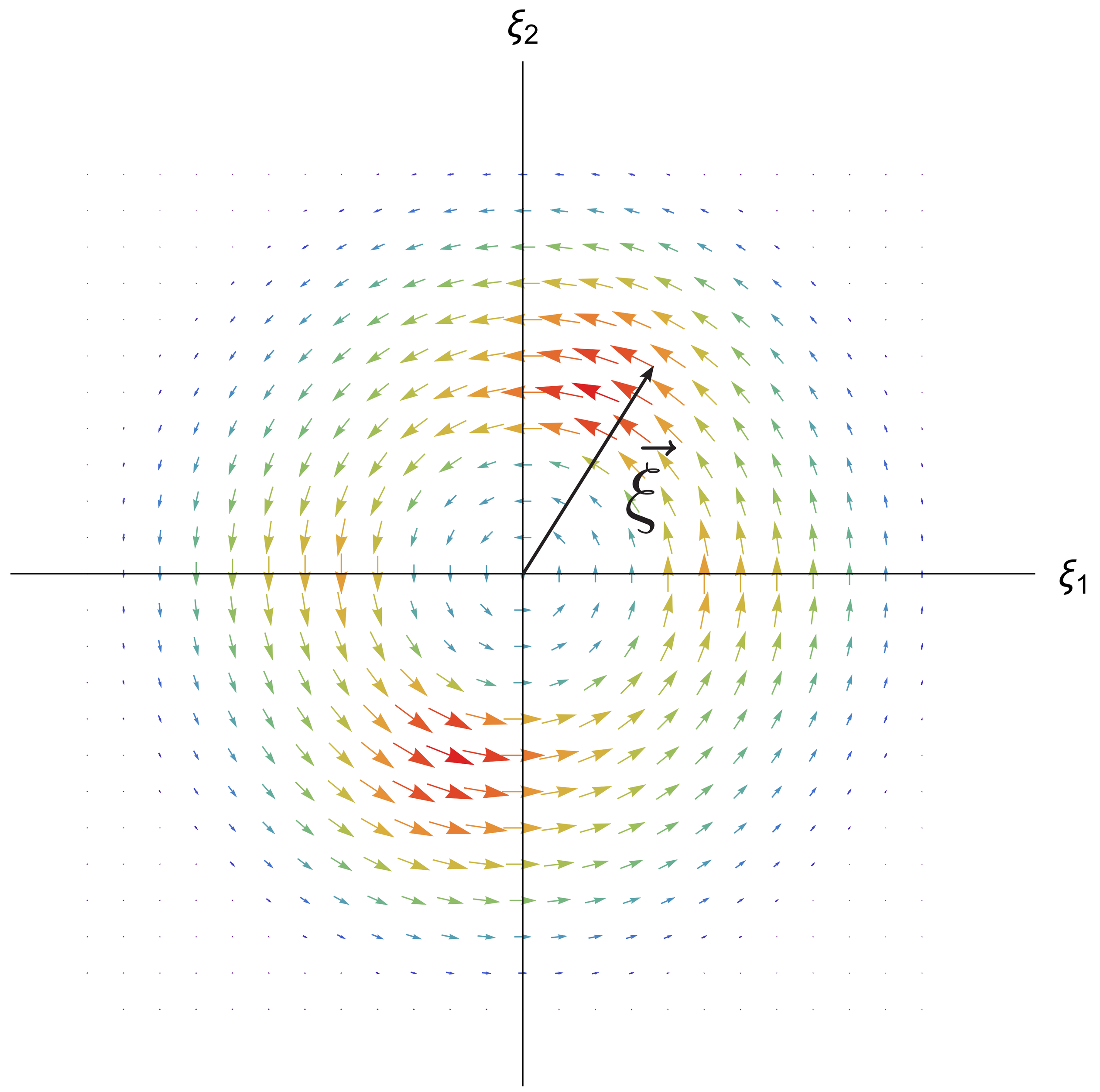

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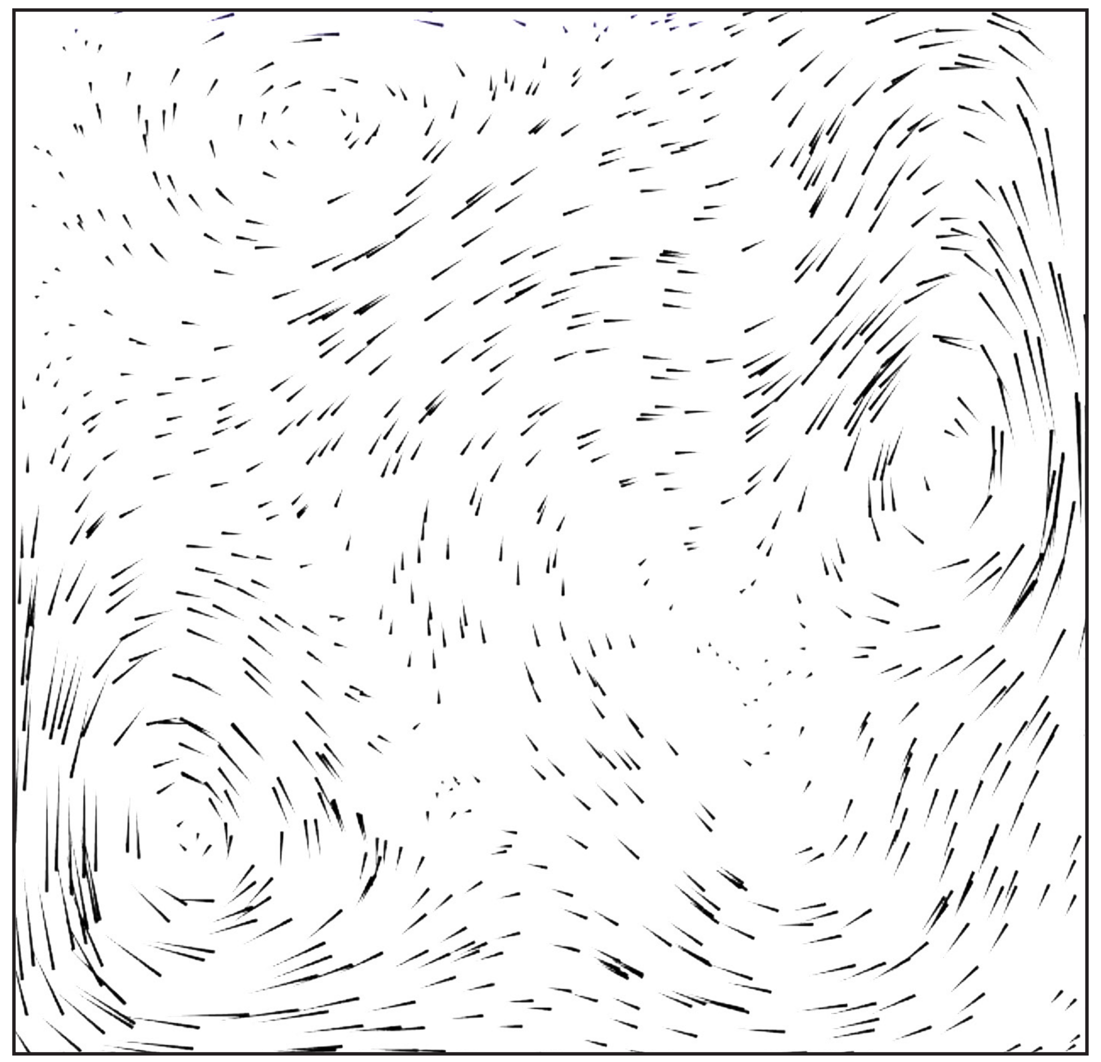

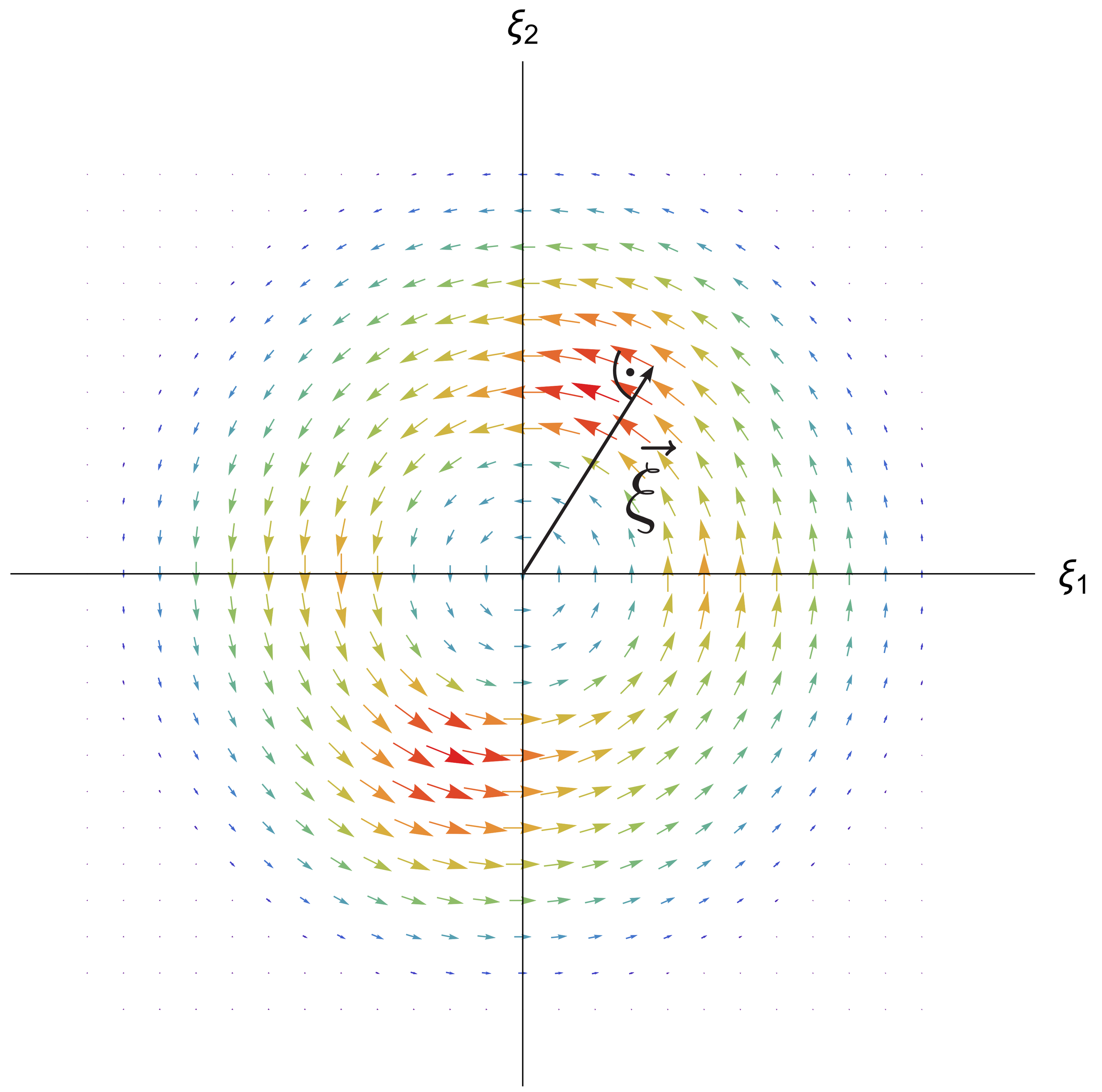

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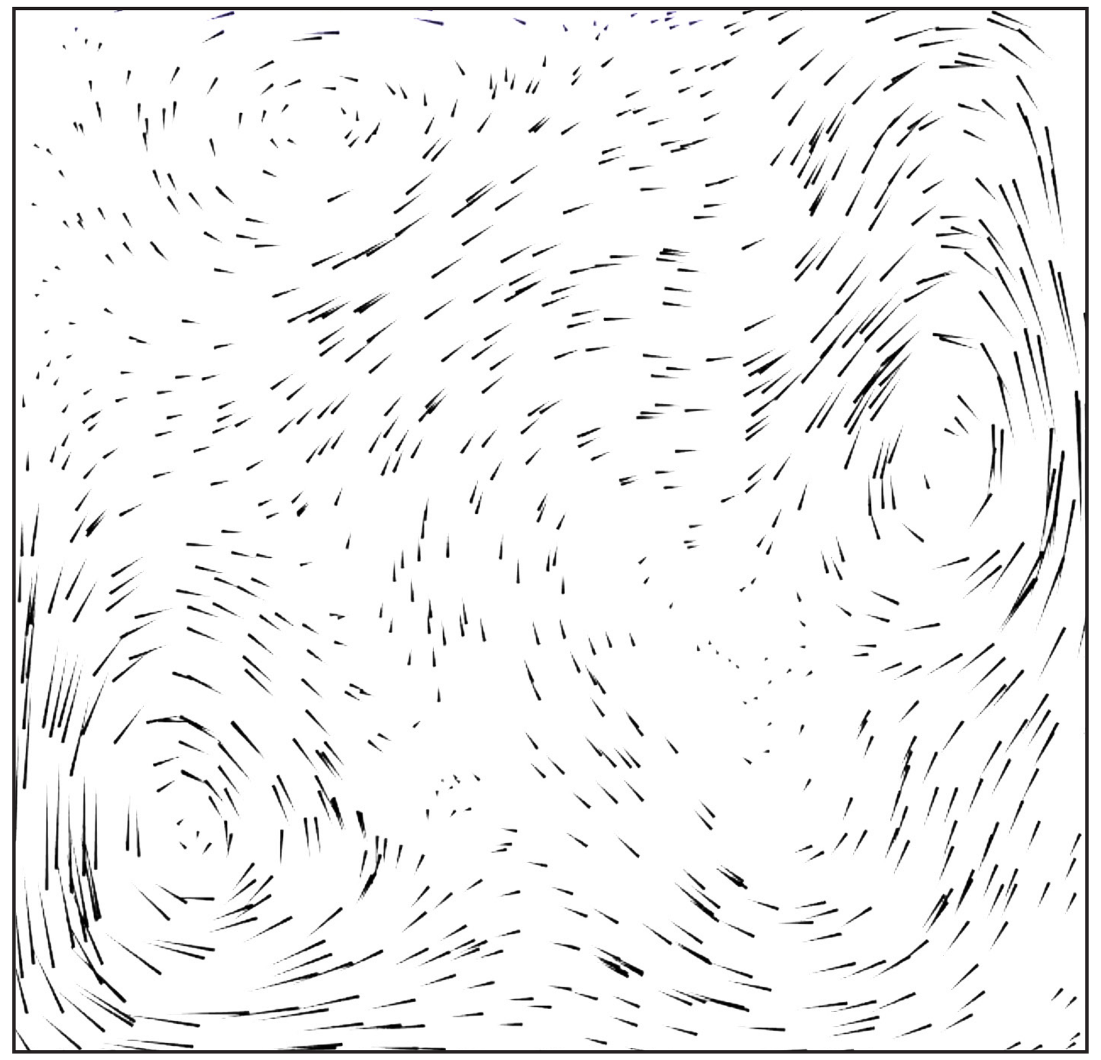

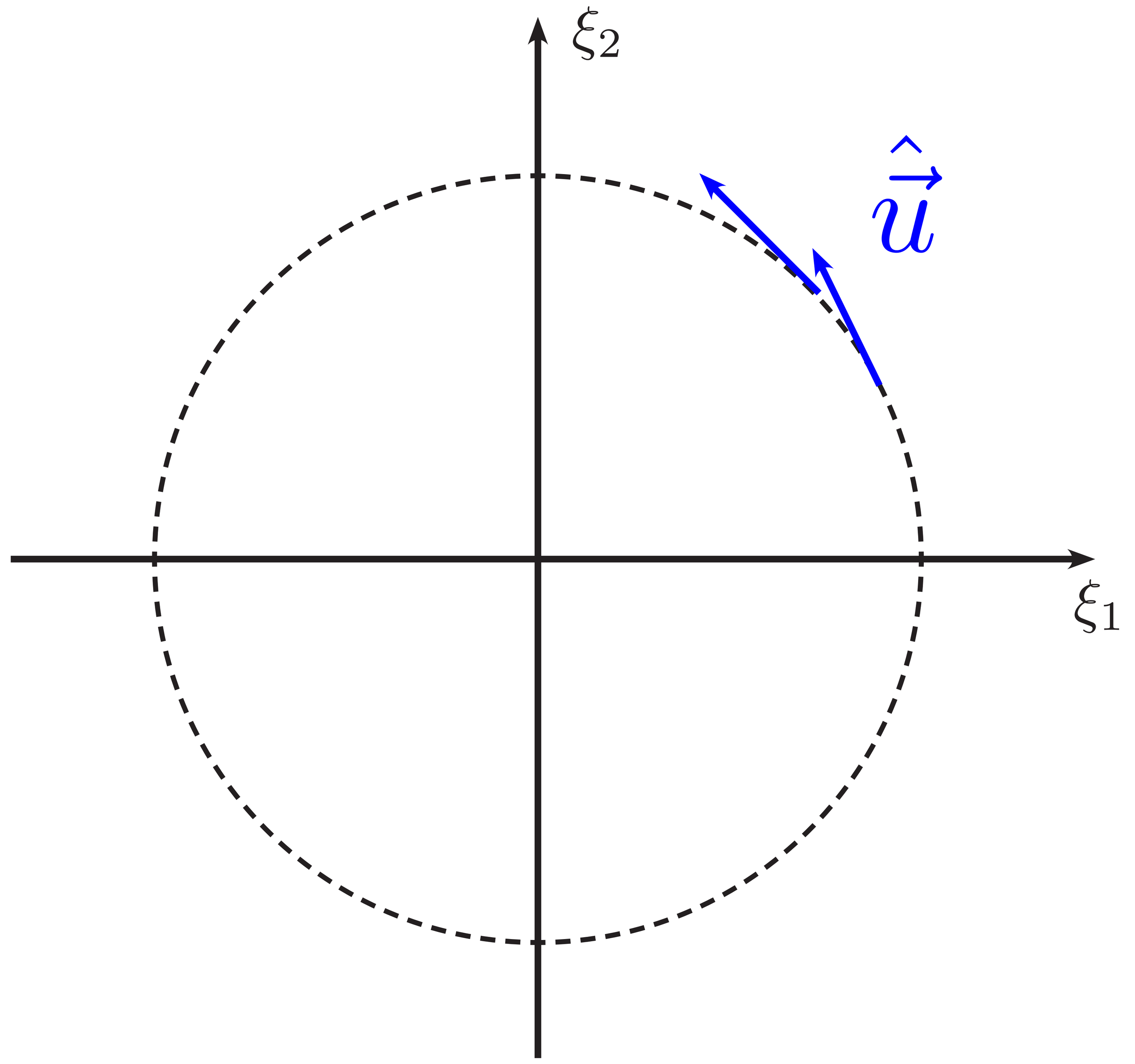

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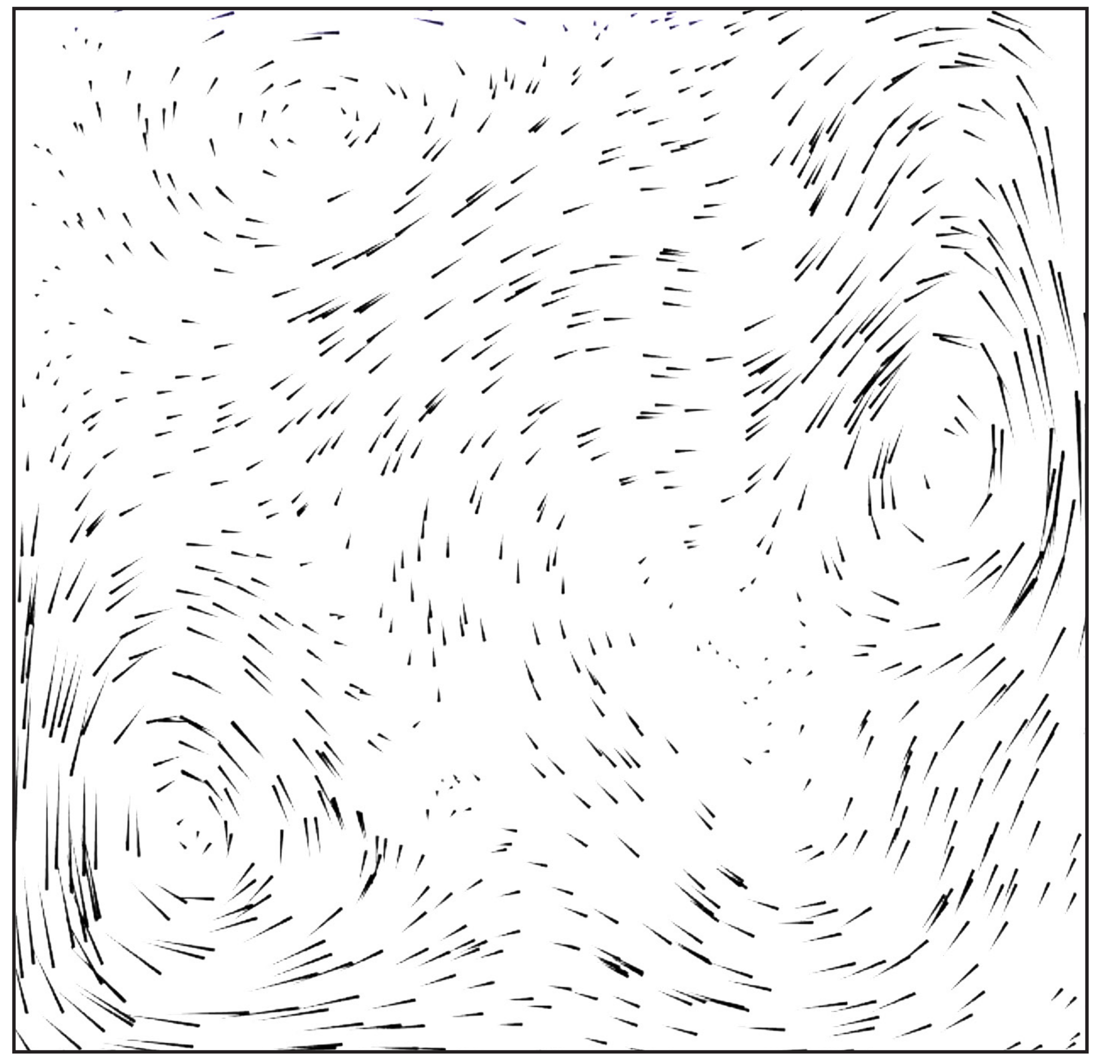

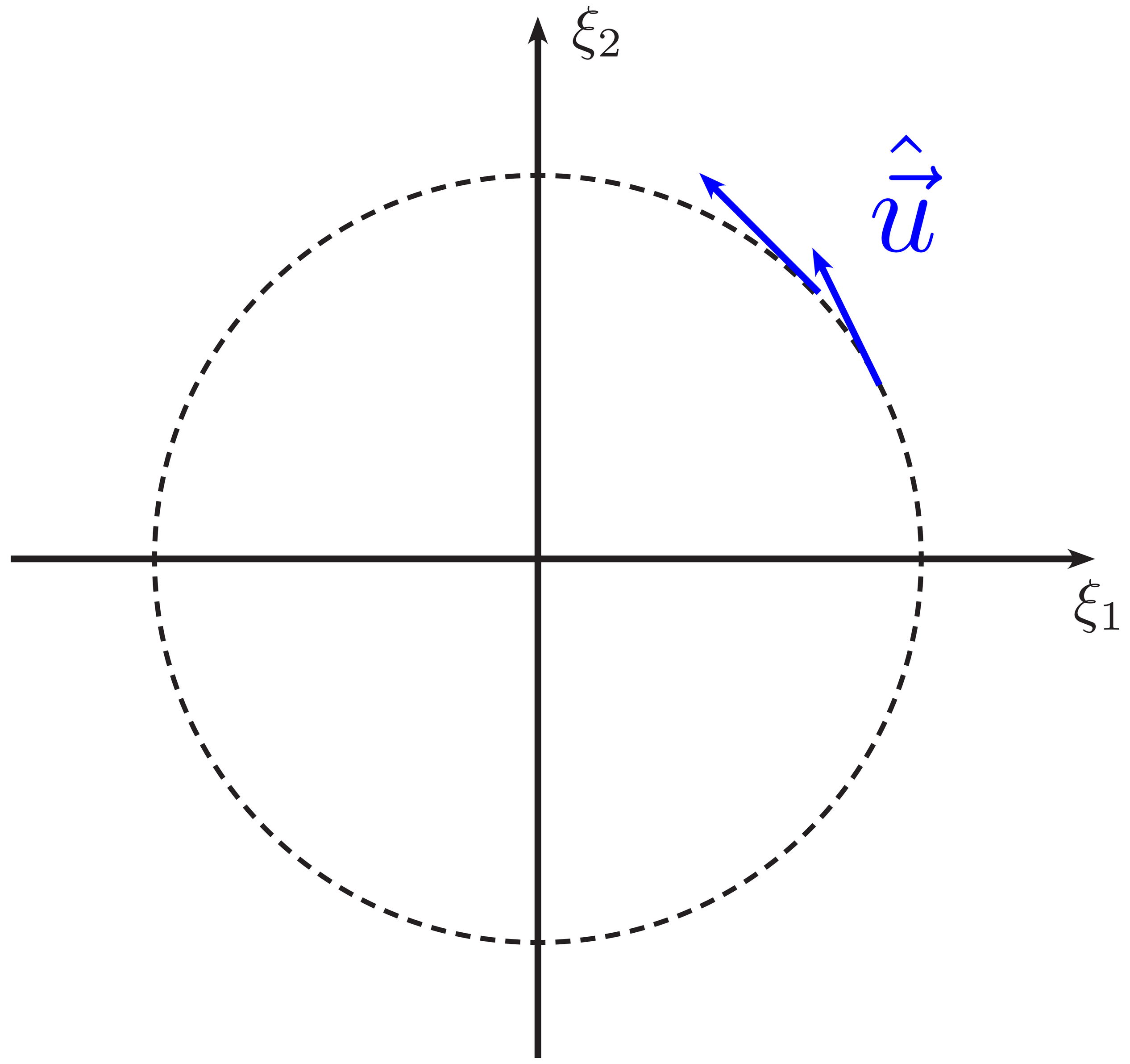

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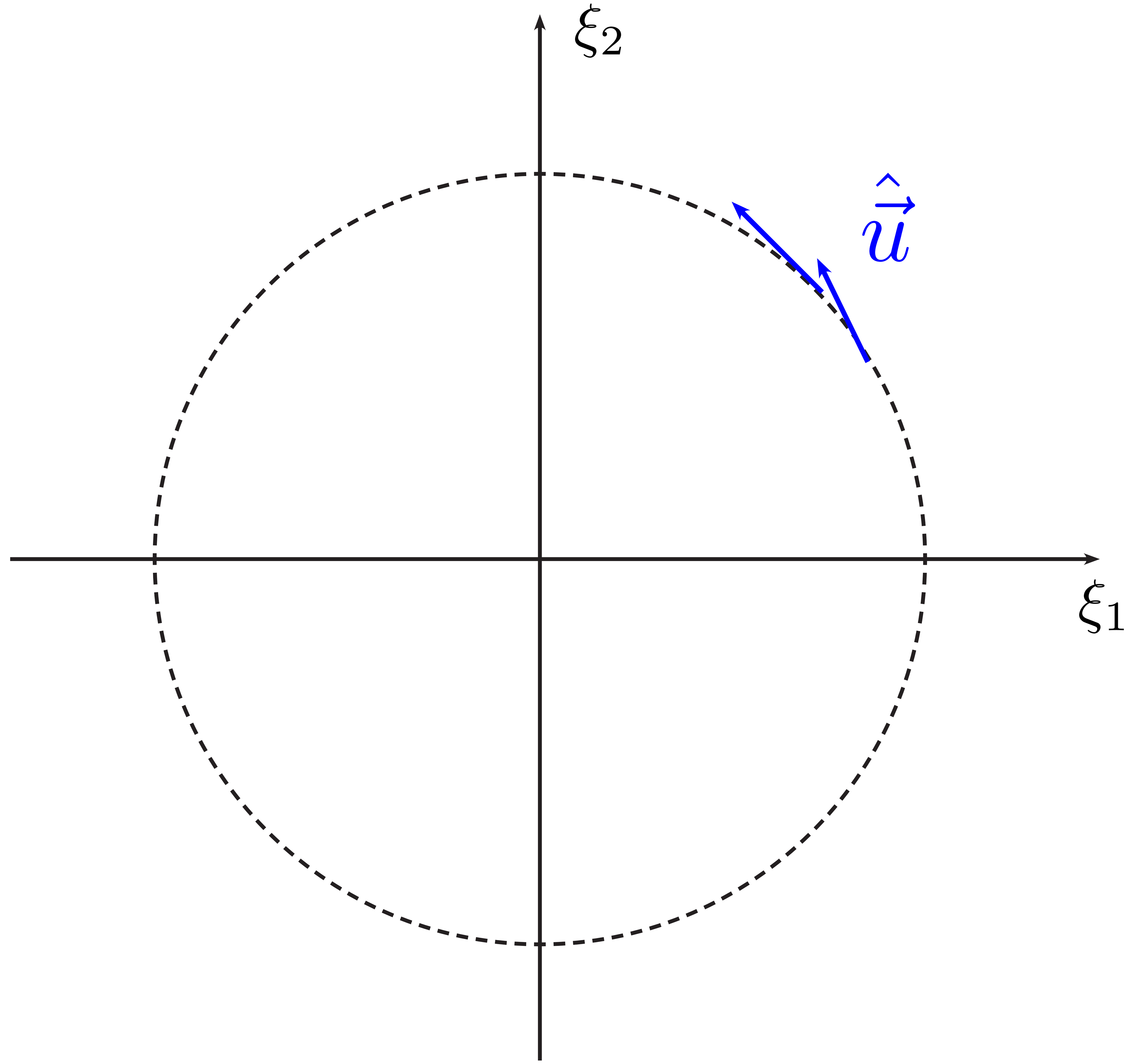

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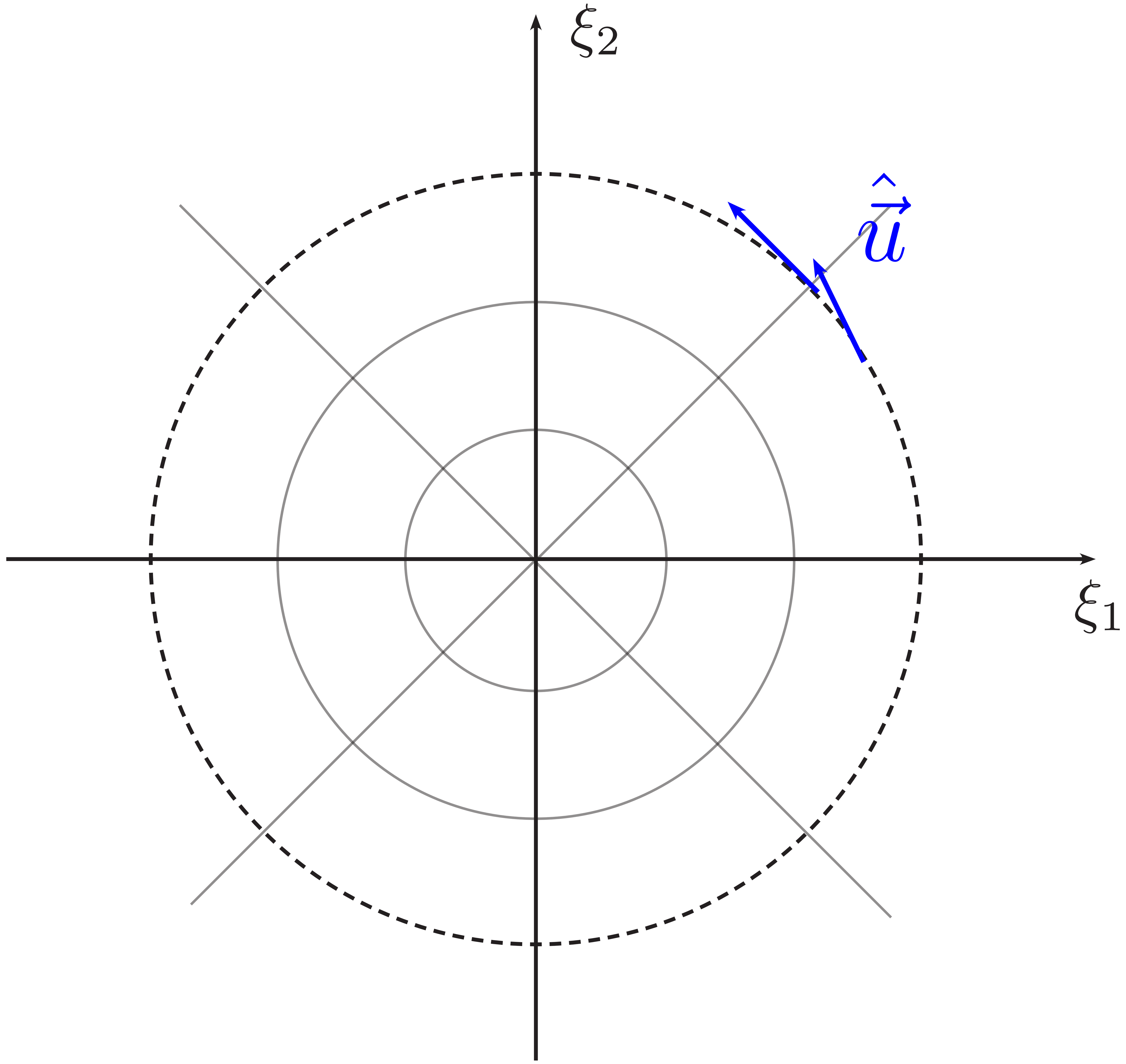

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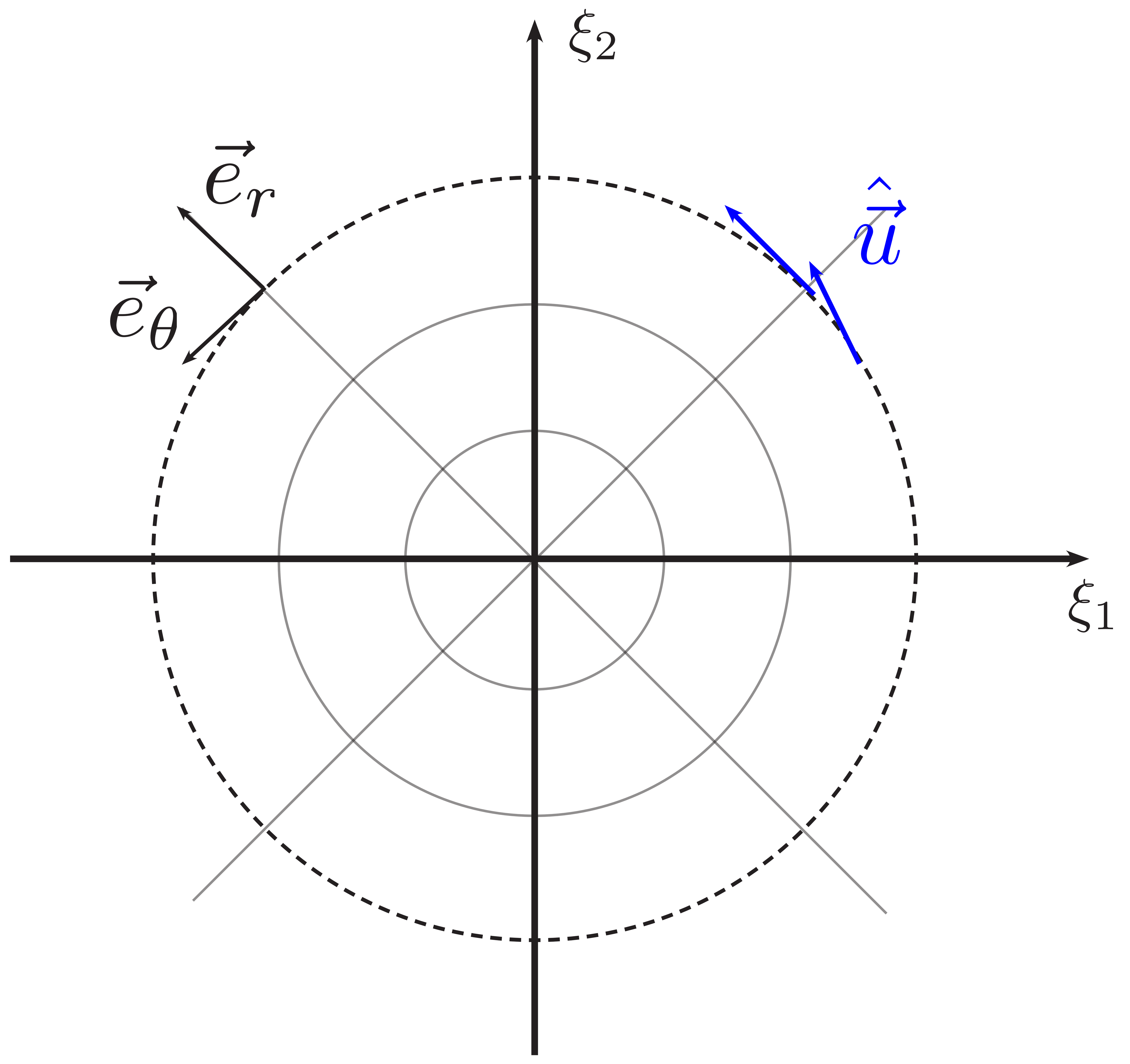

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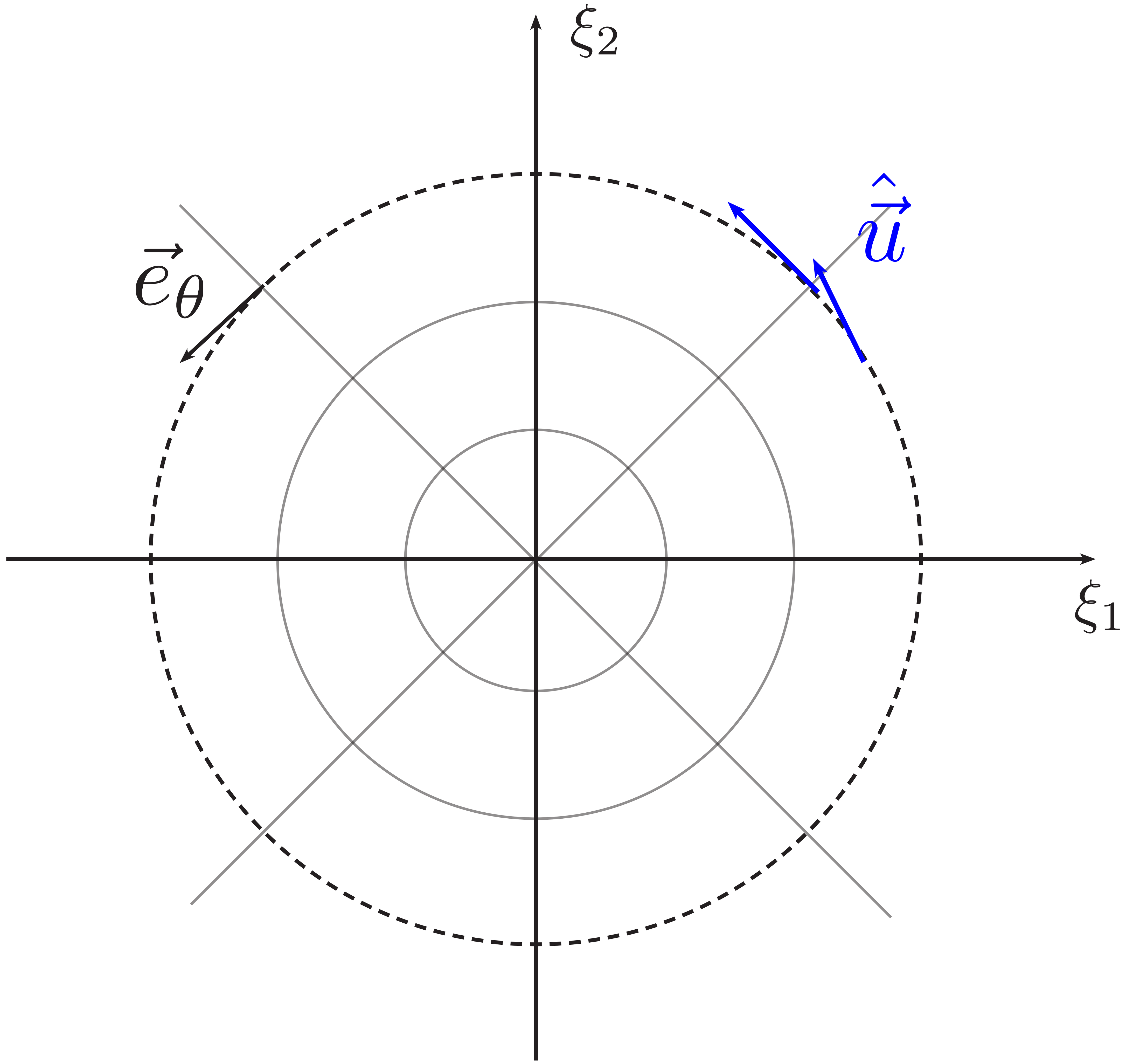

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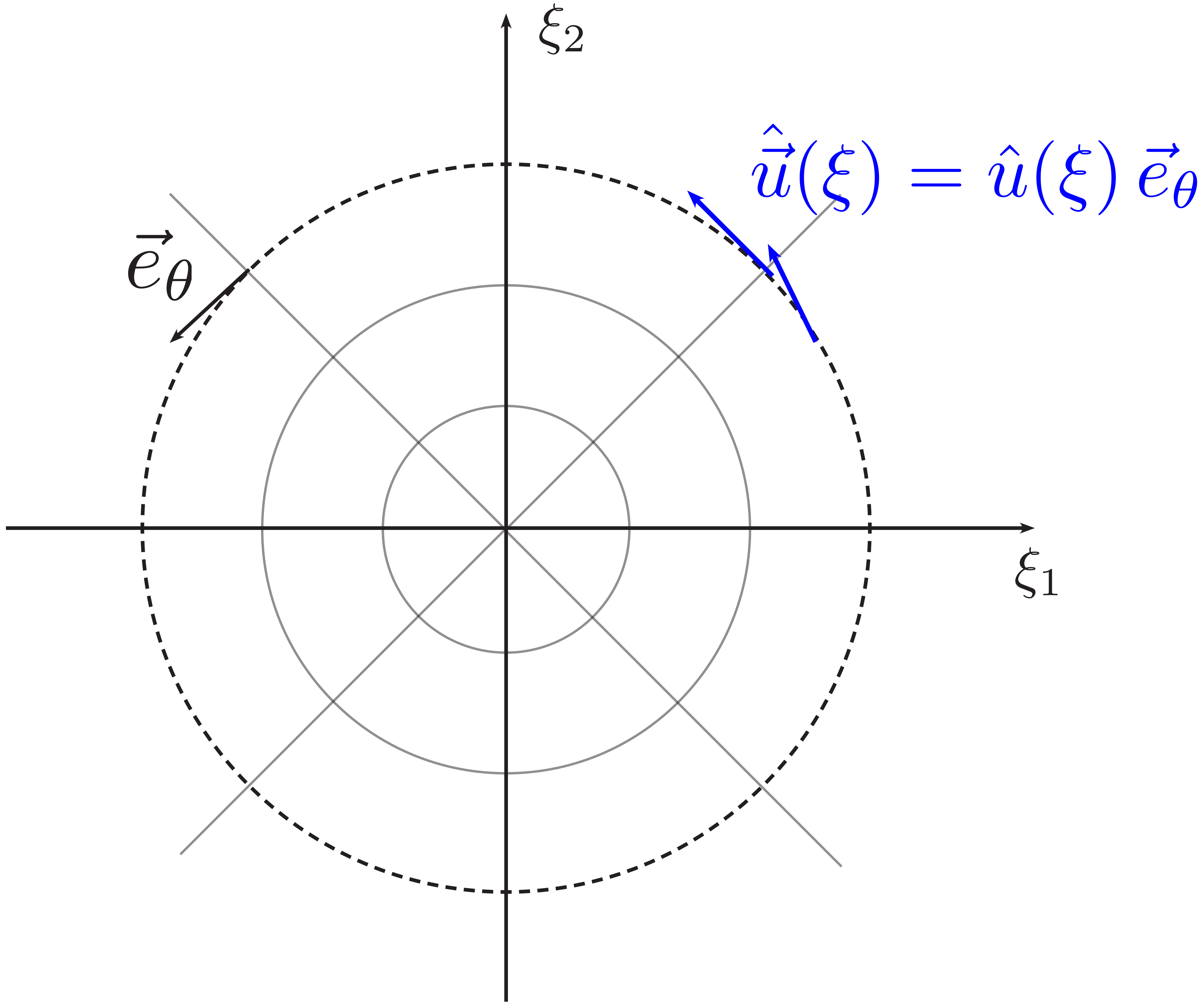
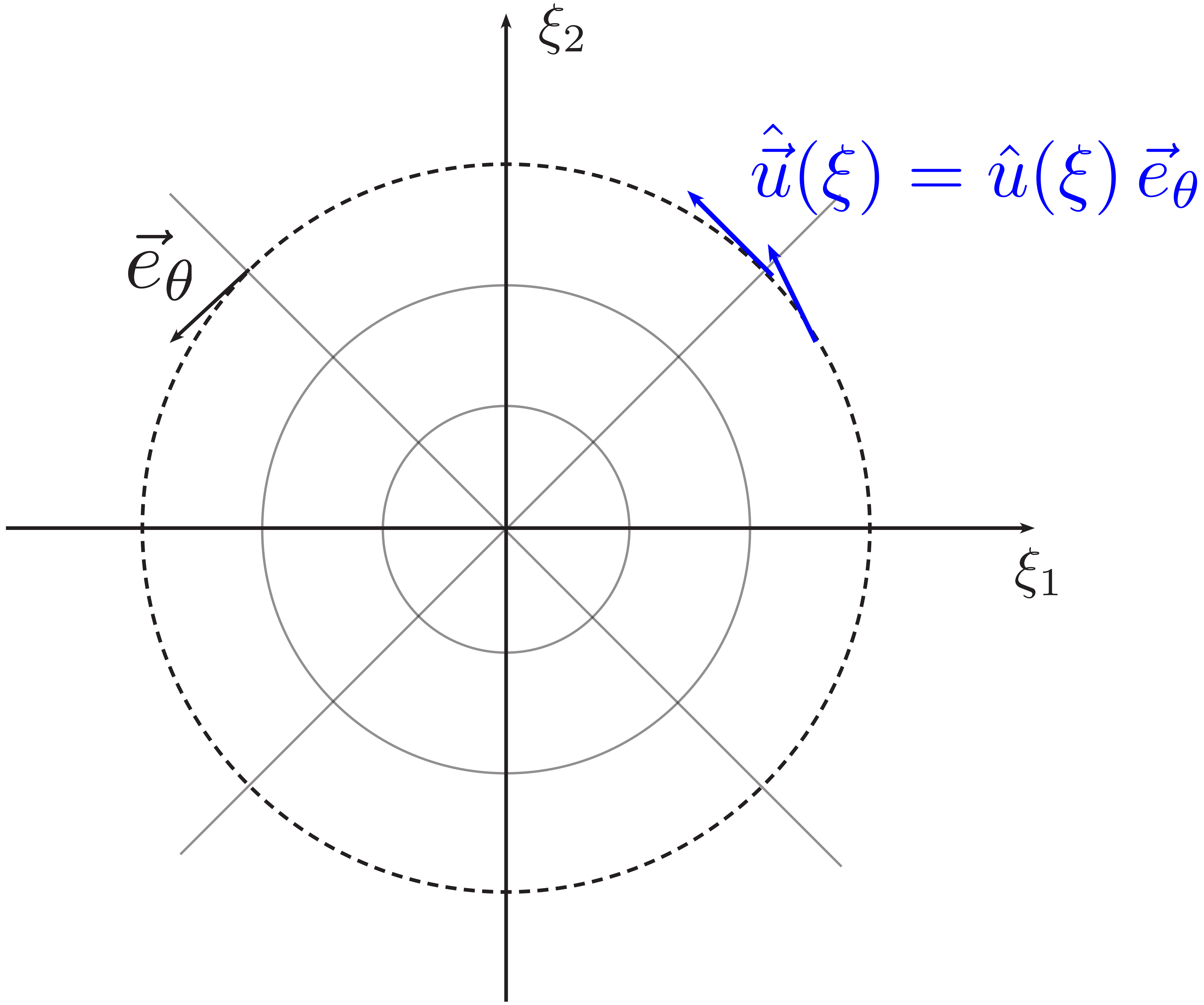


# Divergence freedom

- Divergence free basis:

$$\hat{\vec{\psi}}(\xi) = \quad \dots \quad \vec{e}_\theta$$



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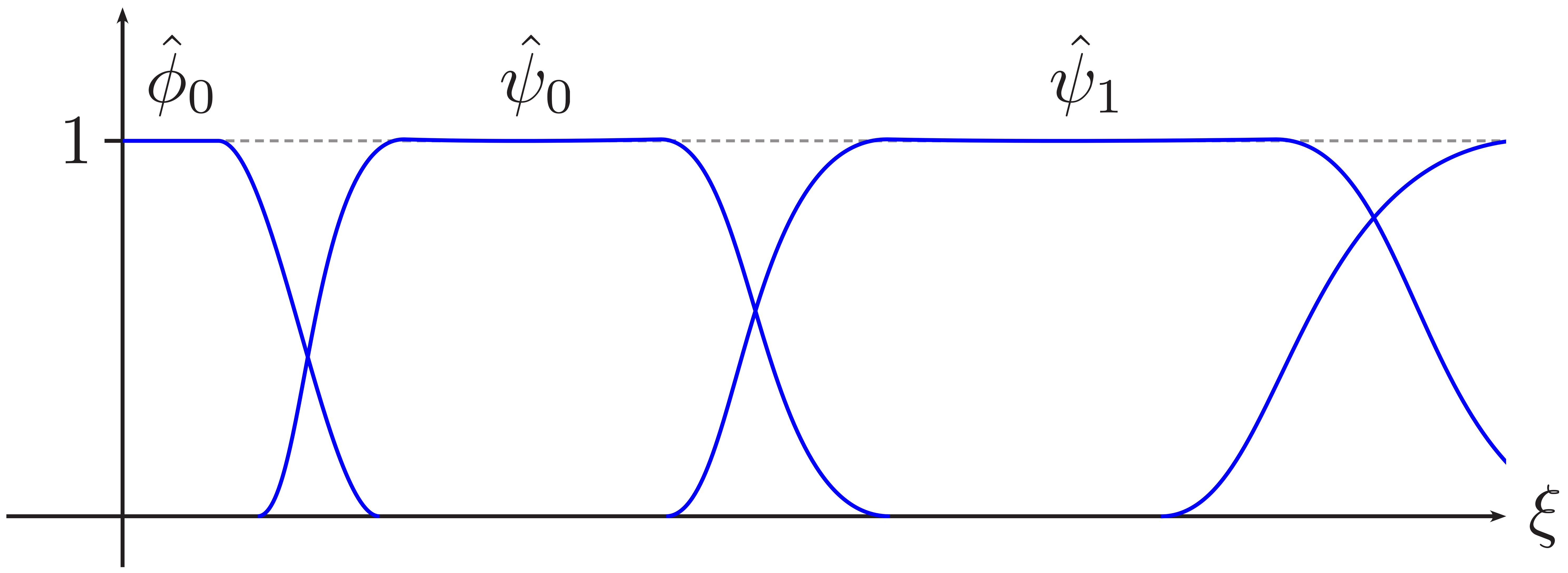
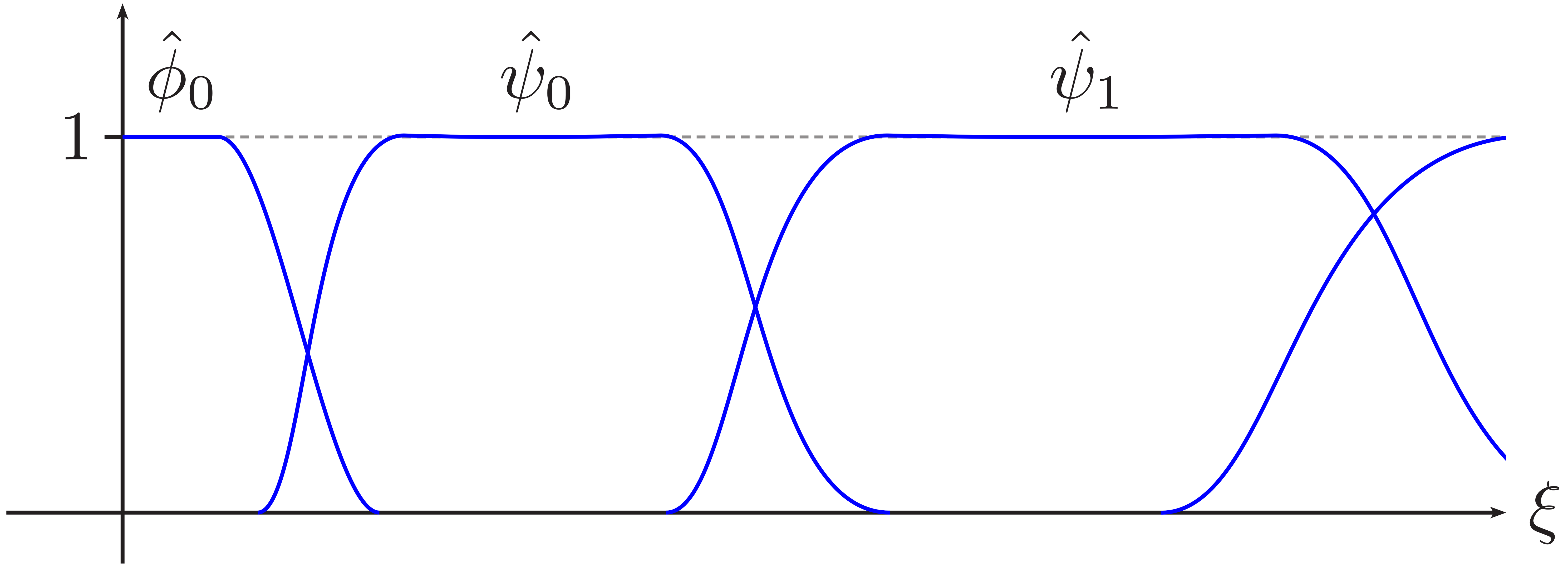


# Wavelets

$$\sum_{j=-1}^{\infty} \left|\hat{\psi}_j(\xi)\right|^2 = 1$$

$$\forall \xi \in \mathbb{R}$$



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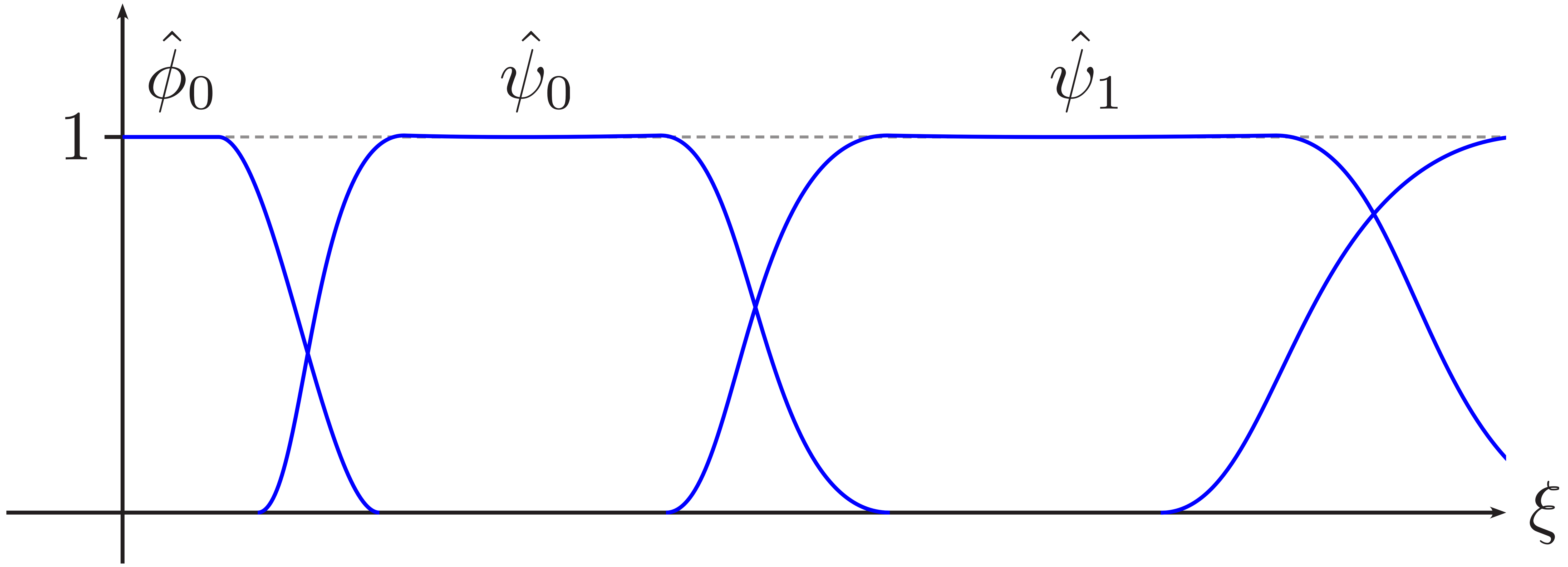
$$\Downarrow$$

forms a tight frame in the spatial domain



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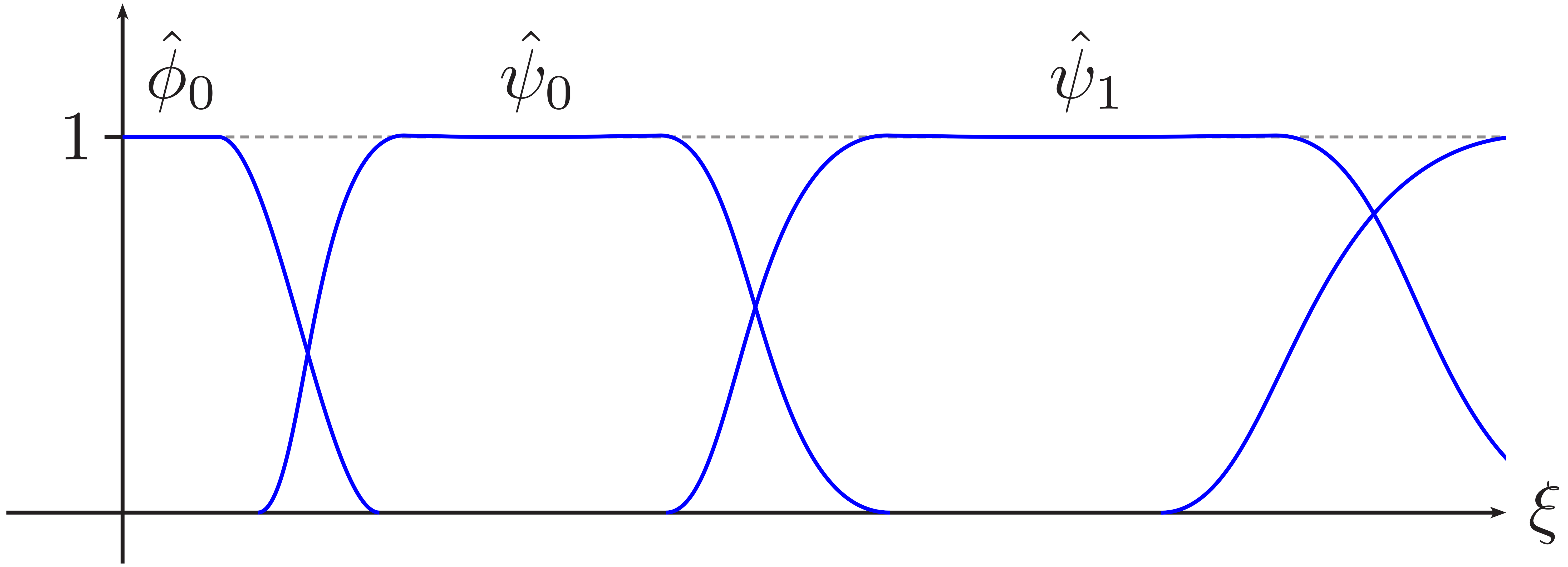
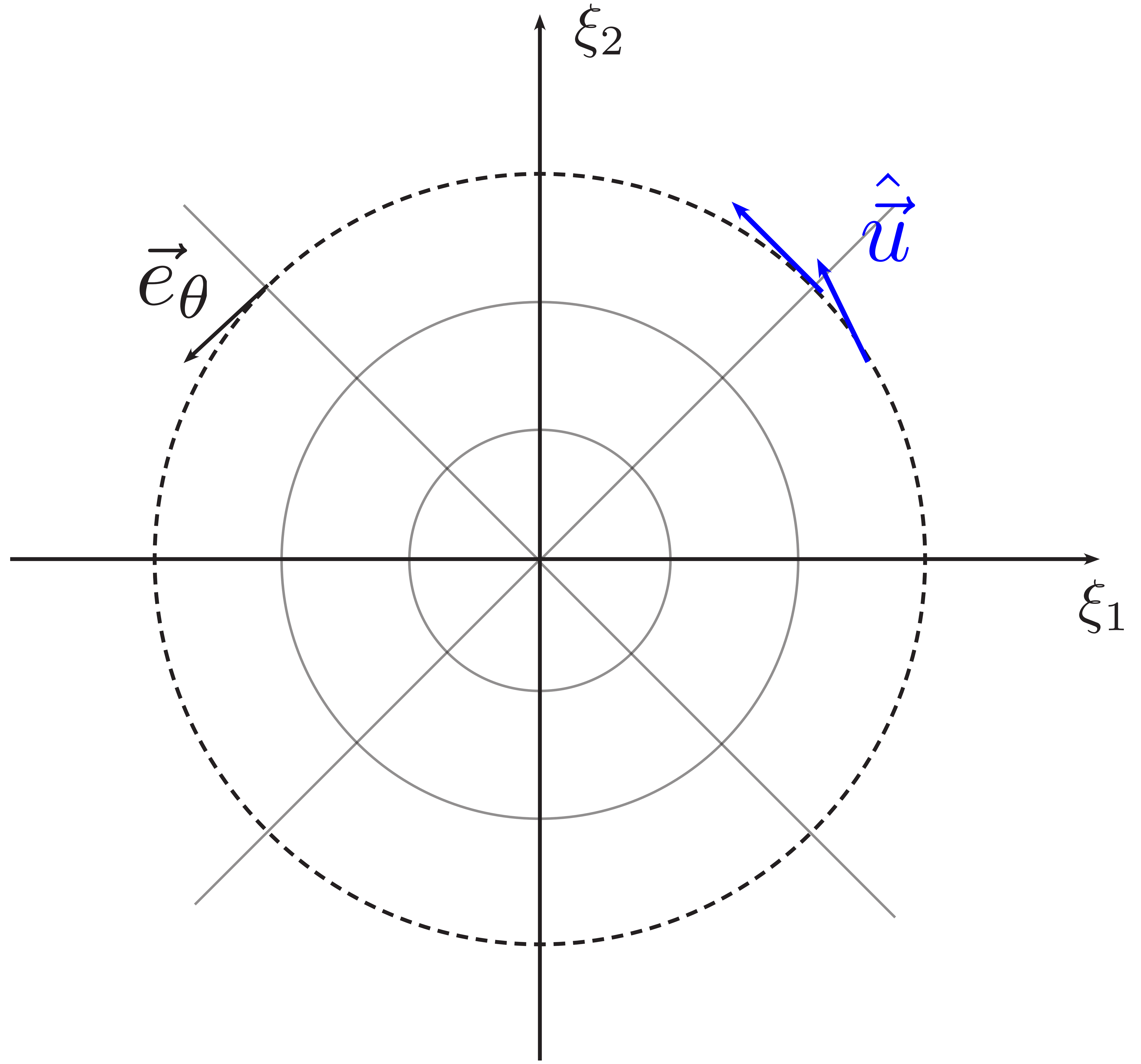



like an orthonormal
basis but redundant

# Divergence free wavelets

$$\hat{\vec{\psi}}(\xi) = \quad \ldots \quad \vec{e}_\theta$$



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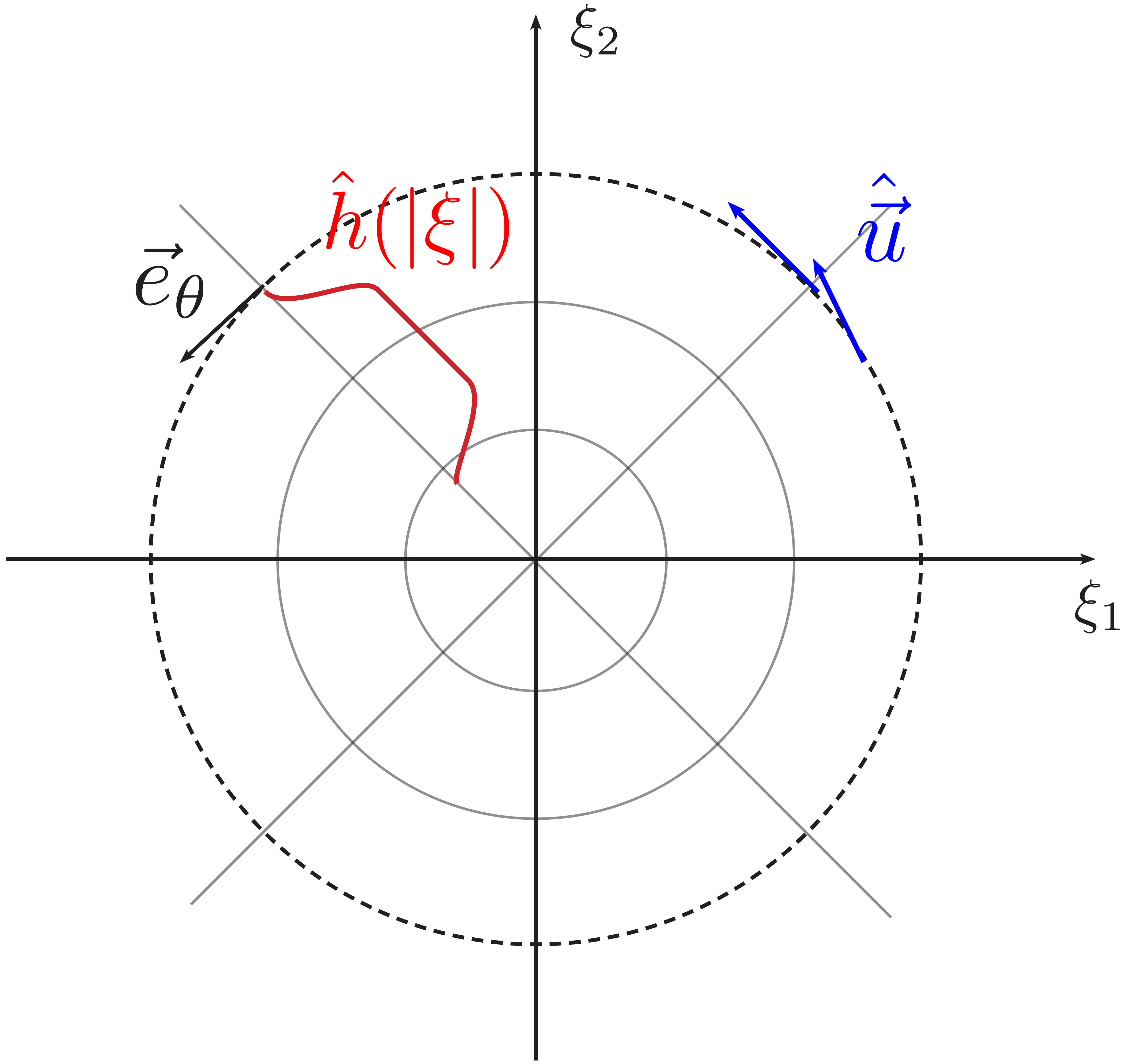
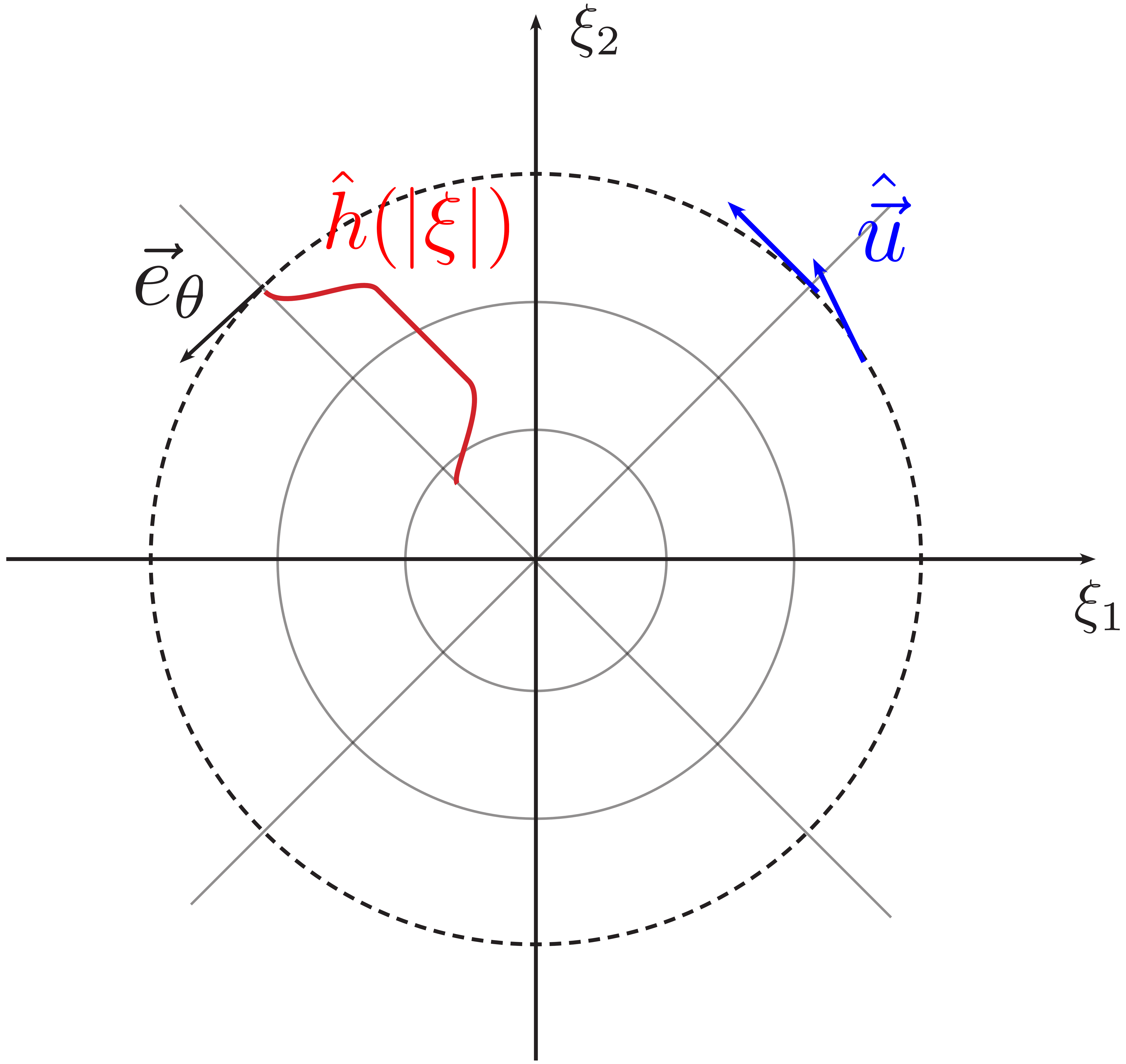


# Divergence free wavelets

$$\hat{\vec{\psi}}(\xi) = -i\, h(|\xi|)\, \vec{e}_\theta$$



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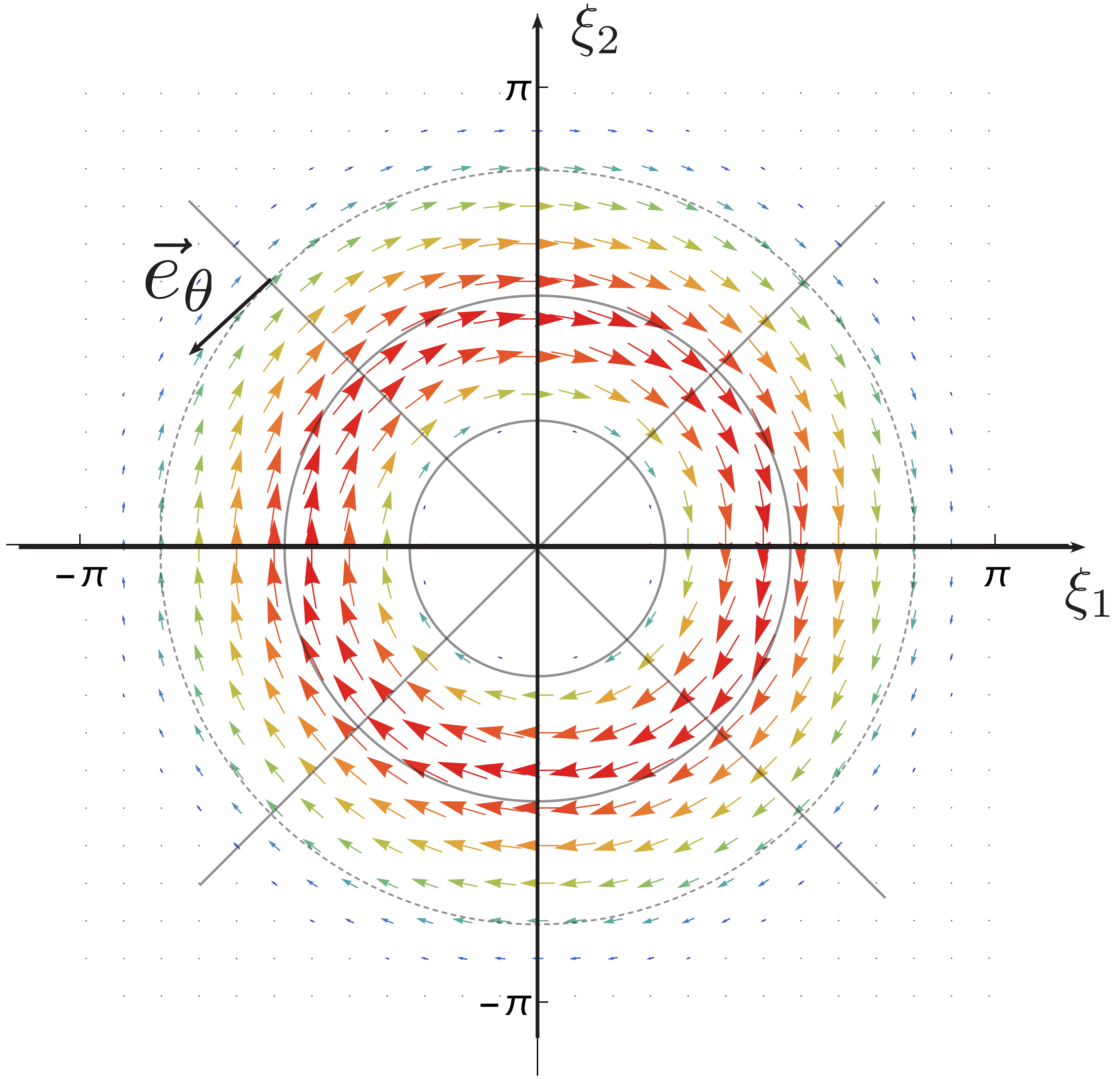

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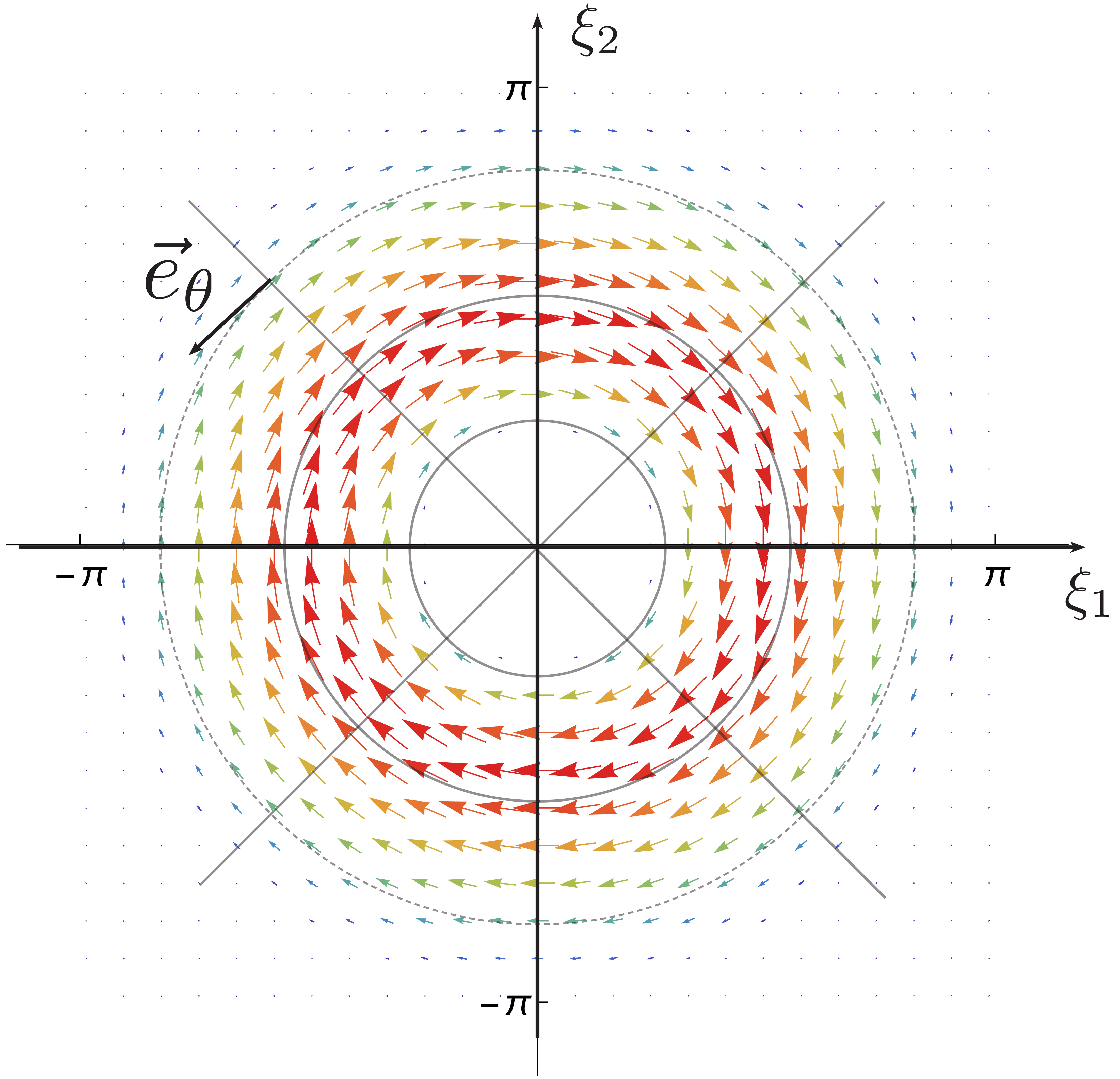

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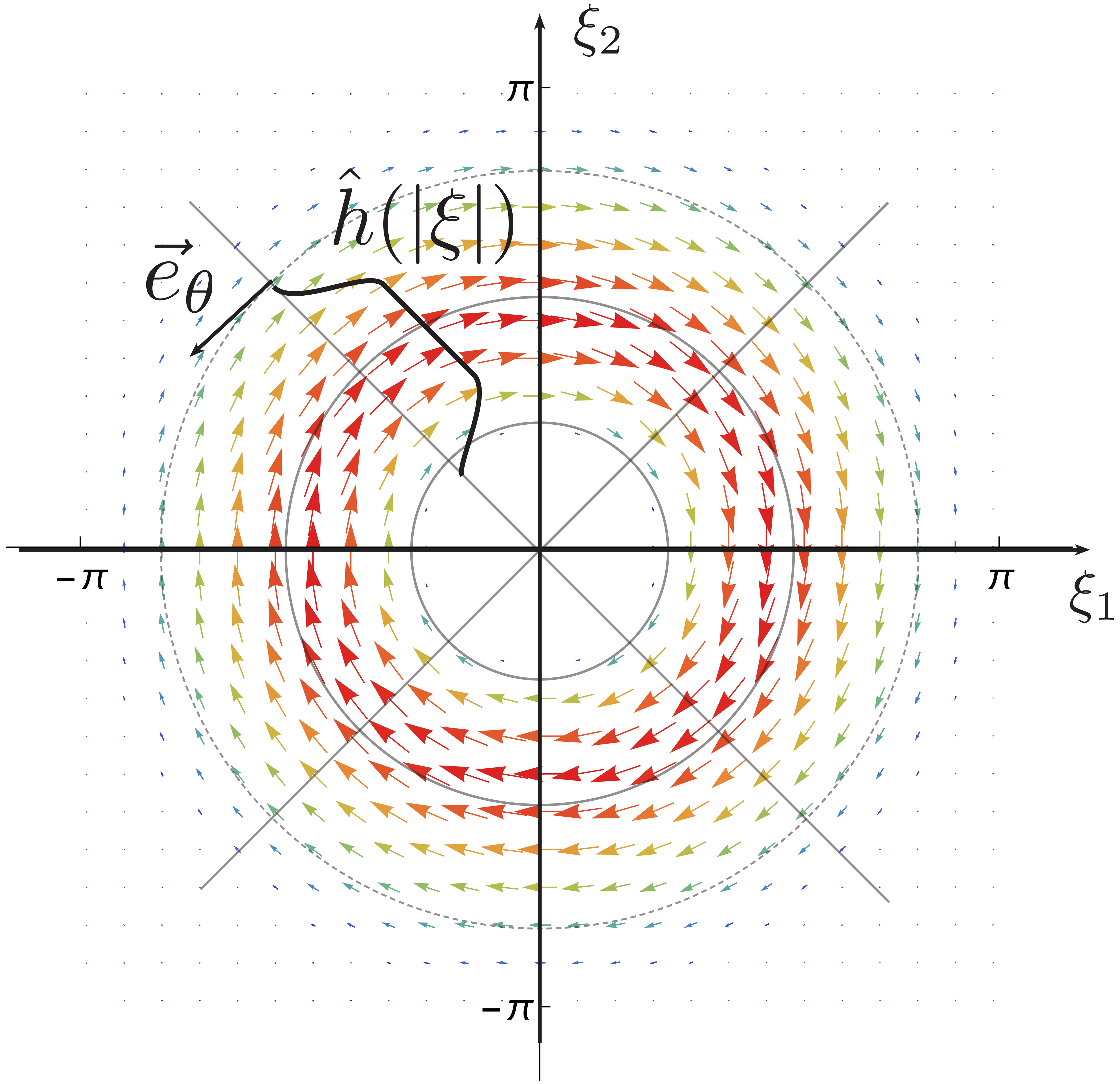

# Divergence free wavelets

- Spatial representation:

$$\vec{\psi}(x) = -\frac{i}{2\pi} \int_{\mathbb{R}^2} h(|\xi|)\, \vec{e}_\theta \, e^{i\langle \xi , x \rangle} \, d\xi$$

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$$\vec{\psi}(x) = -\frac{i}{2\pi} \int_{\mathbb{R}^2} h(|\xi|) \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} e^{i\langle \xi , x \rangle} \, d\xi$$

# Divergence free wavelets

- Spatial representation:

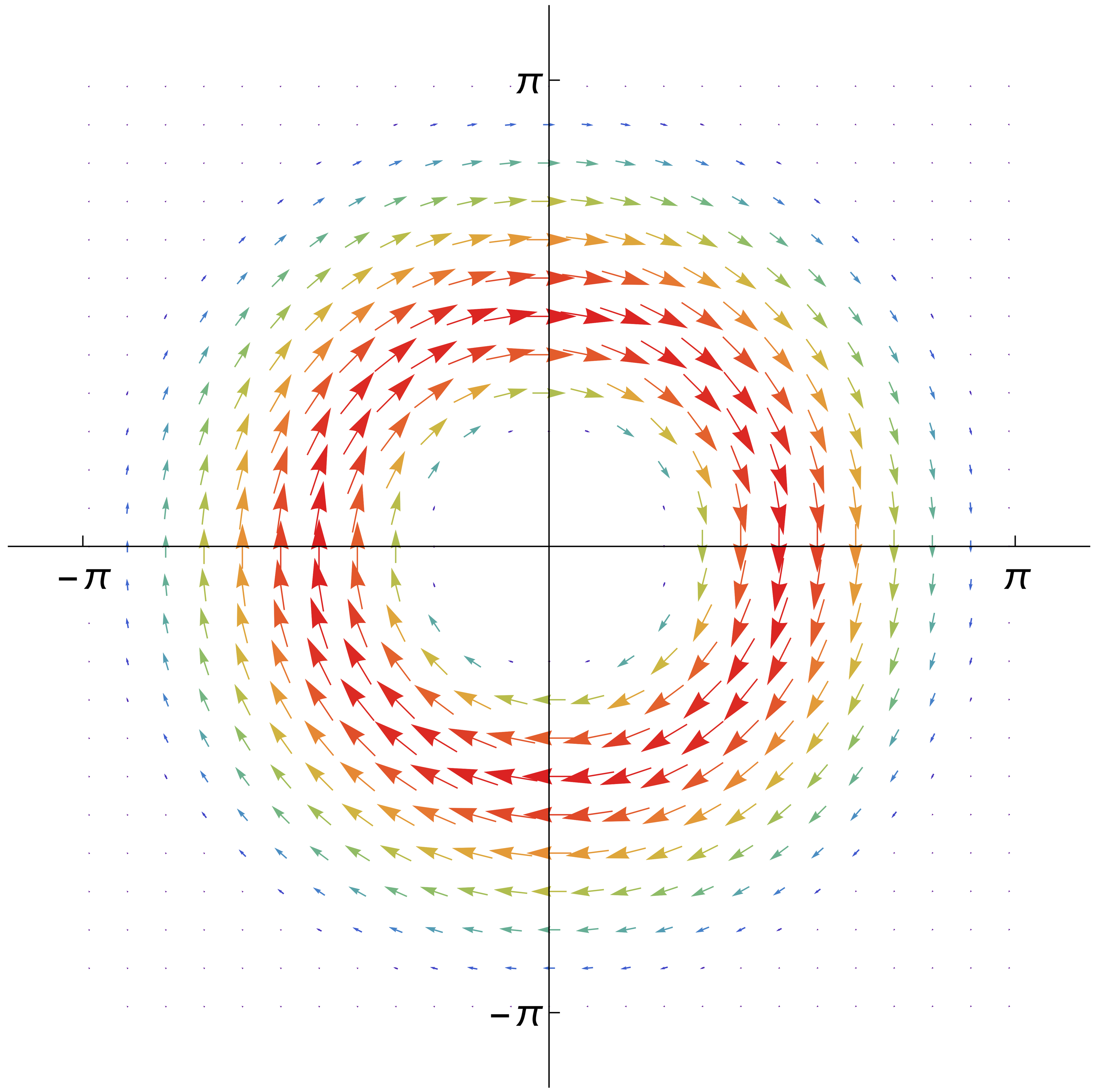
$$\vec{\psi}(x) = -\frac{i}{2\pi} \int_{\mathbb{R}^2} h(|\xi|) \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \underbrace{e^{i\langle \xi, x\rangle}} \, d\xi$$

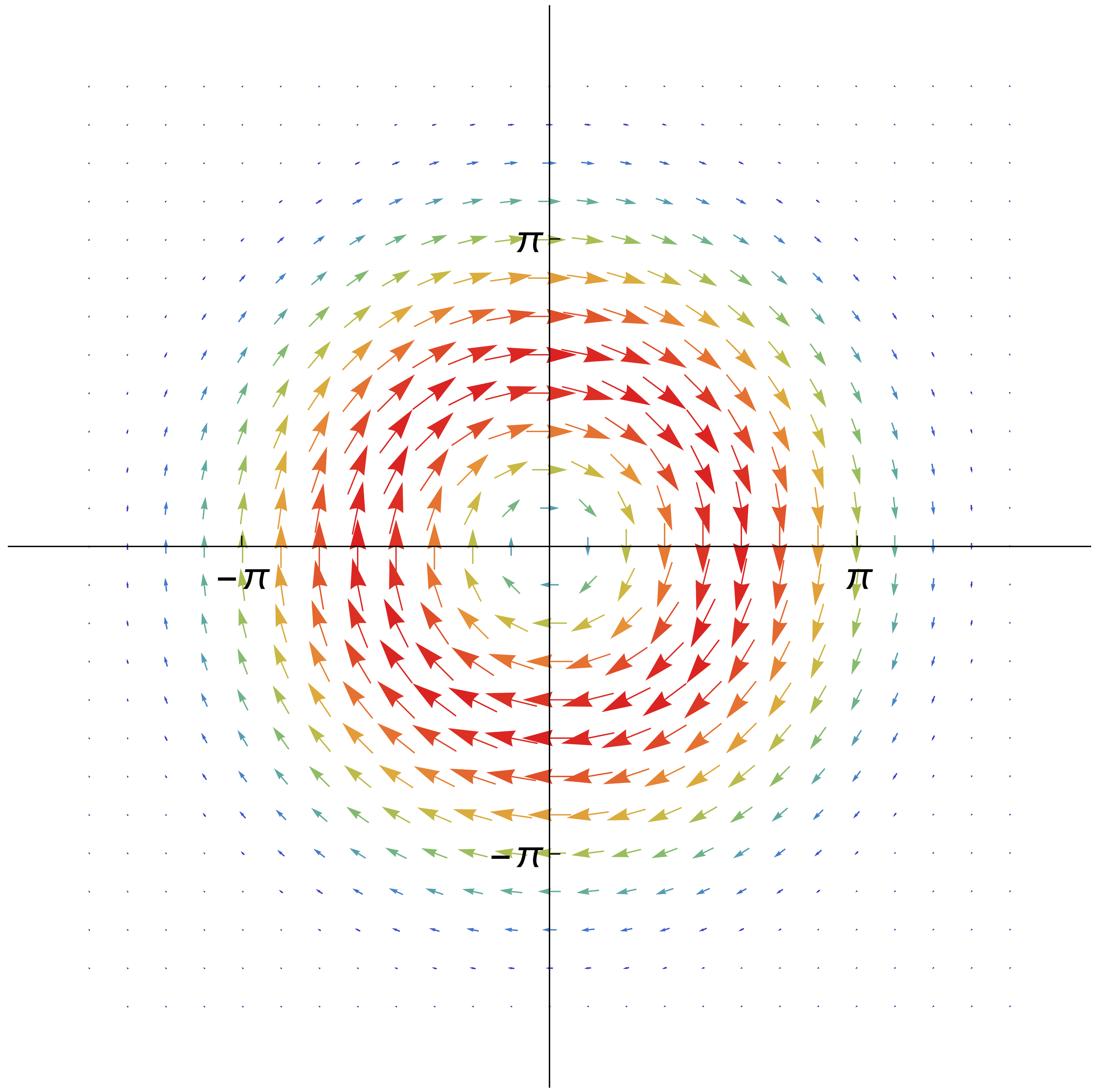
Jacobi-Anger formula:

$$e^{i\langle \xi, x\rangle} = \sum_{m \in \mathbb{Z}} i^m e^{im(\theta_\xi - \theta_x)} J_m(|\xi| \, |x|)$$

# Divergence free wavelets

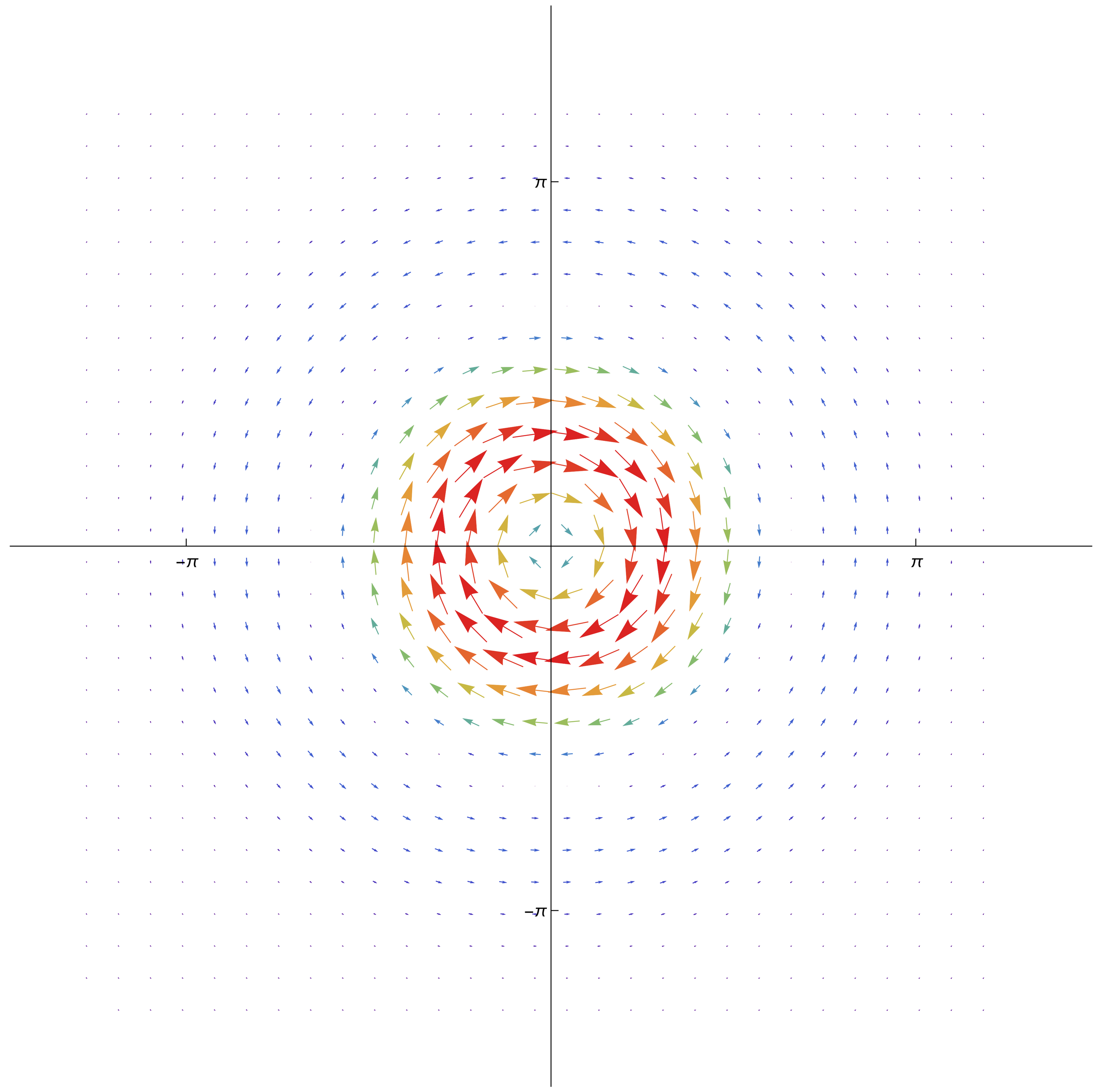
$$\hat{\vec{\psi}}(\xi) = -i\,h(|\xi|)\,\vec{e}_\theta \qquad \vec{\psi}(x) = h_1(|x|)\vec{e}_\theta$$



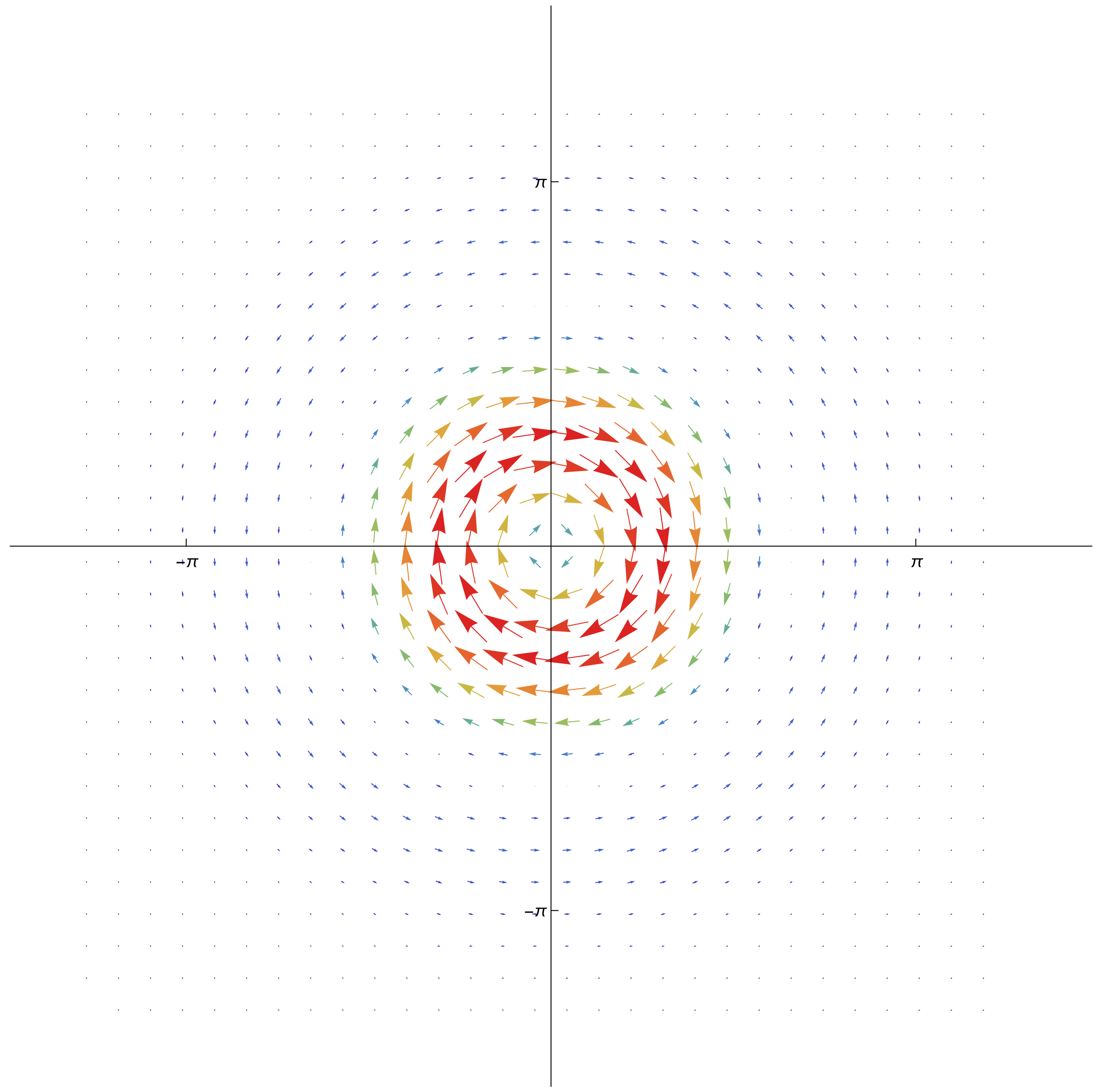
# Divergence free wavelets

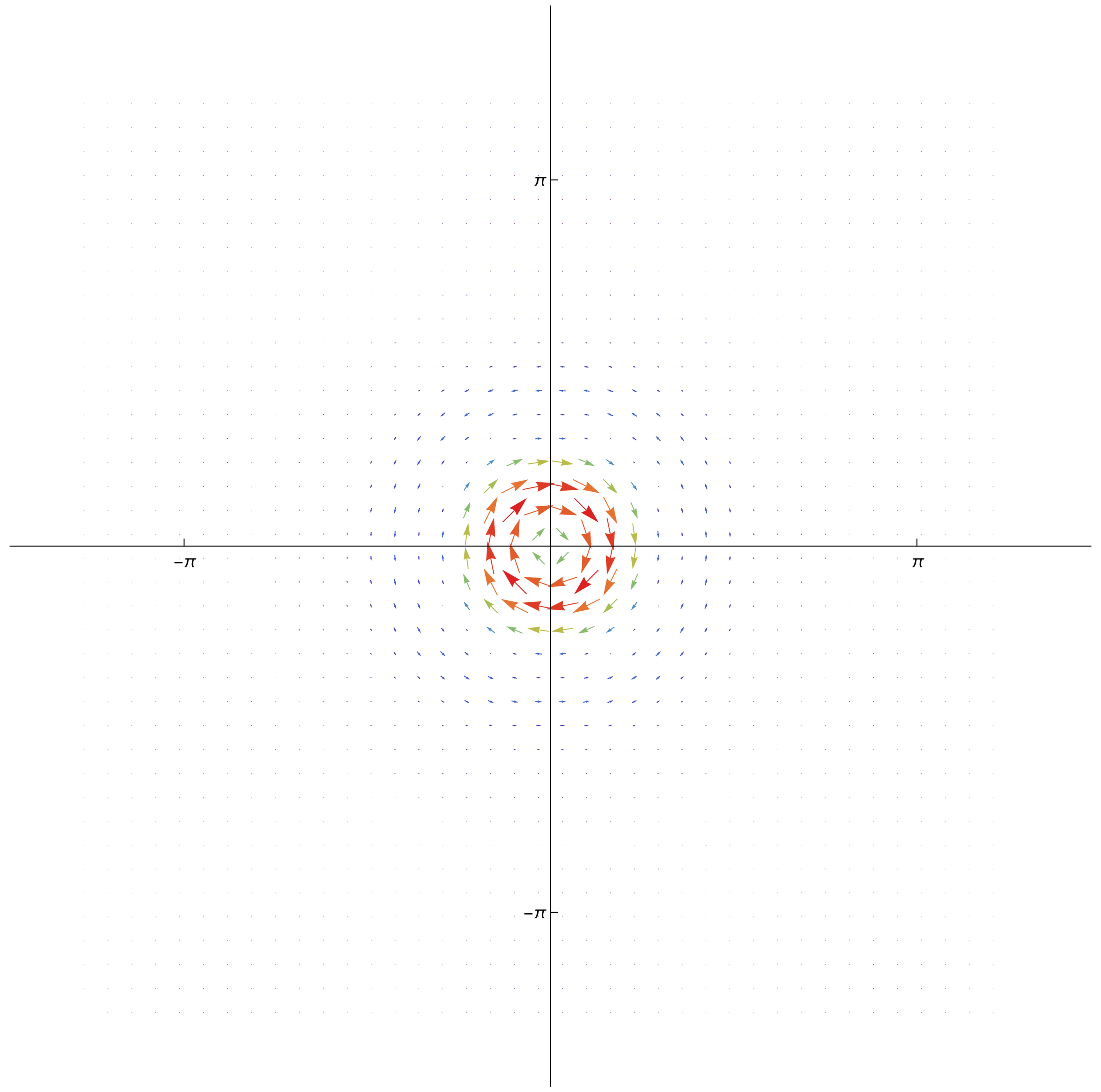
$$\vec{\psi}_0(x) = \vec{\psi}(2^0 x)$$



# Divergence free wavelets

$$\vec{\psi}_0(x) = \vec{\psi}(2^0 x) \qquad \vec{\psi}_1(x) = \vec{\psi}(2^1 x)$$



# Divergence free wavelets

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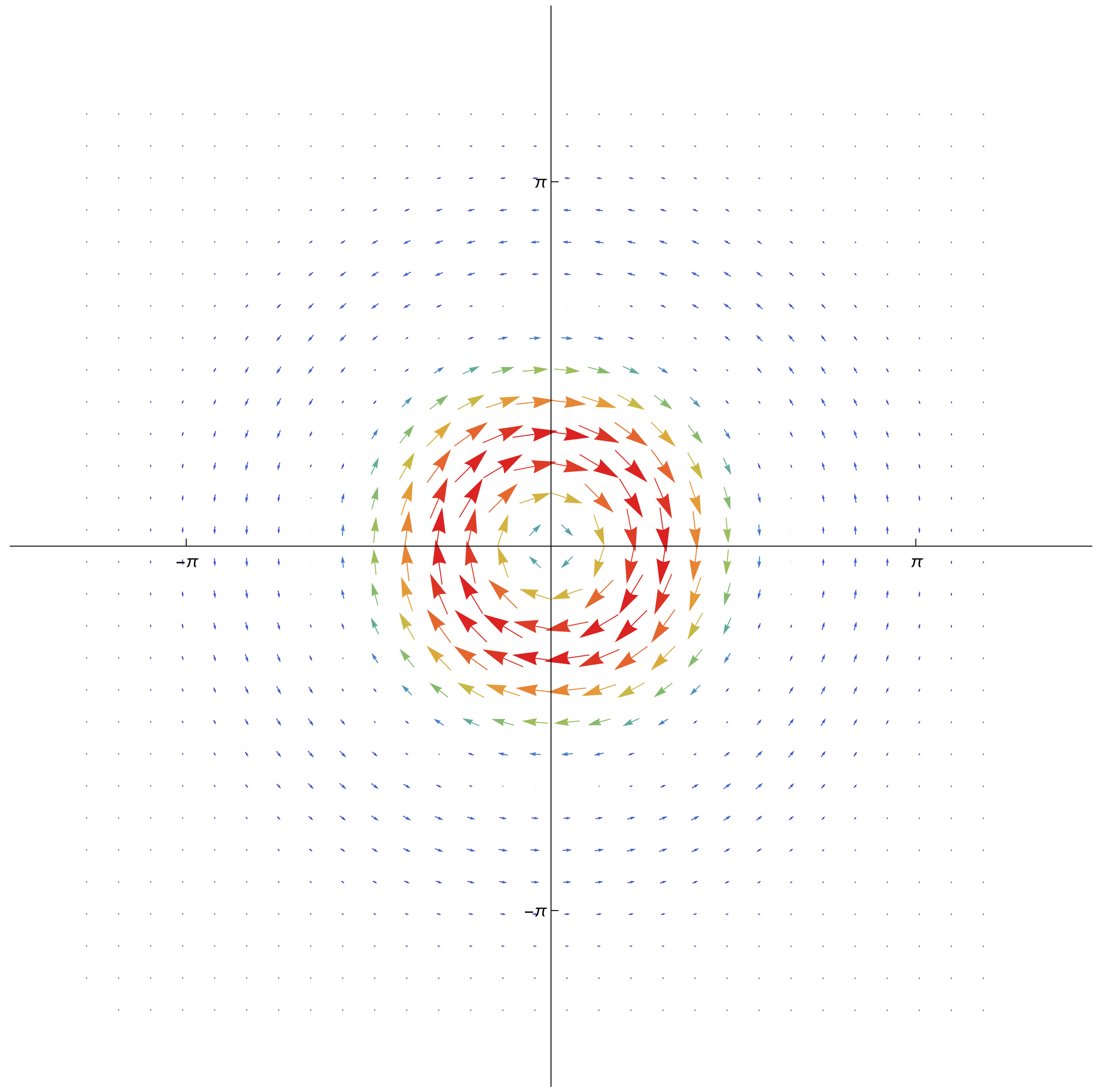
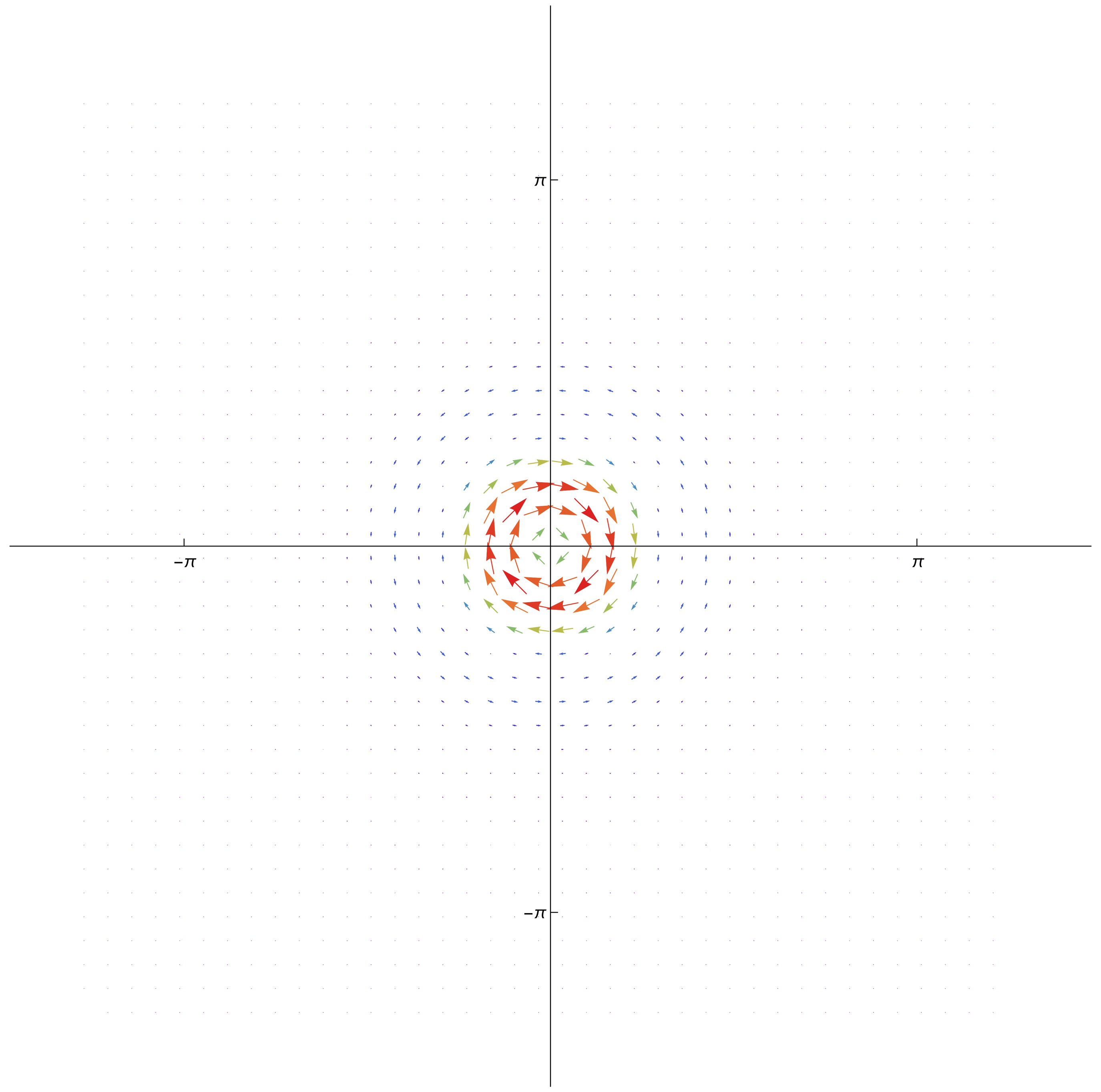
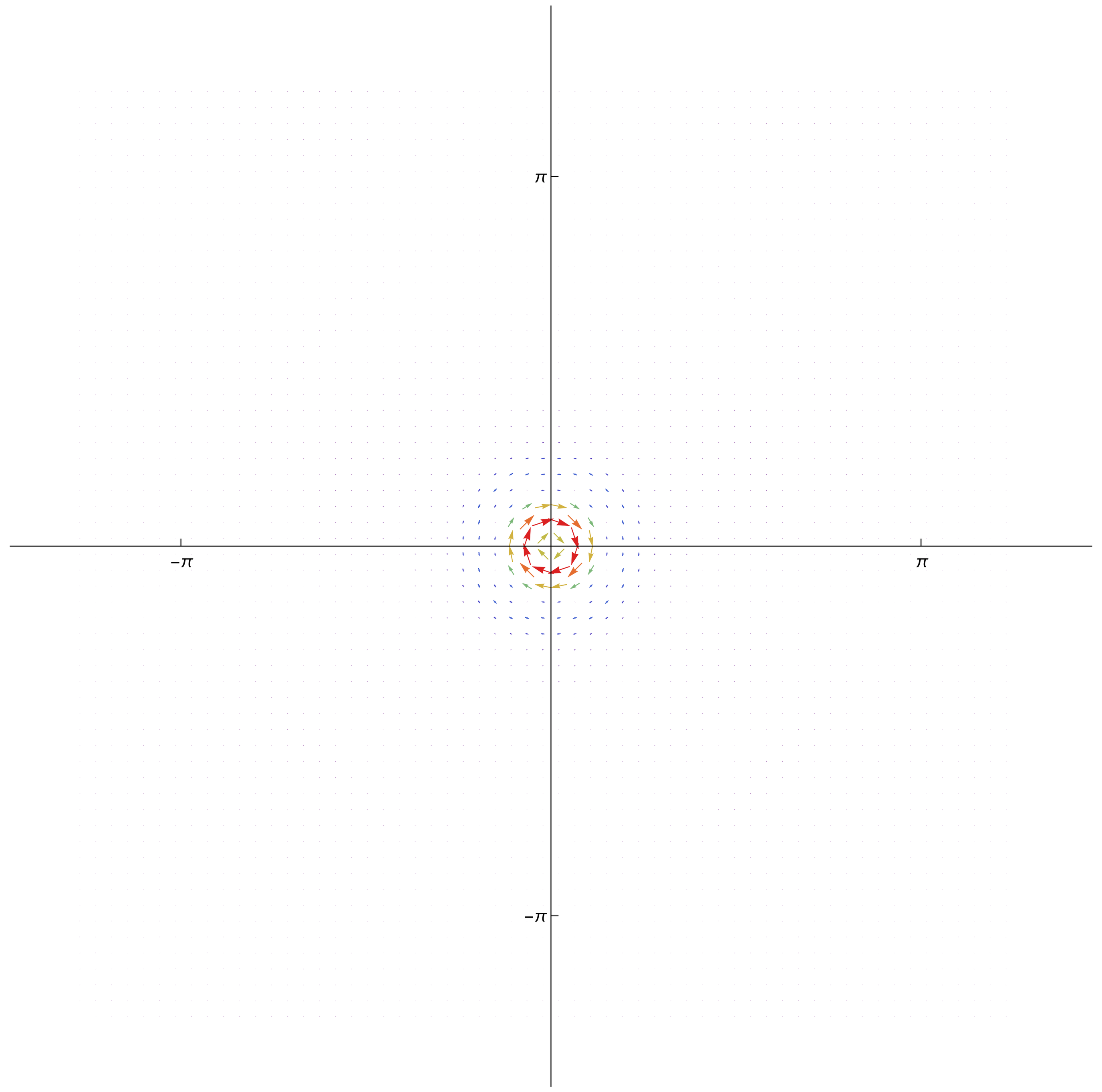


$$\vec{\psi}_1(x) = \vec{\psi}(2^1 x)$$



$$\vec{\psi}_2(x) = \vec{\psi}(2^2 x)$$



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# Divergence free wavelets

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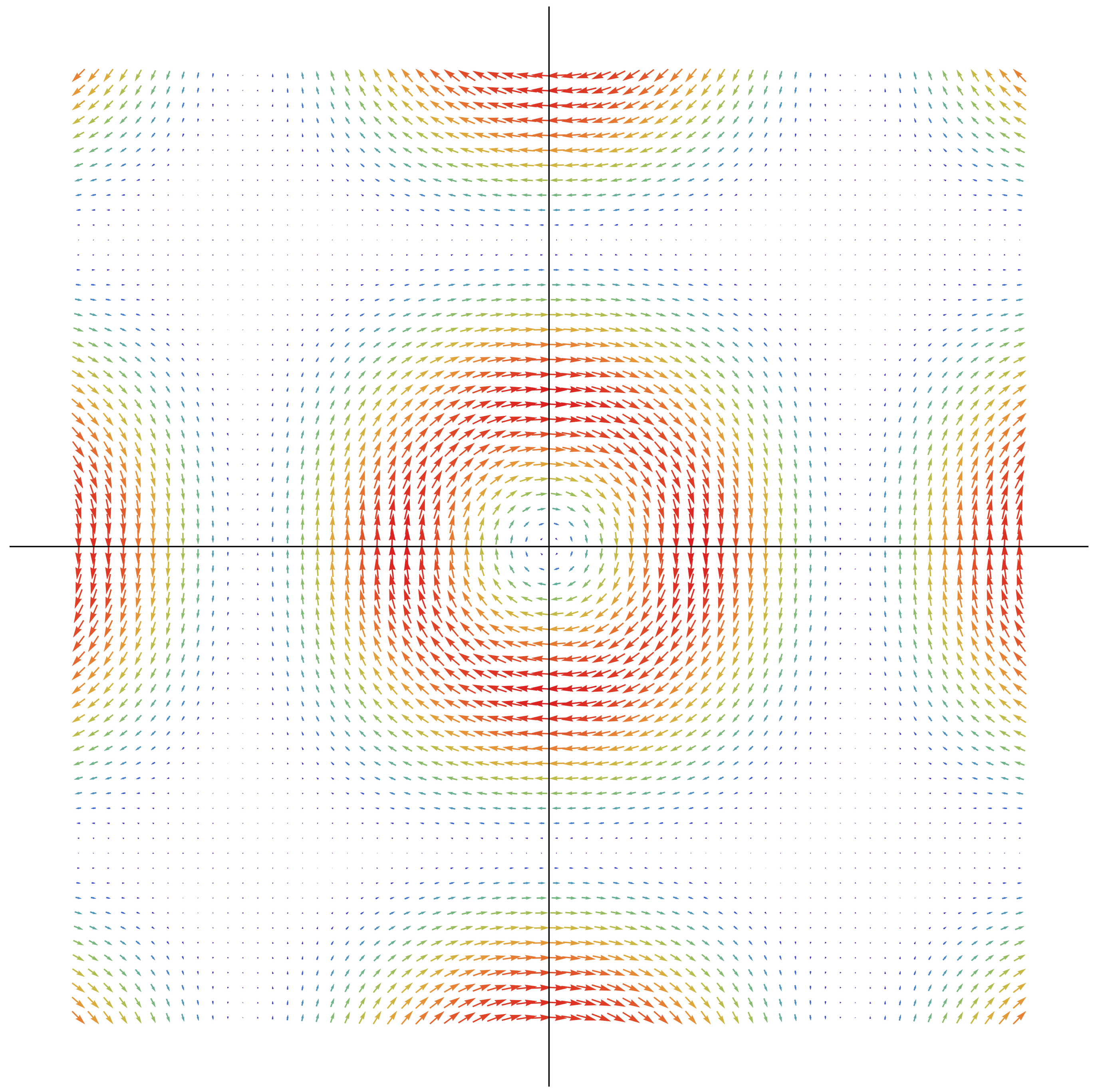
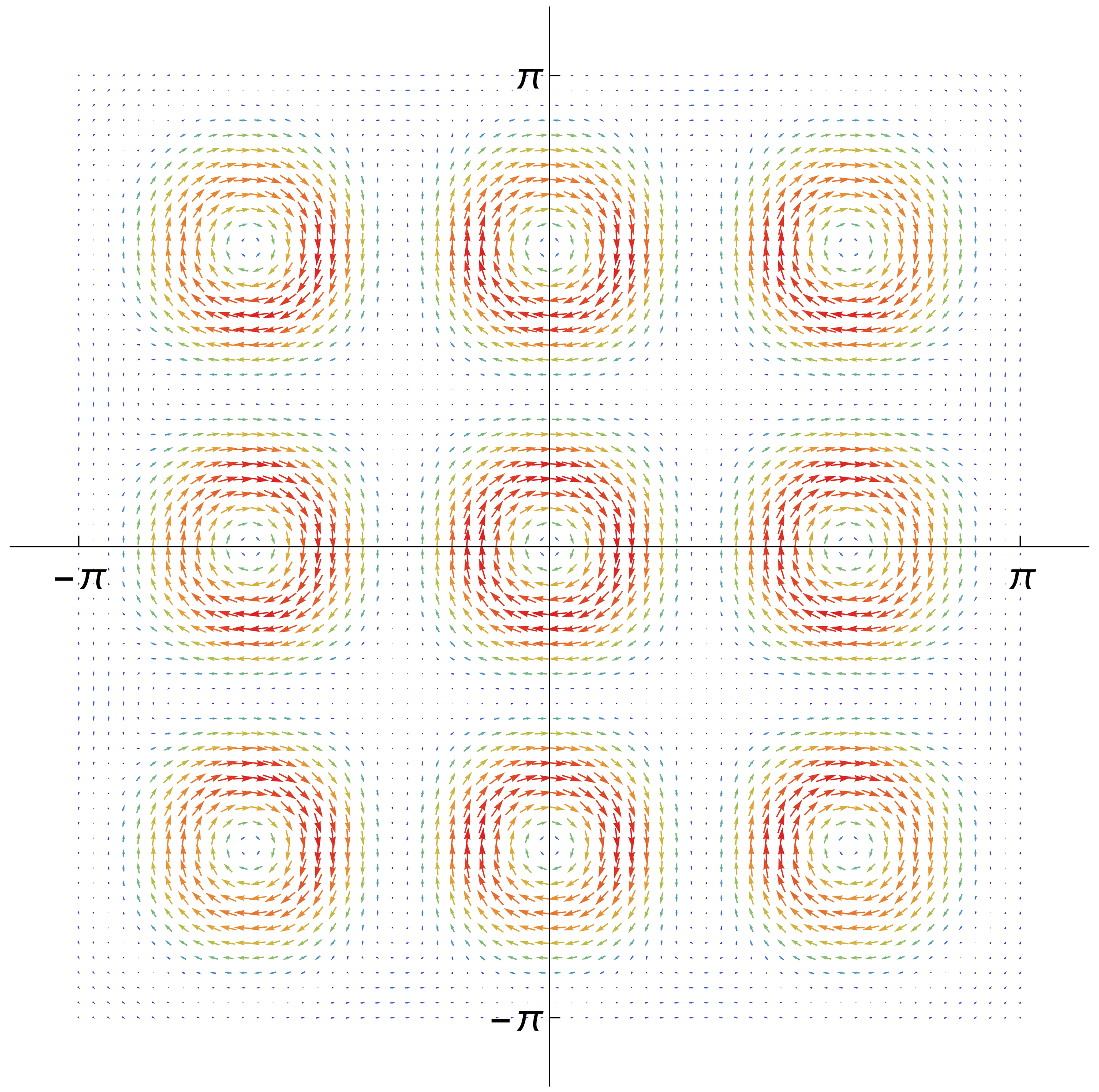
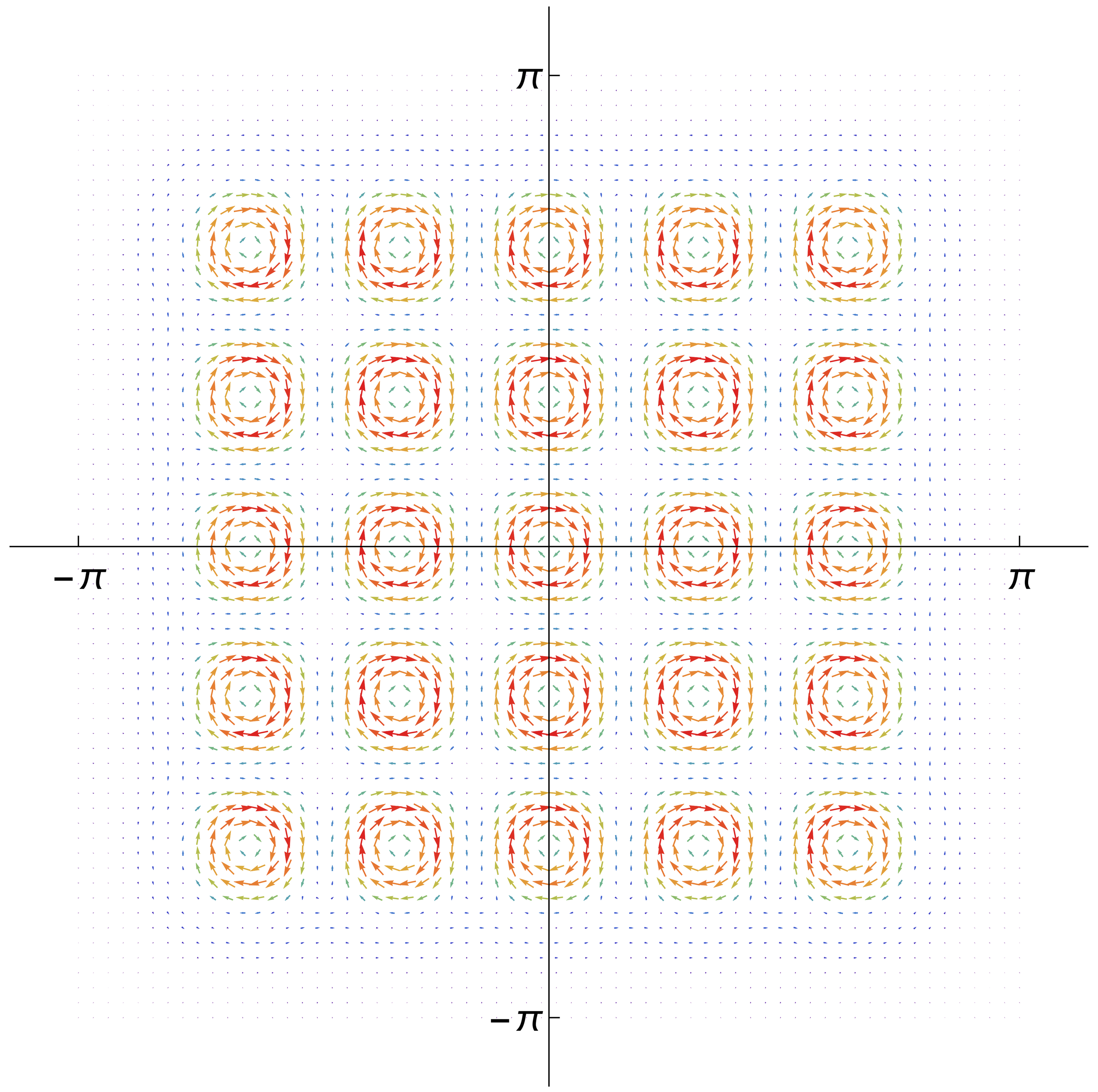


$$\vec{\psi}_1(x) = \vec{\psi}(2^1 x)$$




$$\vec{\psi}_2(x) = \vec{\psi}(2^2 x)$$

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# Divergence free wavelets

Proposition: $\vec{u} \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^2)$

$$\vec{u}(x) = \sum_{j=-1}^{\infty} \sum_{k \in \mathbb{Z}^2} \left\langle \vec{u}(y) \, , \, \vec{\psi}_{jk}(y) \right\rangle \vec{\psi}_{jk}(x)$$

$$\vec{\psi}_{jk}(x) = \frac{2^j}{2\pi} \vec{\psi}\big(2^j x - k\big)$$

# Divergence free wavelets

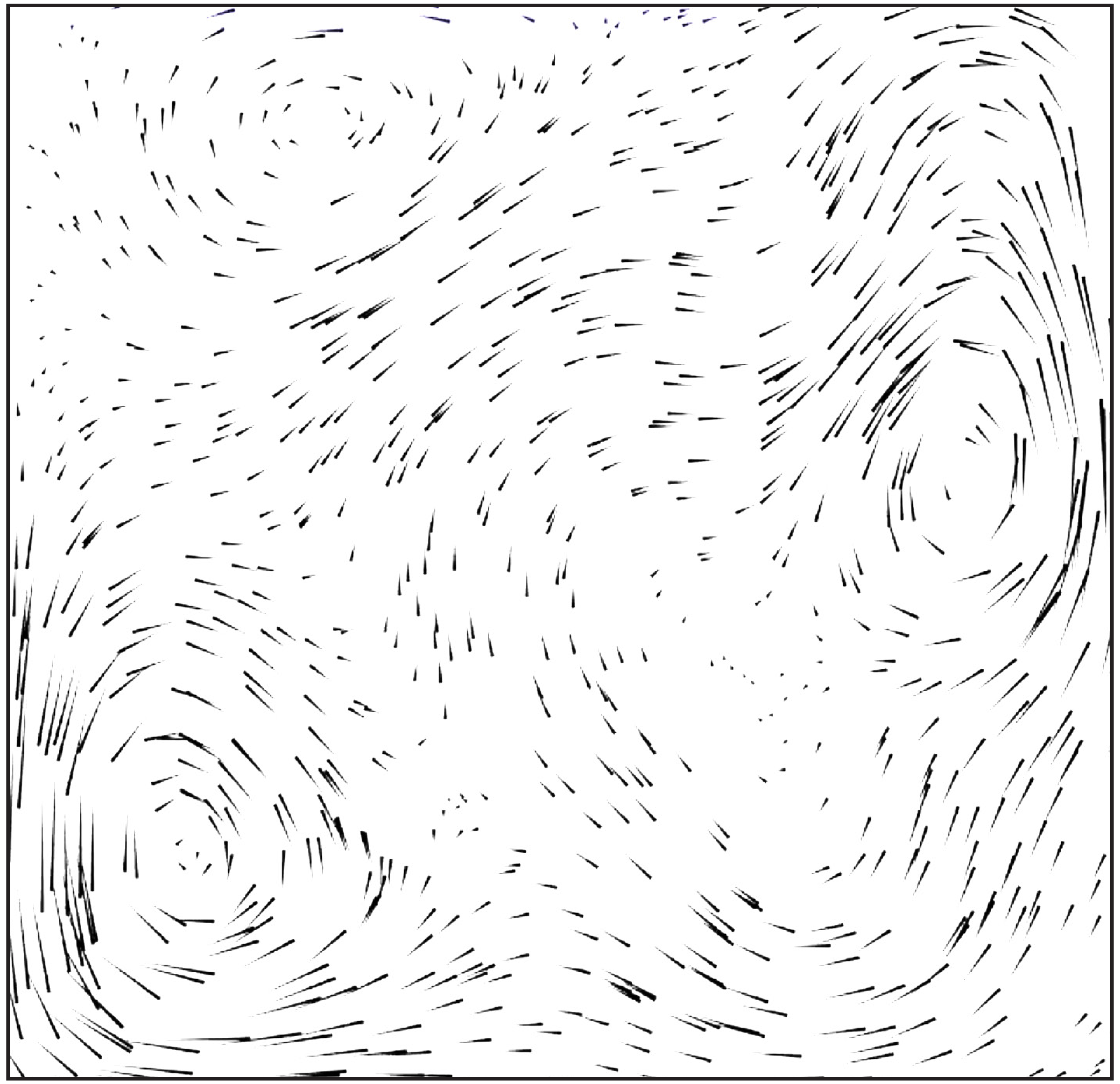
Proposition: $\vec{u} \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^2)$

Finer and finer vortices on finer and finer grids

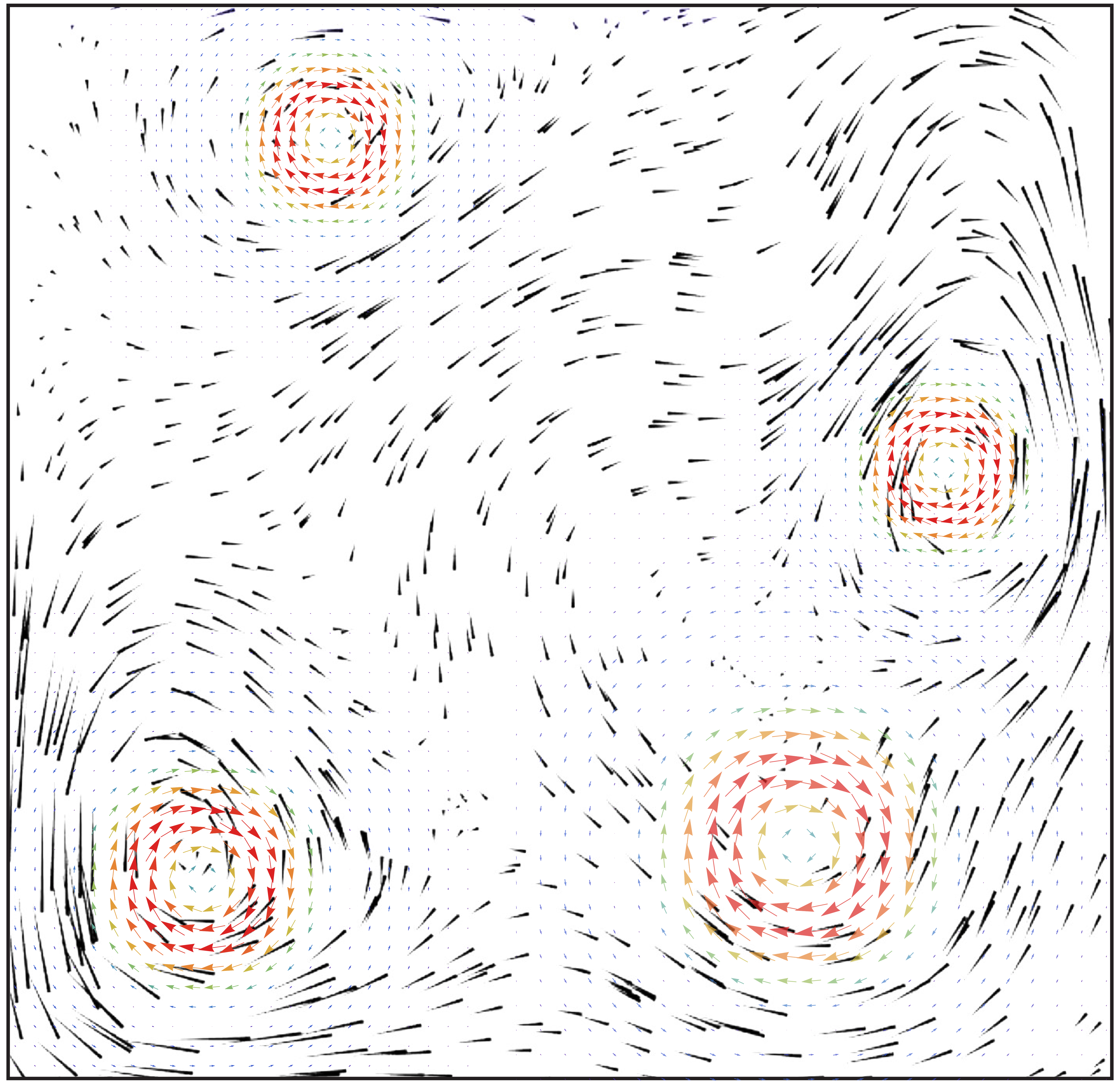
$$\vec{u}(x) = \sum_{j=-1}^{\infty} \sum_{k \in \mathbb{Z}^2} \langle \vec{u}(y) \, , \, \vec{\psi}_{jk}(y) \rangle \boxed{\vec{\psi}_{jk}(x)}$$

$$\vec{\psi}_{jk}(x) = \frac{2^j}{2\pi} \vec{\psi}(2^j x - k)$$

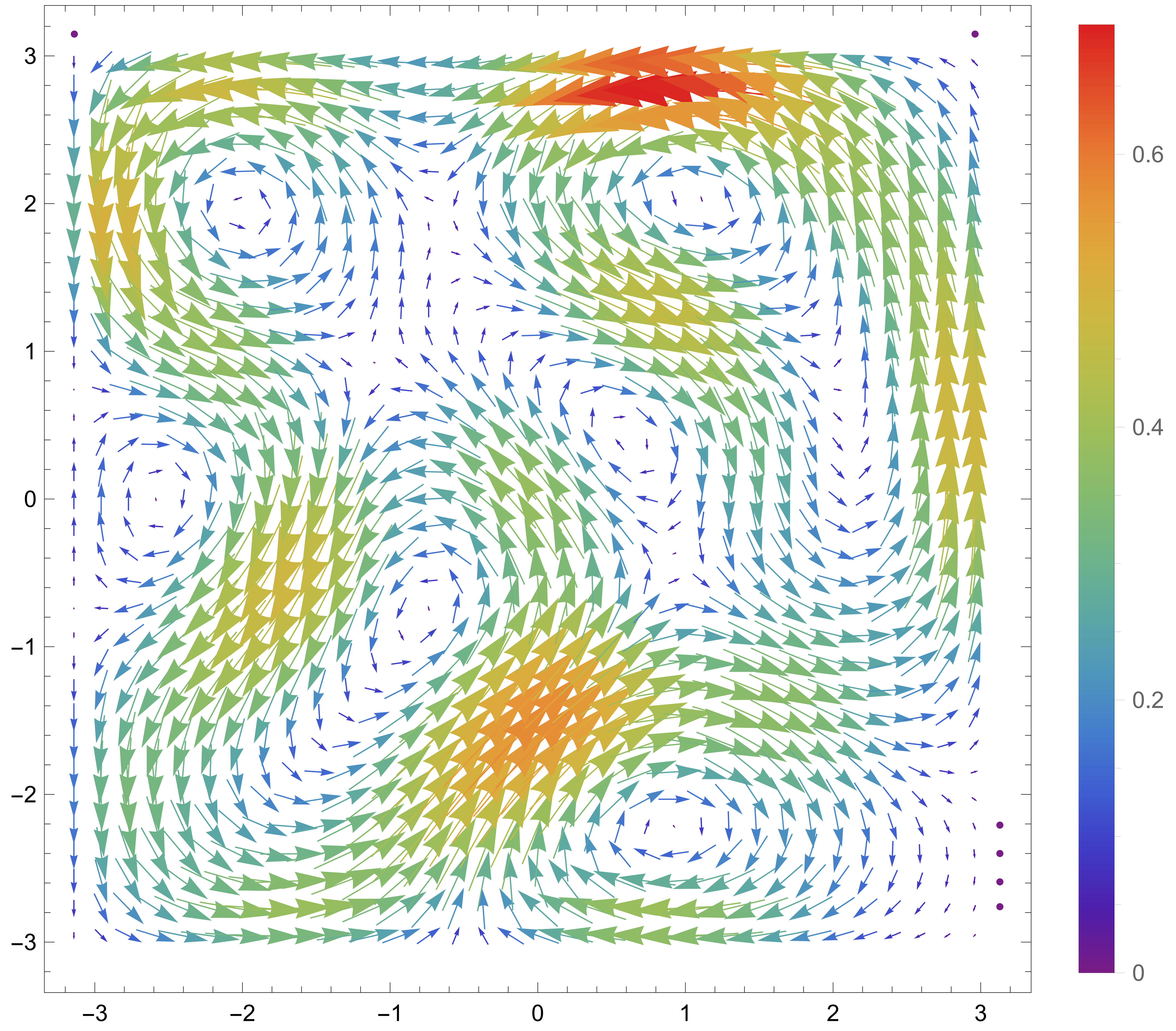
# Divergence free wavelets



# Divergence free wavelets



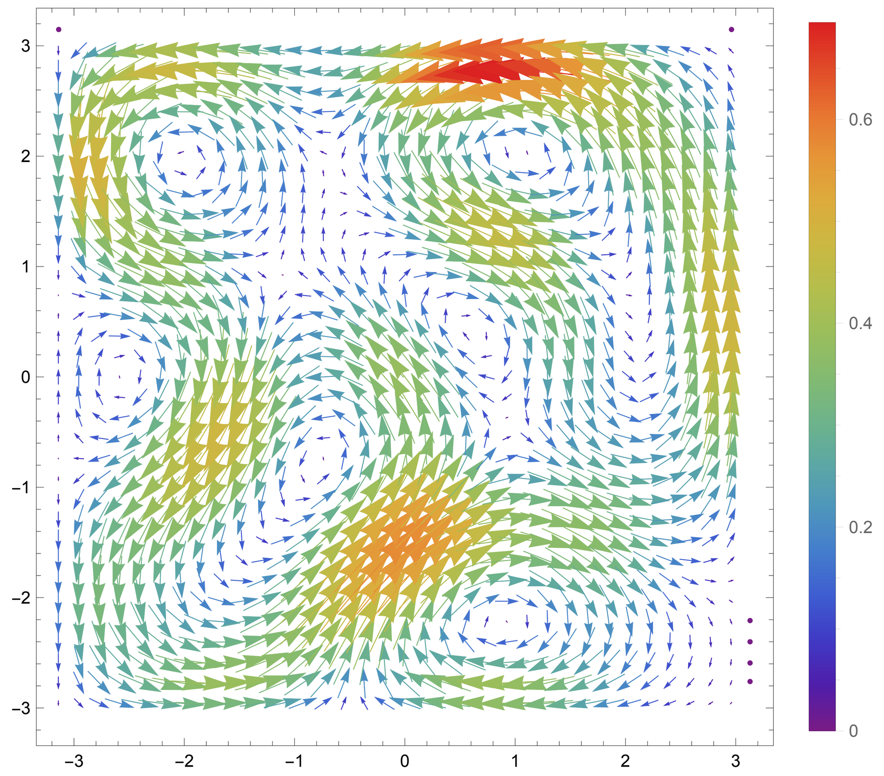
# Divergence free wavelets



# Divergence free wavelets

Error

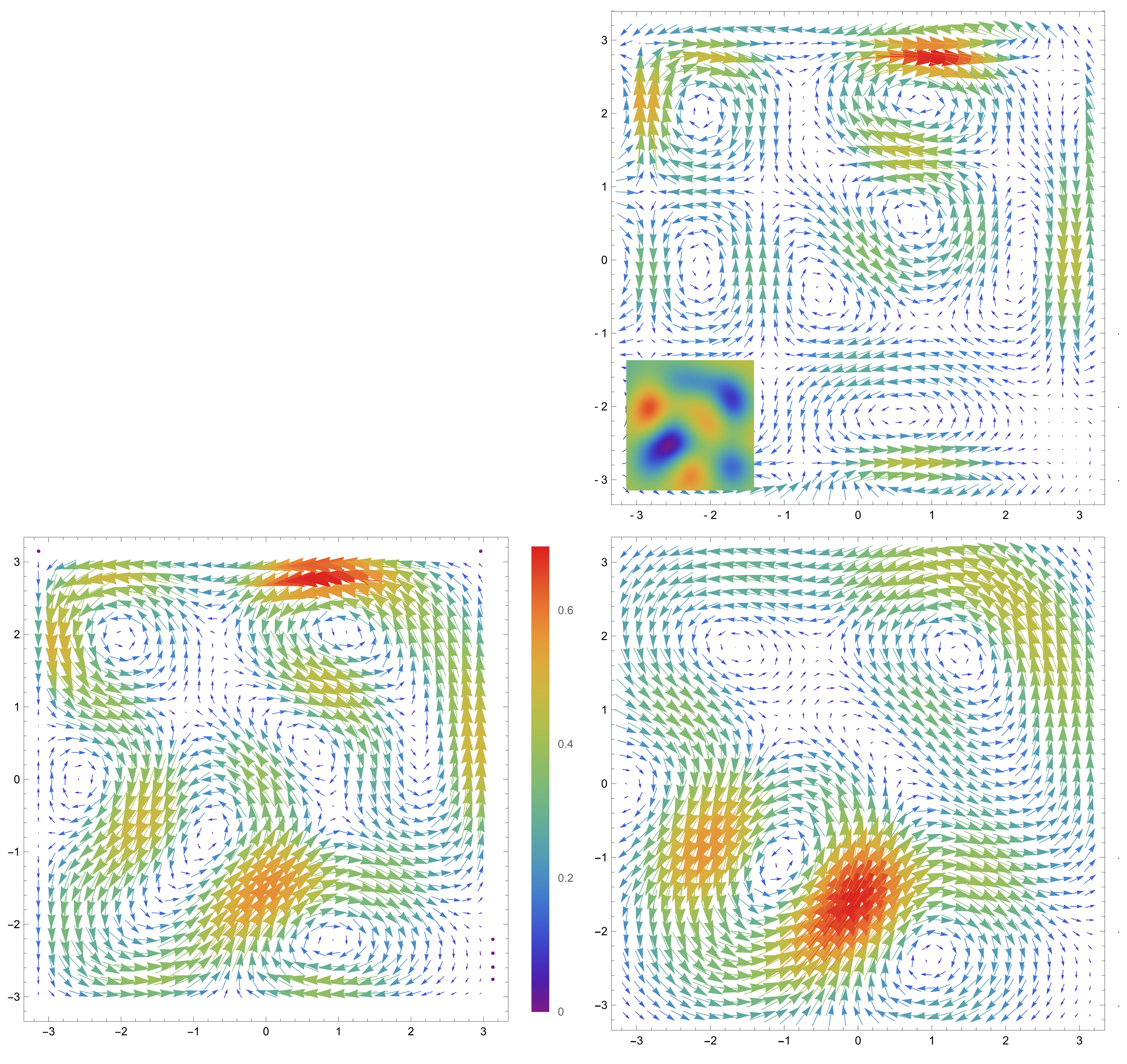
Reconstruction



# Divergence free wavelets

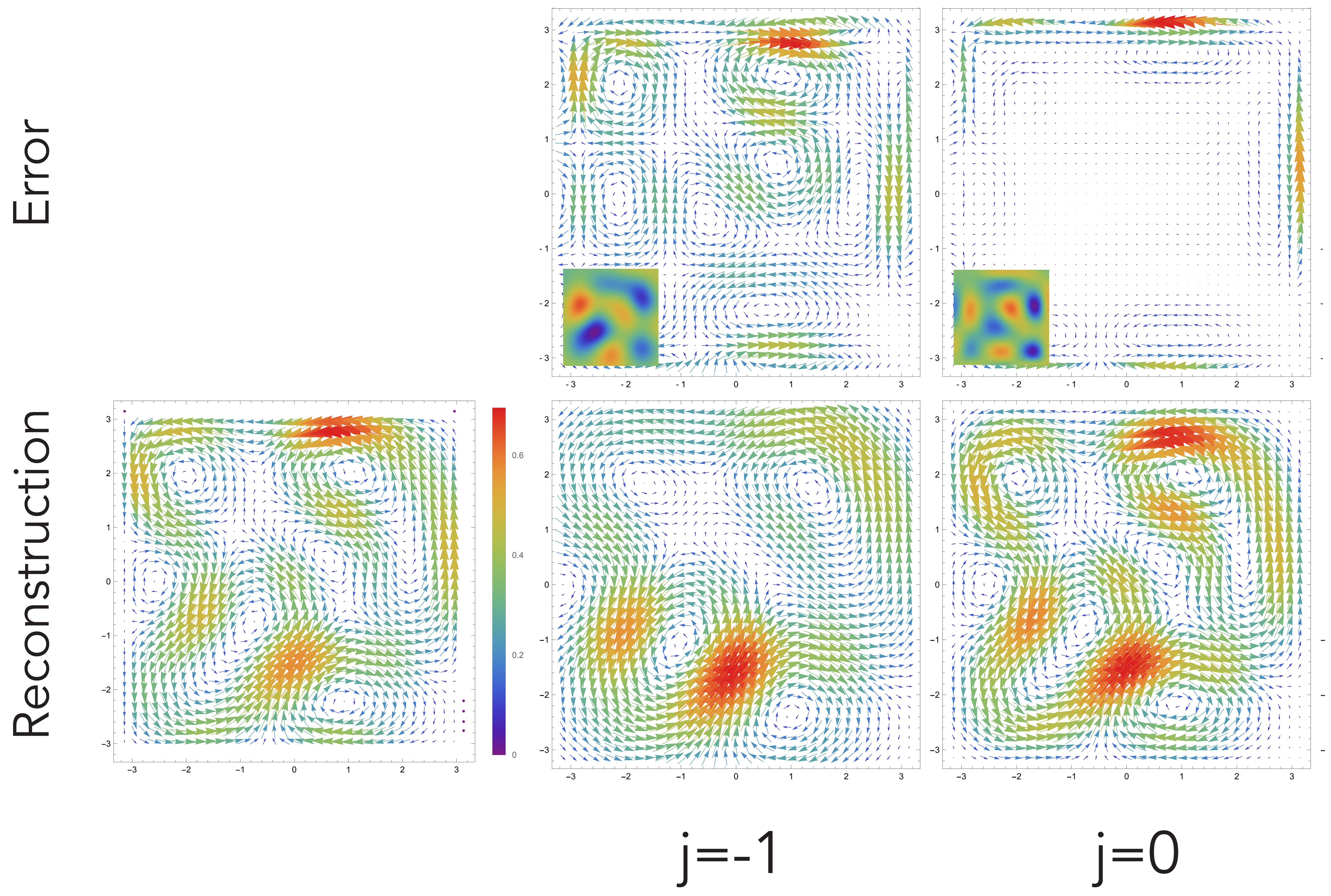
Error

Reconstruction



j=-1

# Divergence free wavelets



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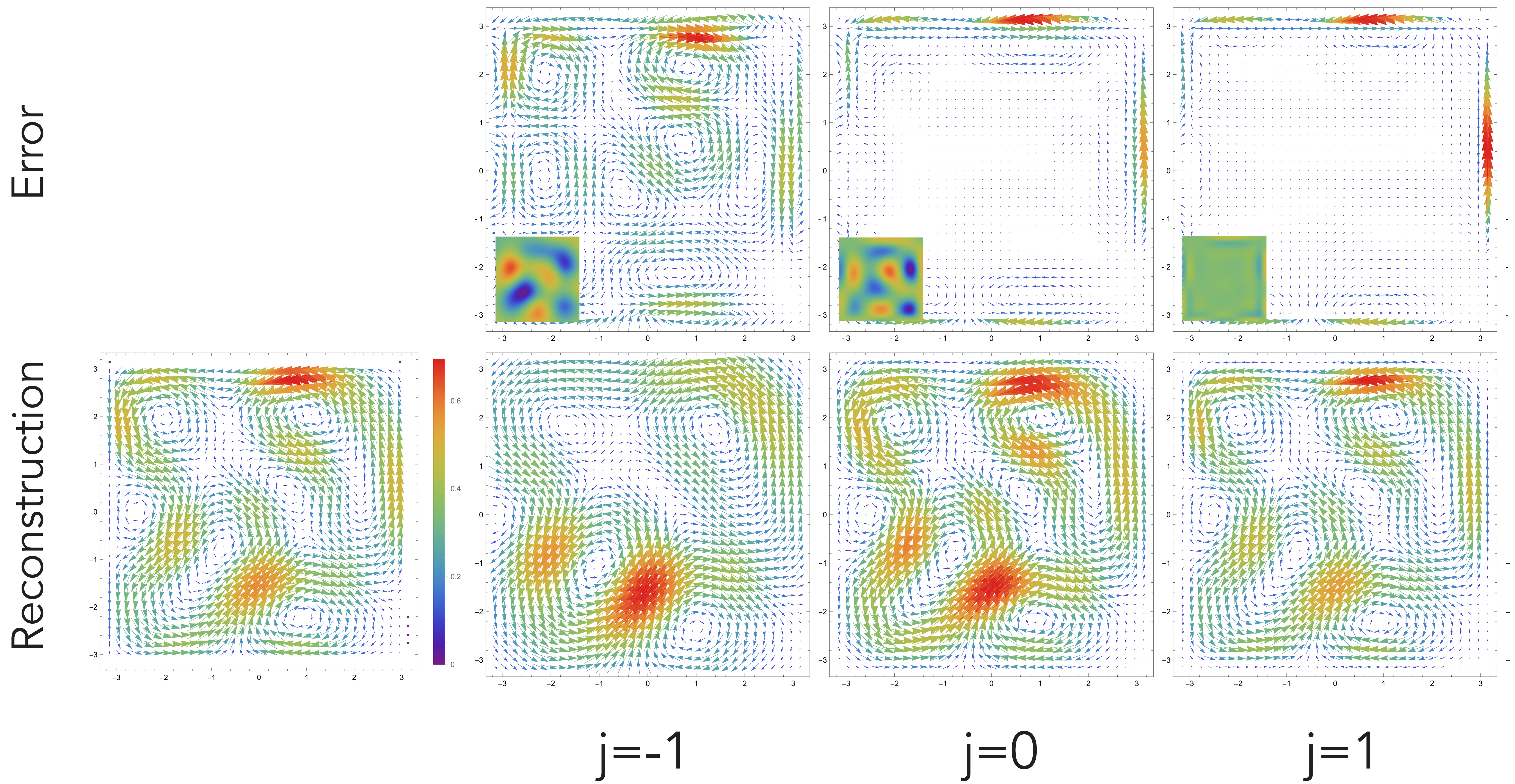

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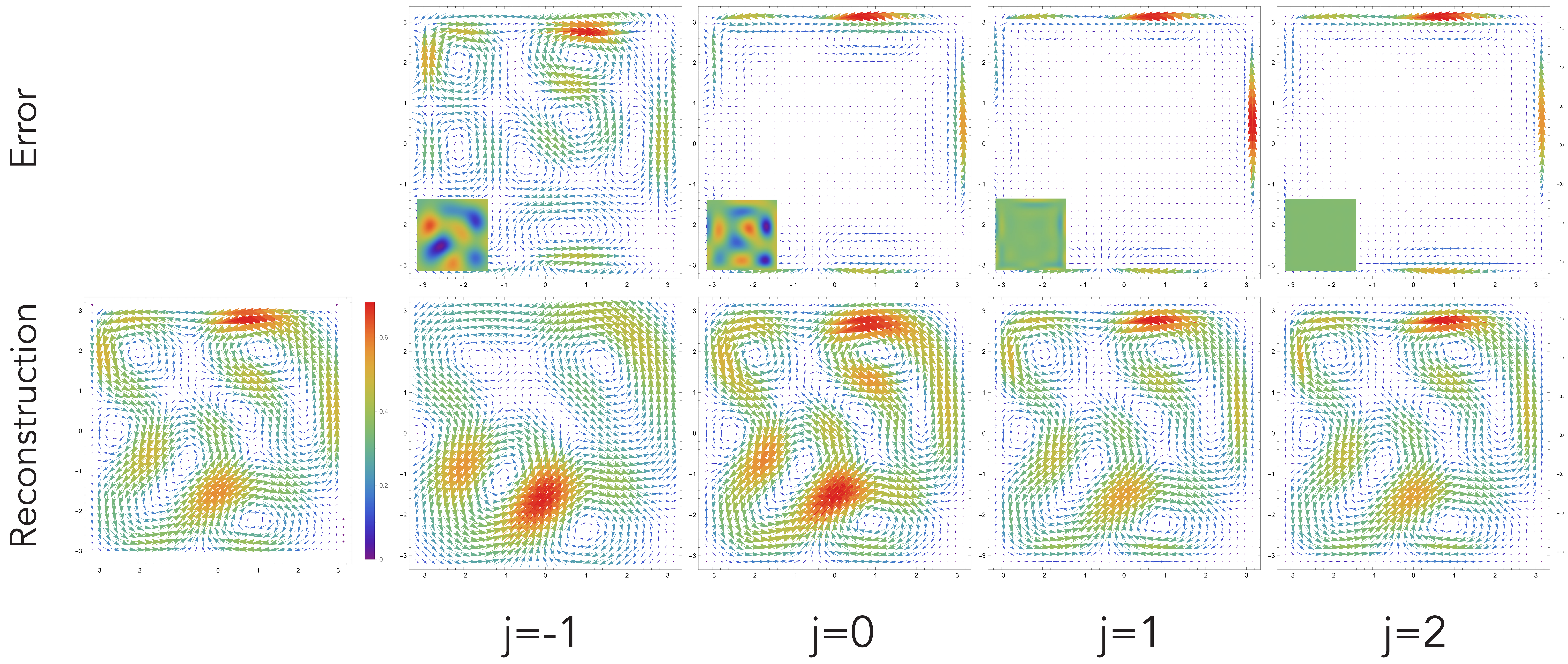


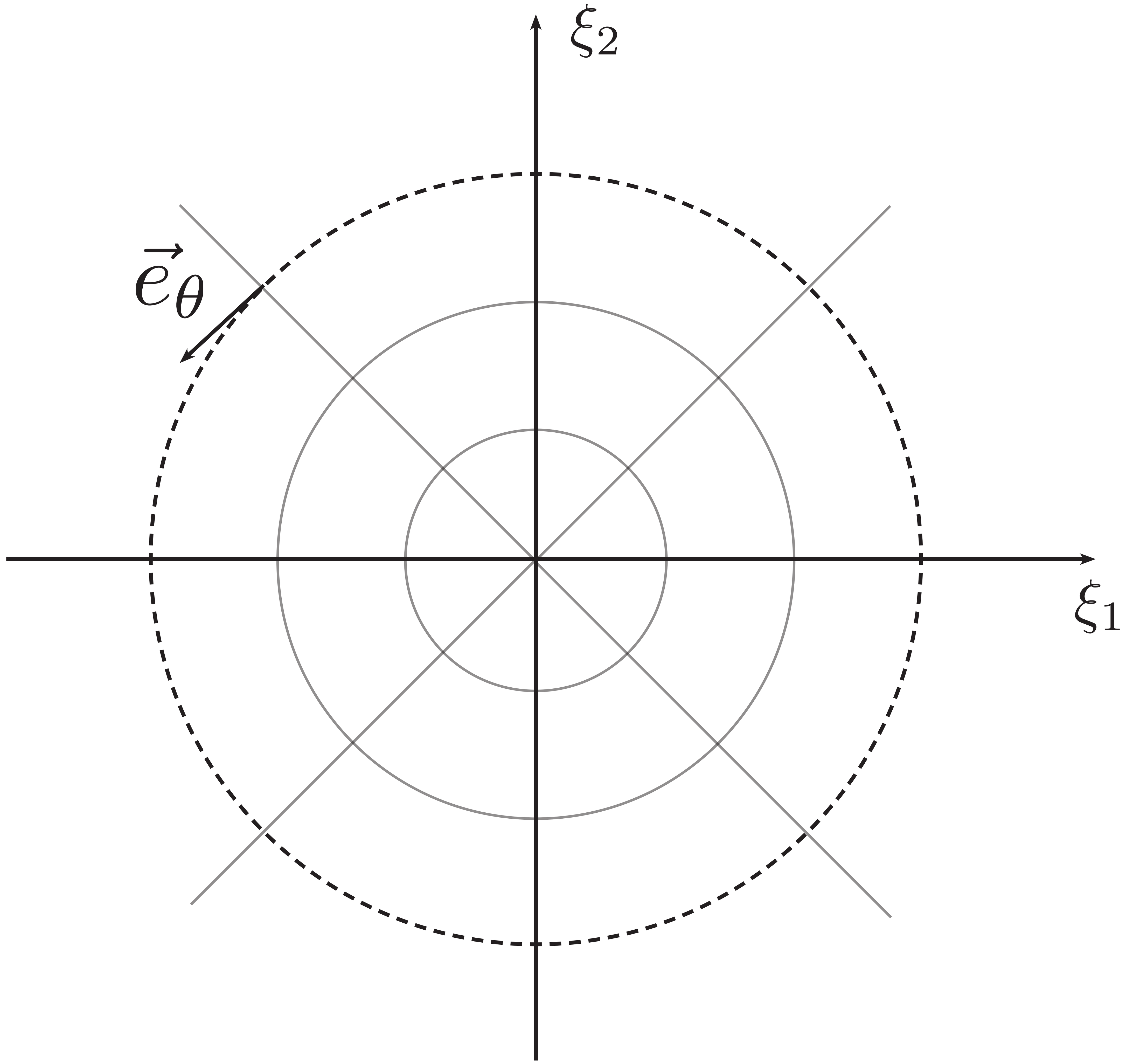
# Divergence free wavelets

# Divergence free wavelets

Isotropic vortices are not well suited for boundary flows

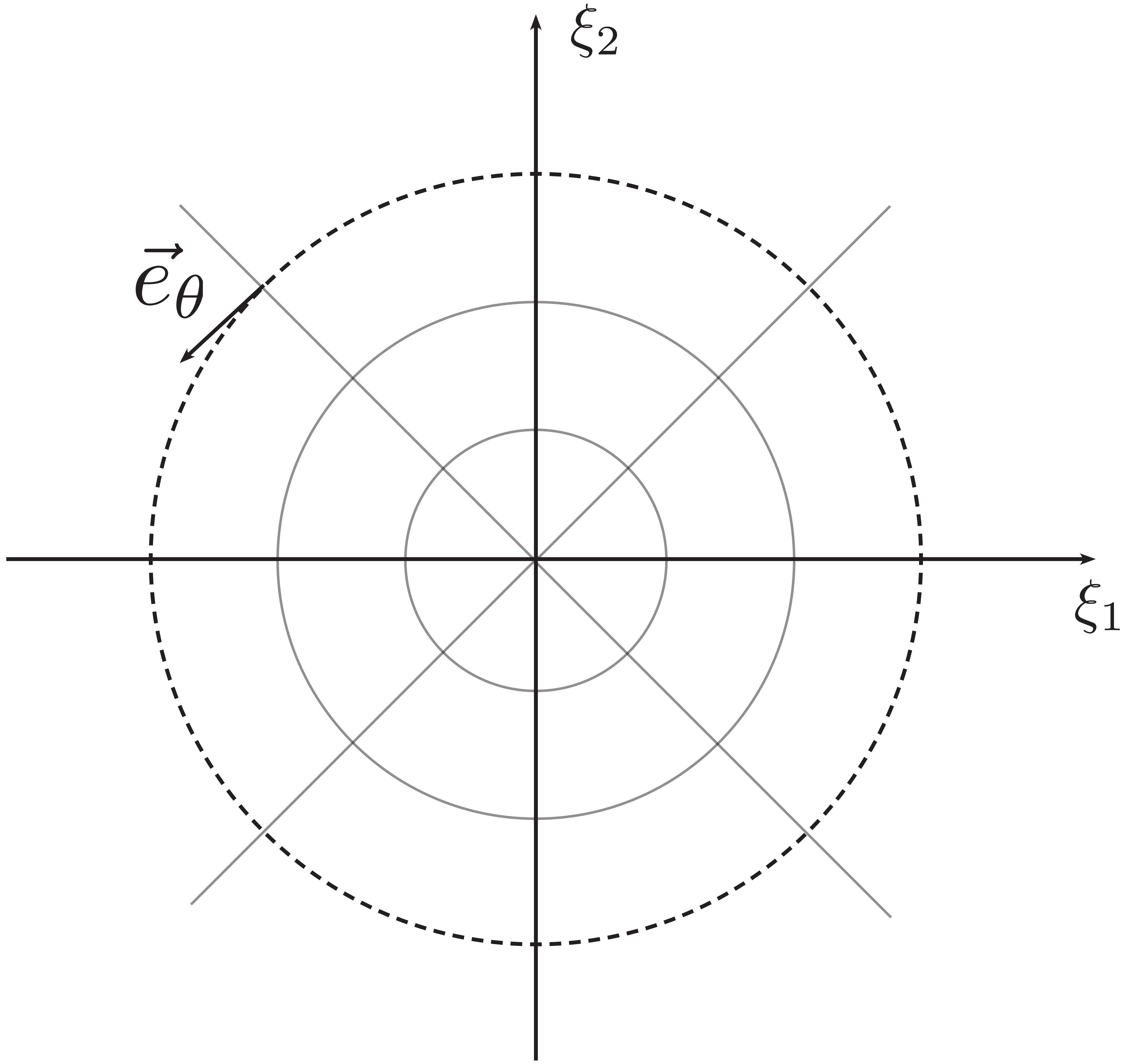
# Divergence free wavelets

$$\hat{\vec{\psi}}(\xi) = \quad \dots \quad \vec{e}_\theta$$



# Divergence free wavelets

$$\hat{\vec{\psi}}(\xi) = \underbrace{\quad \dots \quad}_{\text{window function compatible with polar coordinates}} \vec{e}_\theta$$



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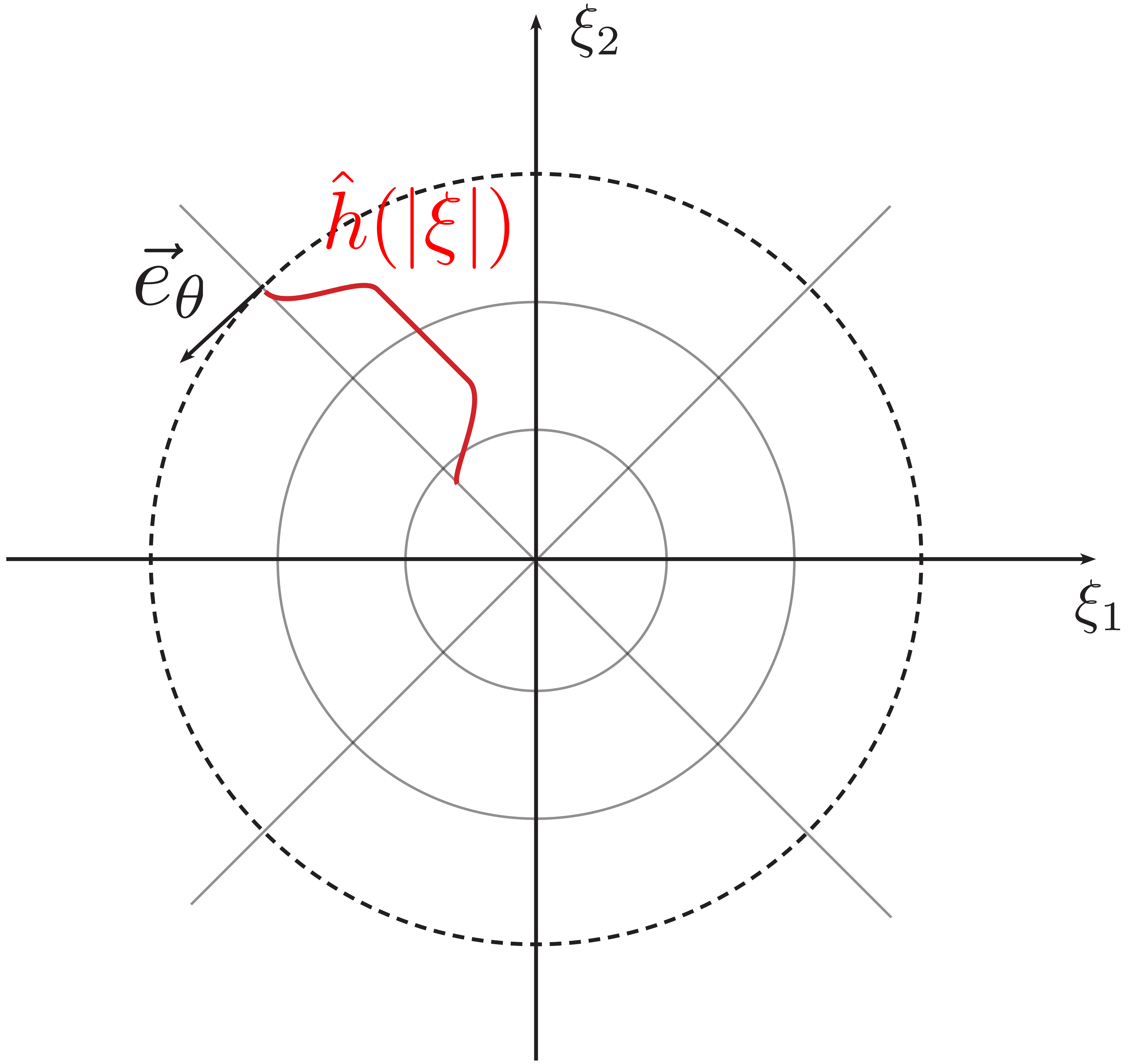

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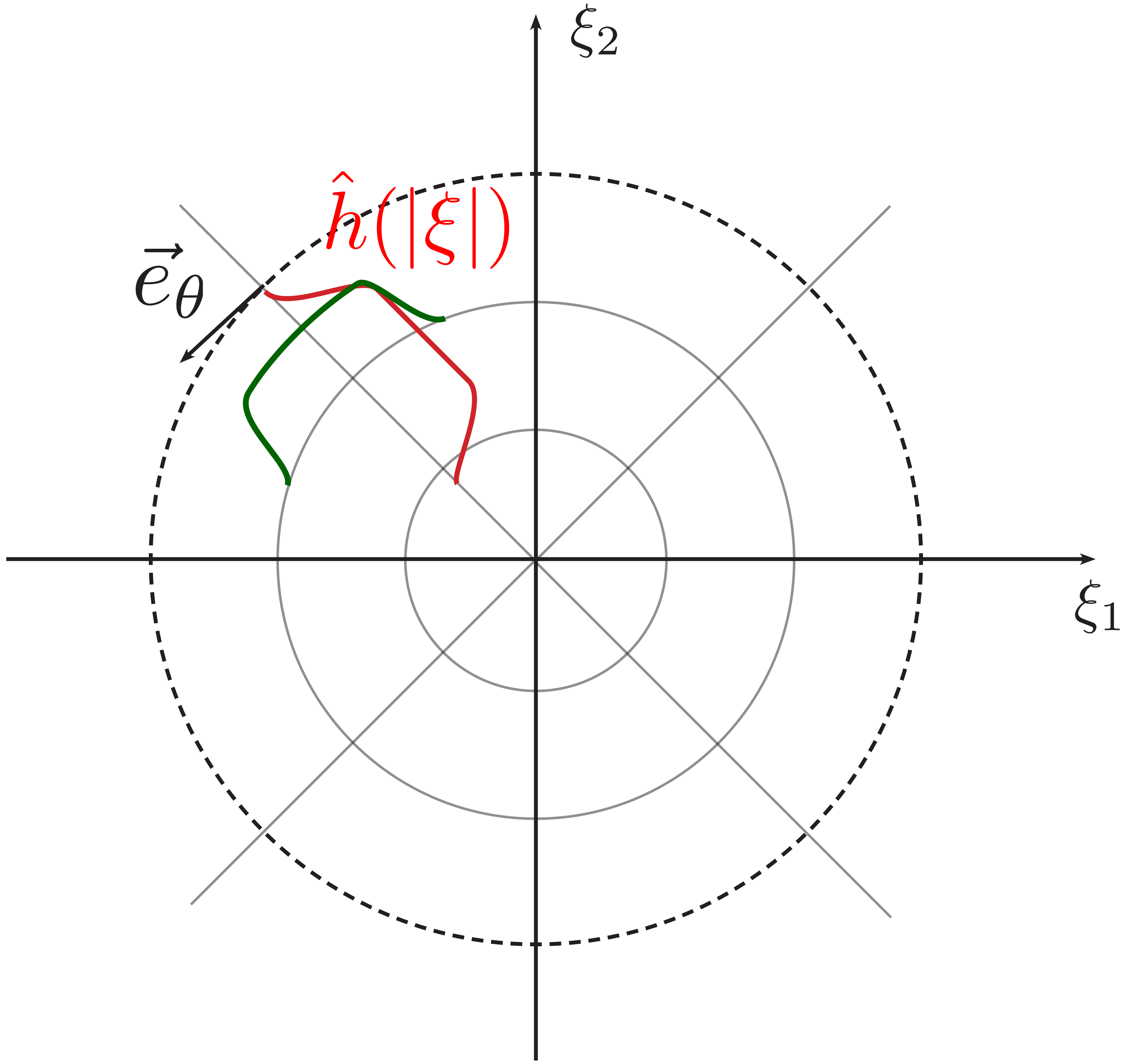
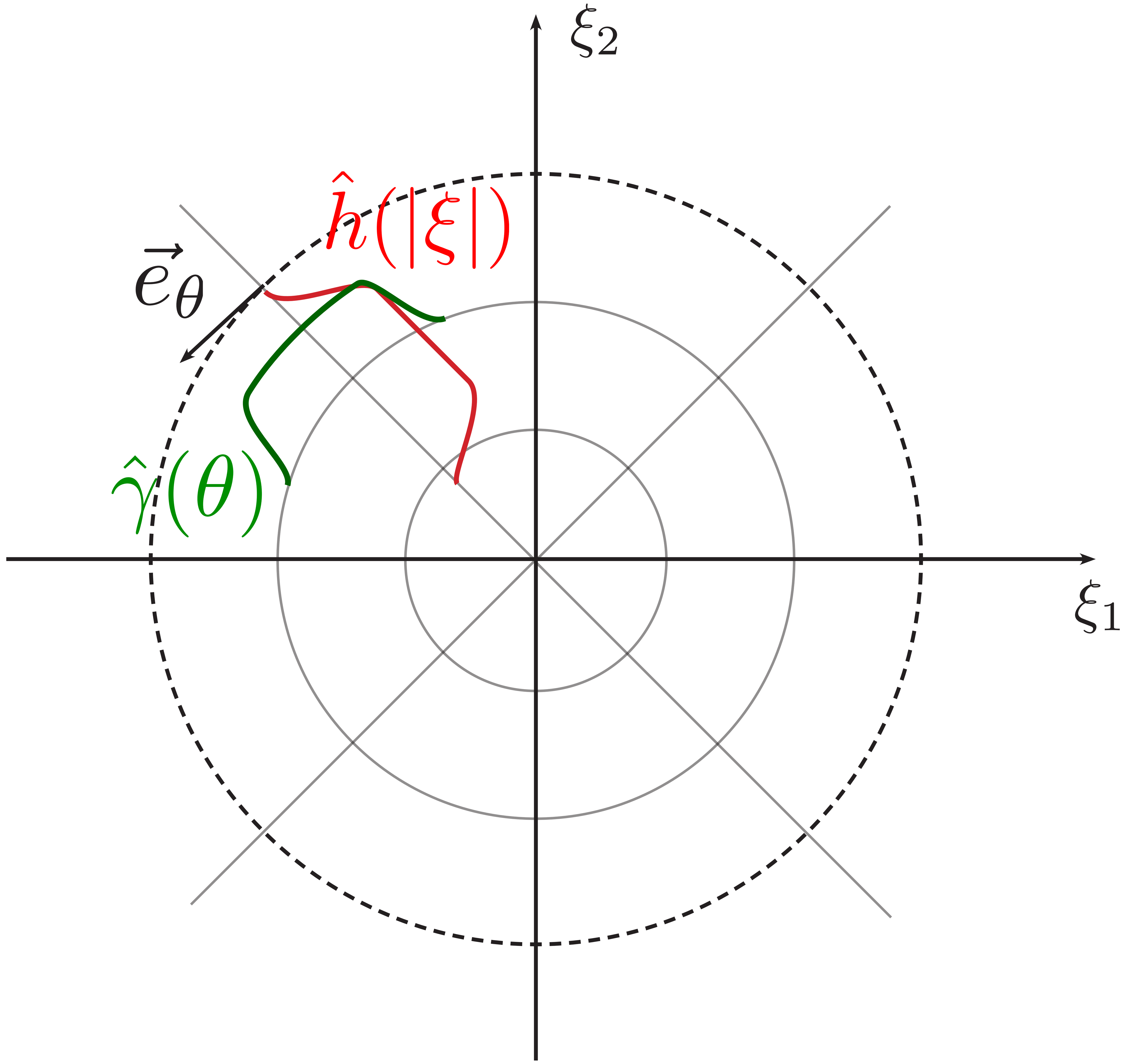


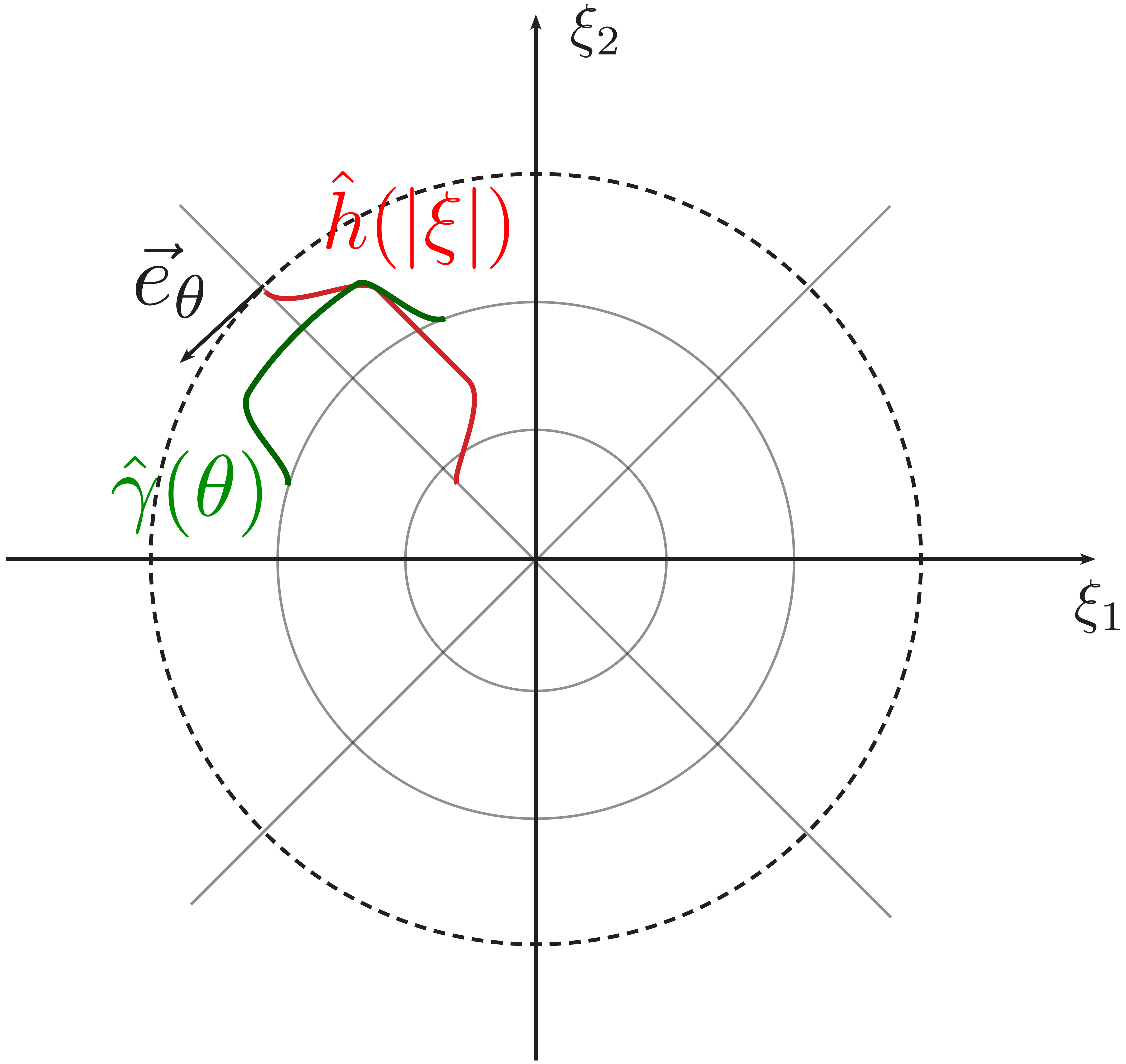
# Divergence free wavelets

$$\hat{\vec{\psi}}(\xi) = -i\,\hat{\gamma}(\theta)\,\hat{h}(|\xi|)\,\vec{e}_\theta$$



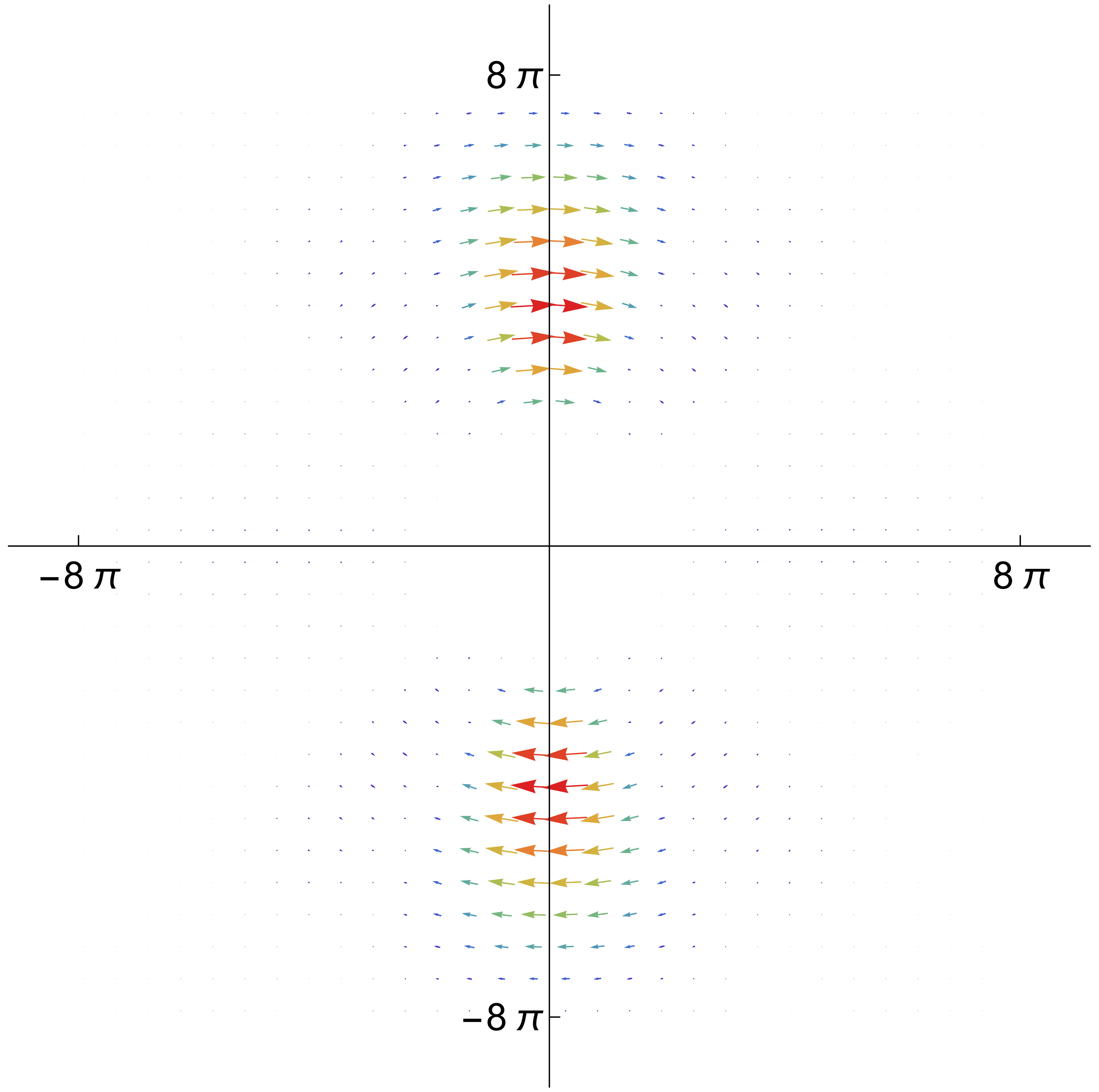
# Divergence free wavelets

$$\hat{\vec{\psi}}(\xi) = -i\,\underbrace{\hat{\gamma}(\theta)}_{\hat{\gamma}(\theta) = \sum_m \beta_m\, e^{im\theta}}\,\hat{h}(|\xi|)\,\vec{e}_\theta$$



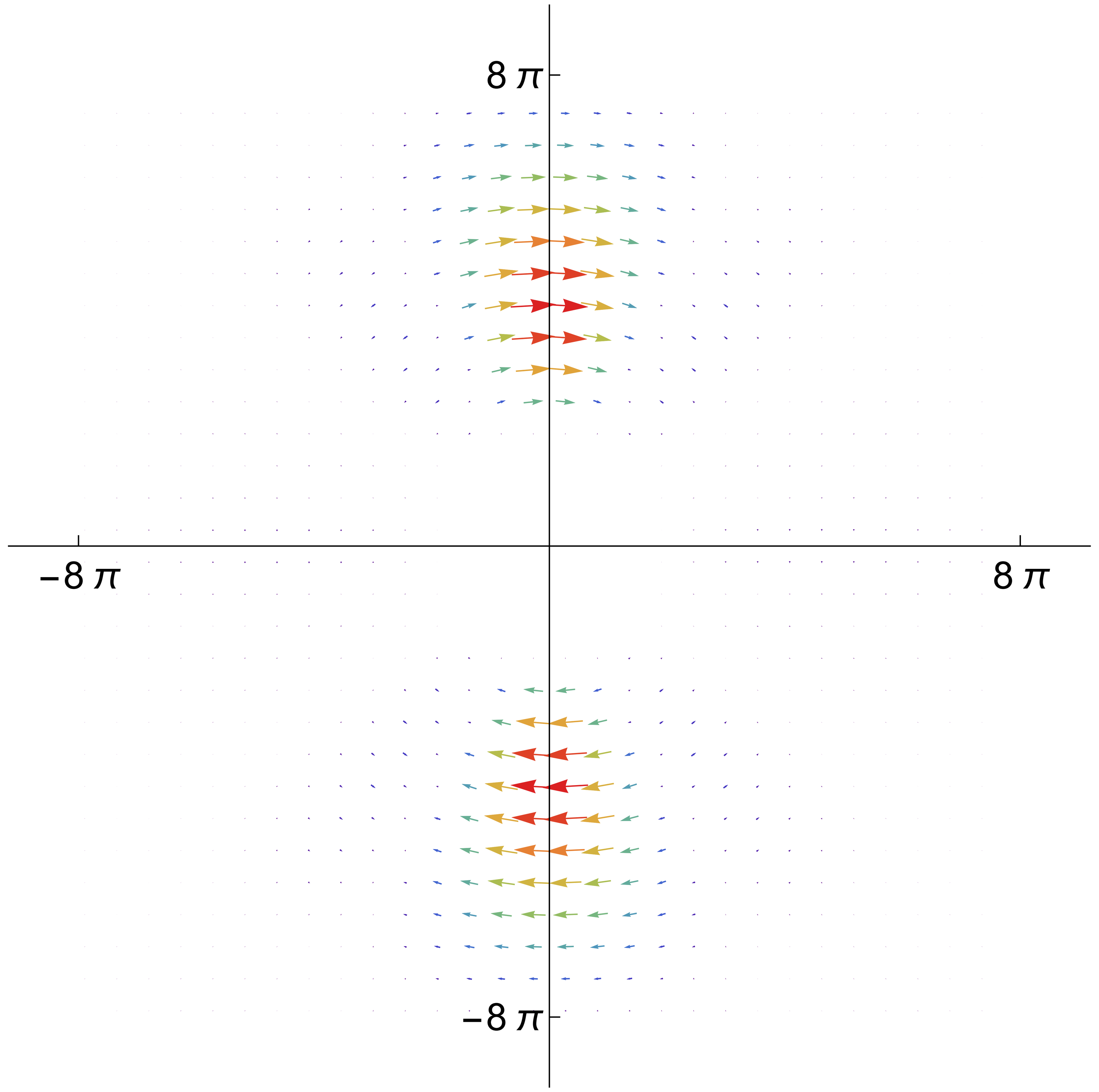
# Divergence free wavelets

$$\hat{\vec{\psi}}(\xi)$$

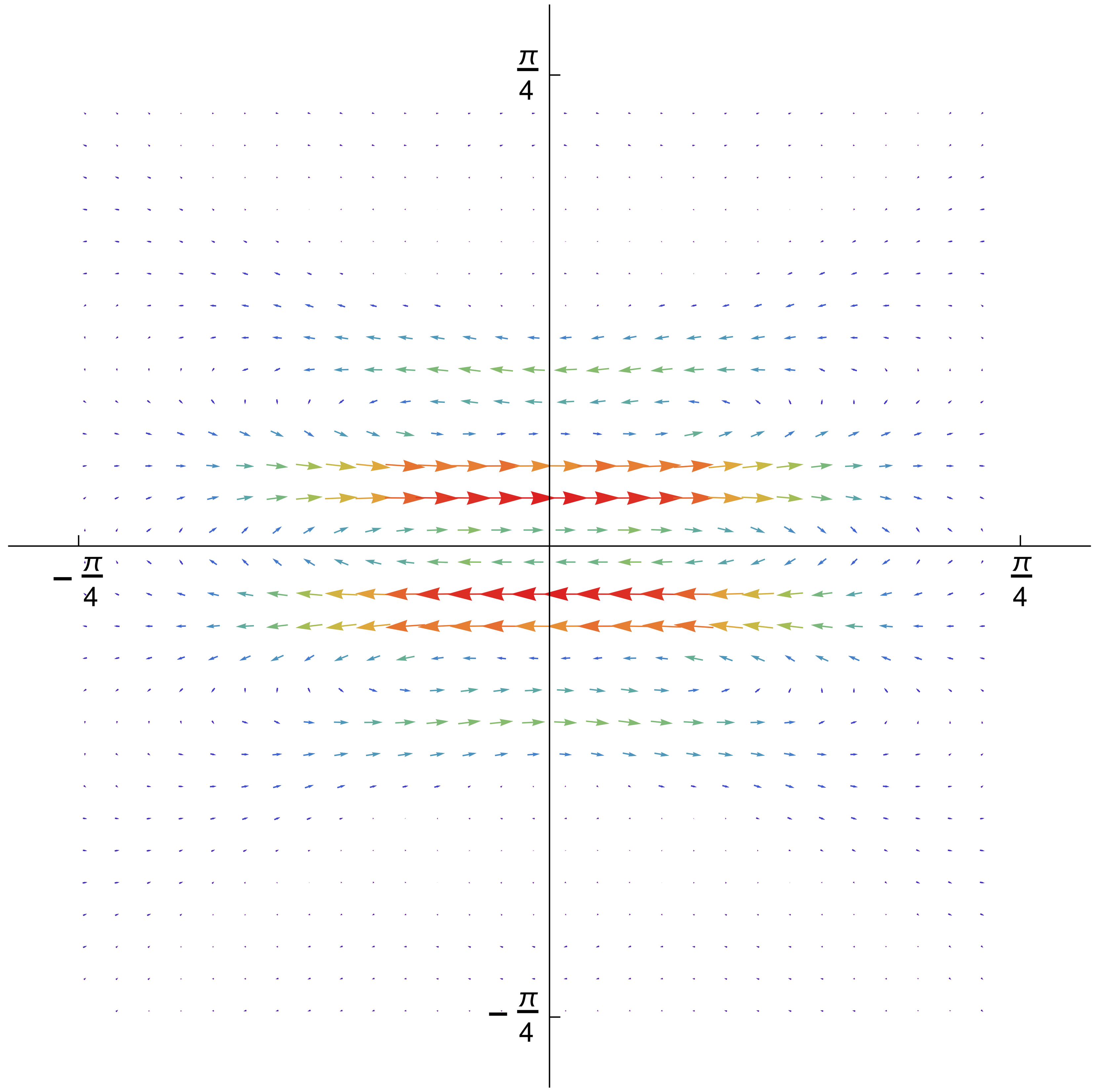


# Divergence free wavelets

$$\hat{\vec{\psi}}(\xi)$$




$$\vec{\psi}(x)$$



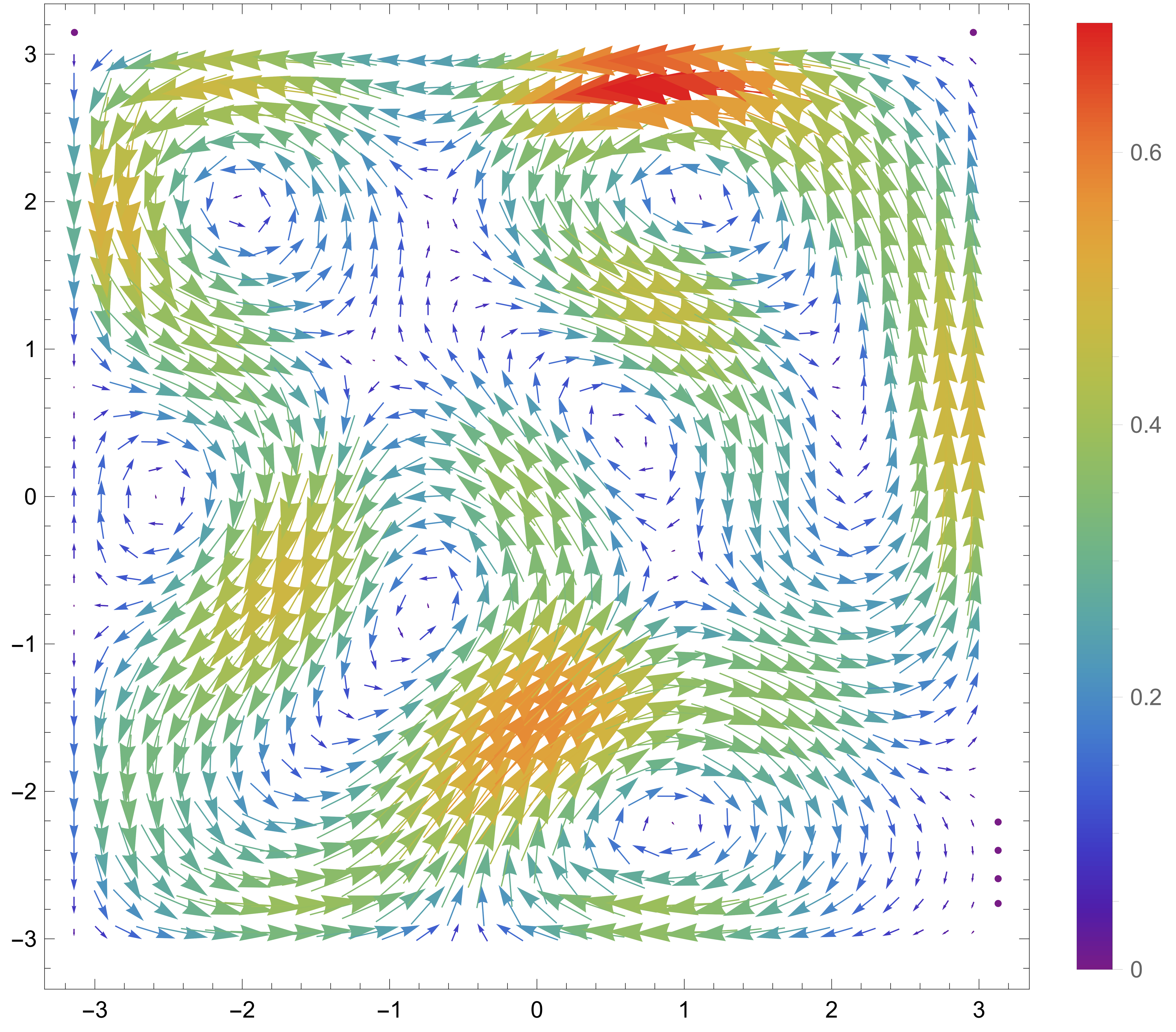
# Divergence free wavelets

# Divergence free wavelets

Properties of directional divergence free wavelets:

- Tight frame for $L_2^{\mathrm{div}}(\mathbb{R}^2)$
- Closed form expression in spatial domain
- Quasi-optimal approximation of discontinuities (under suitable assumptions)

# Divergence free wavelets



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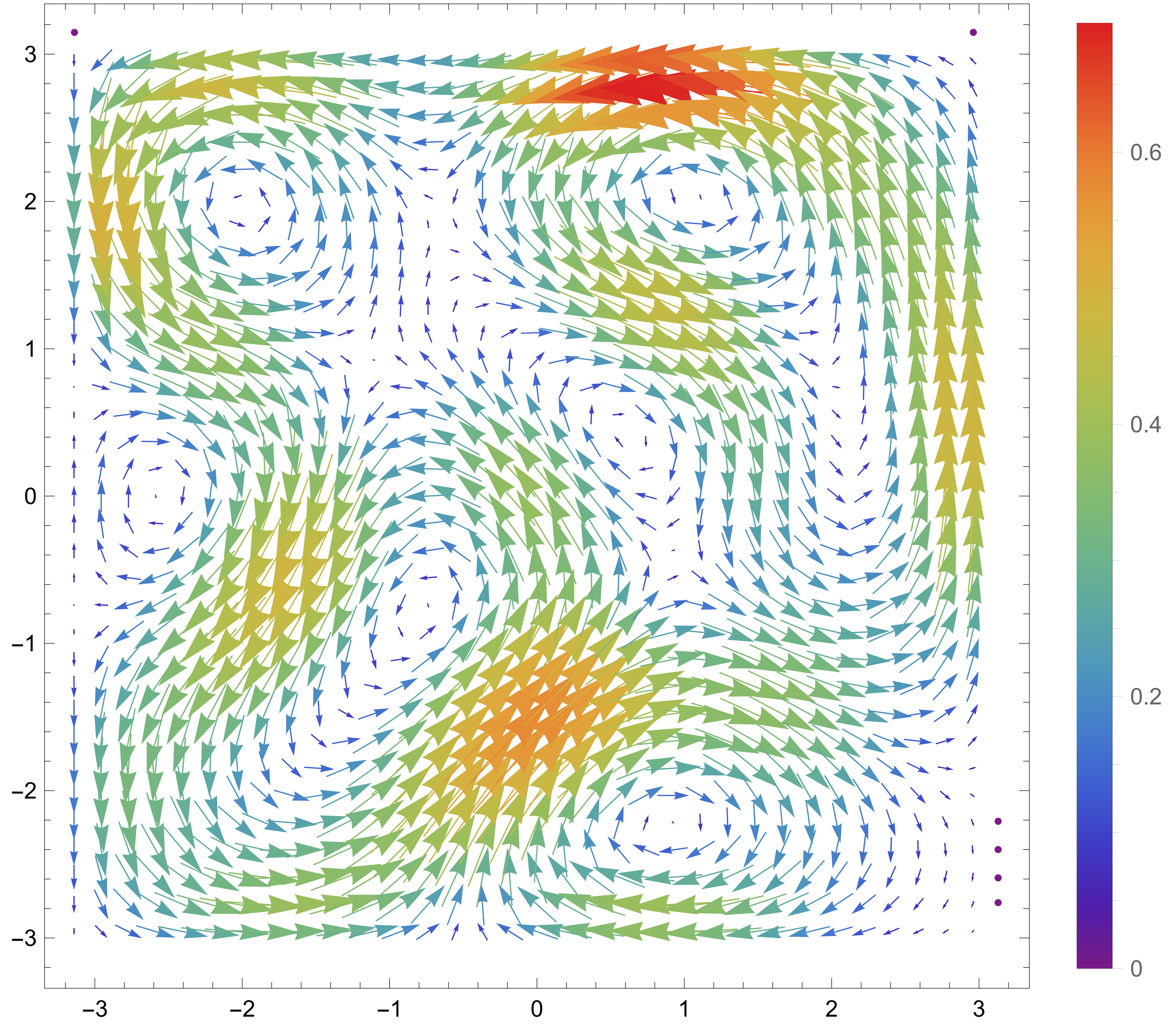
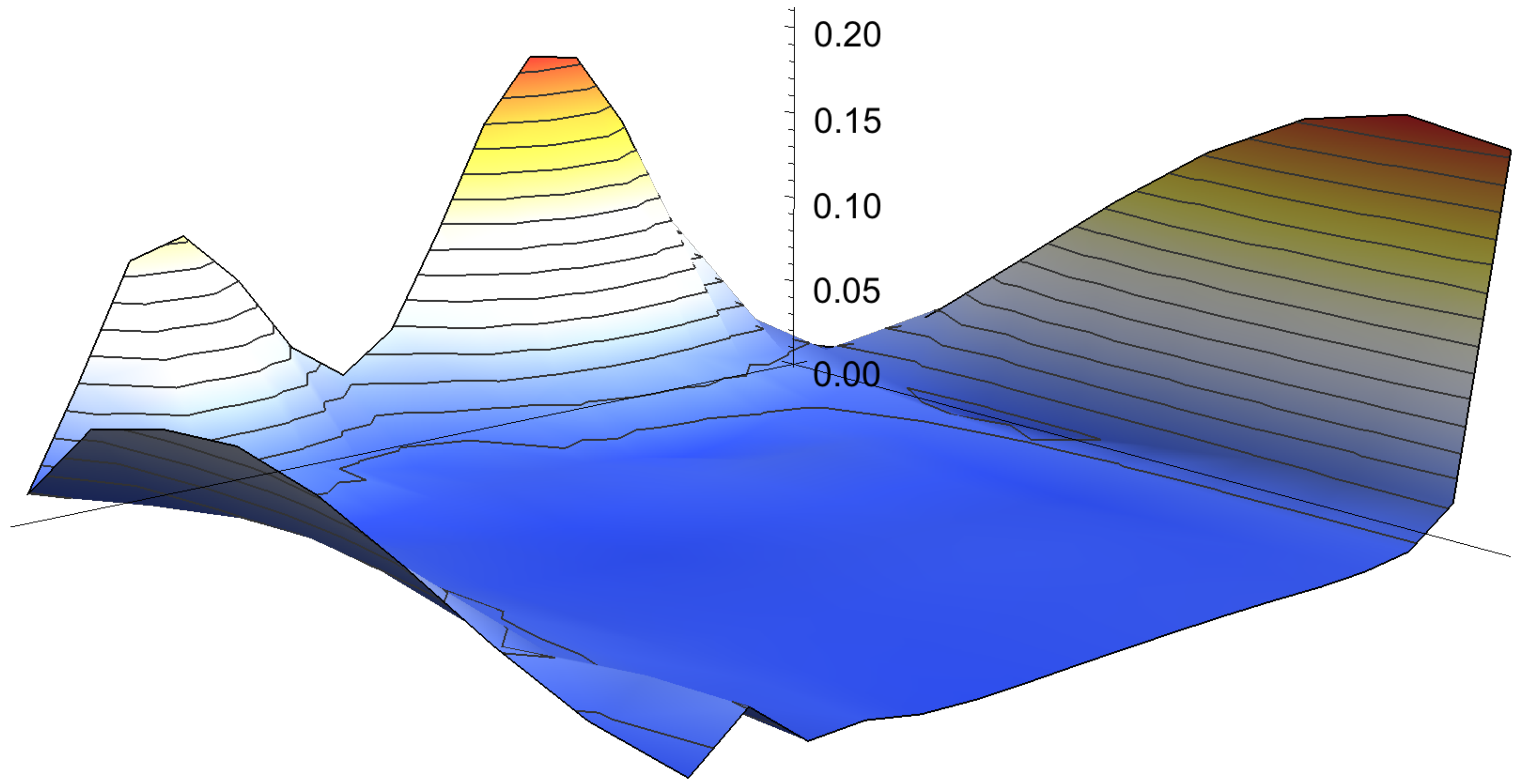
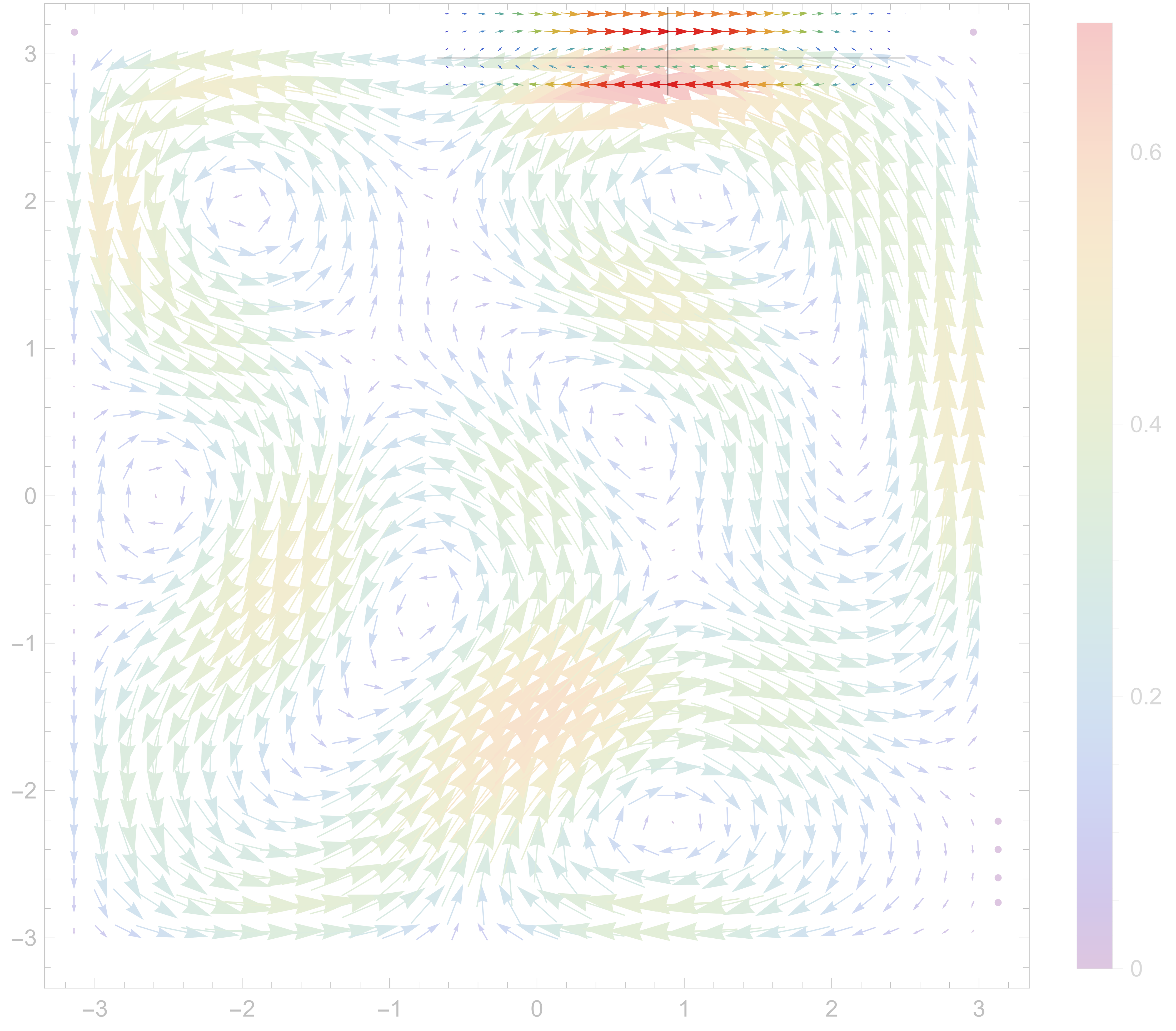


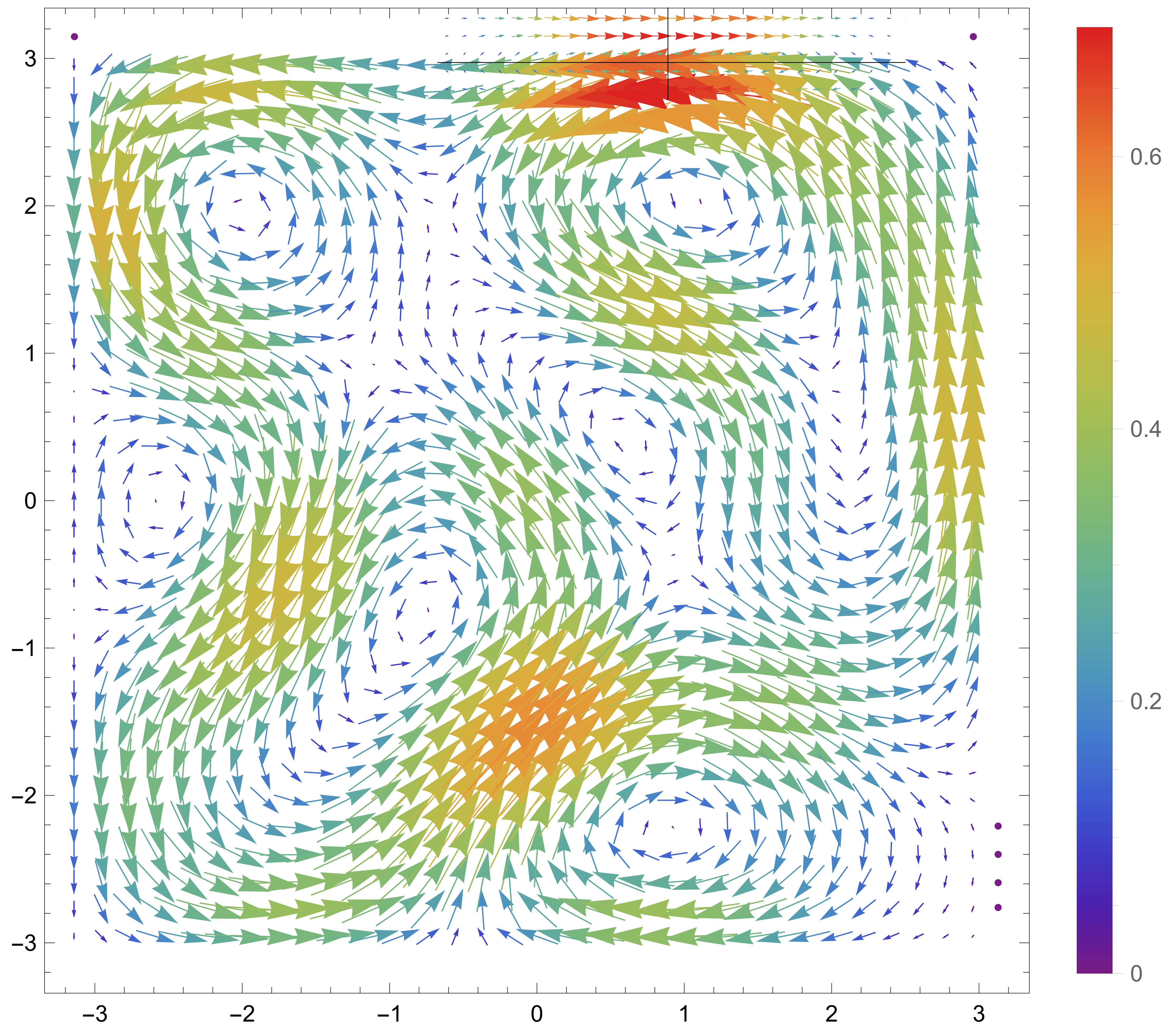



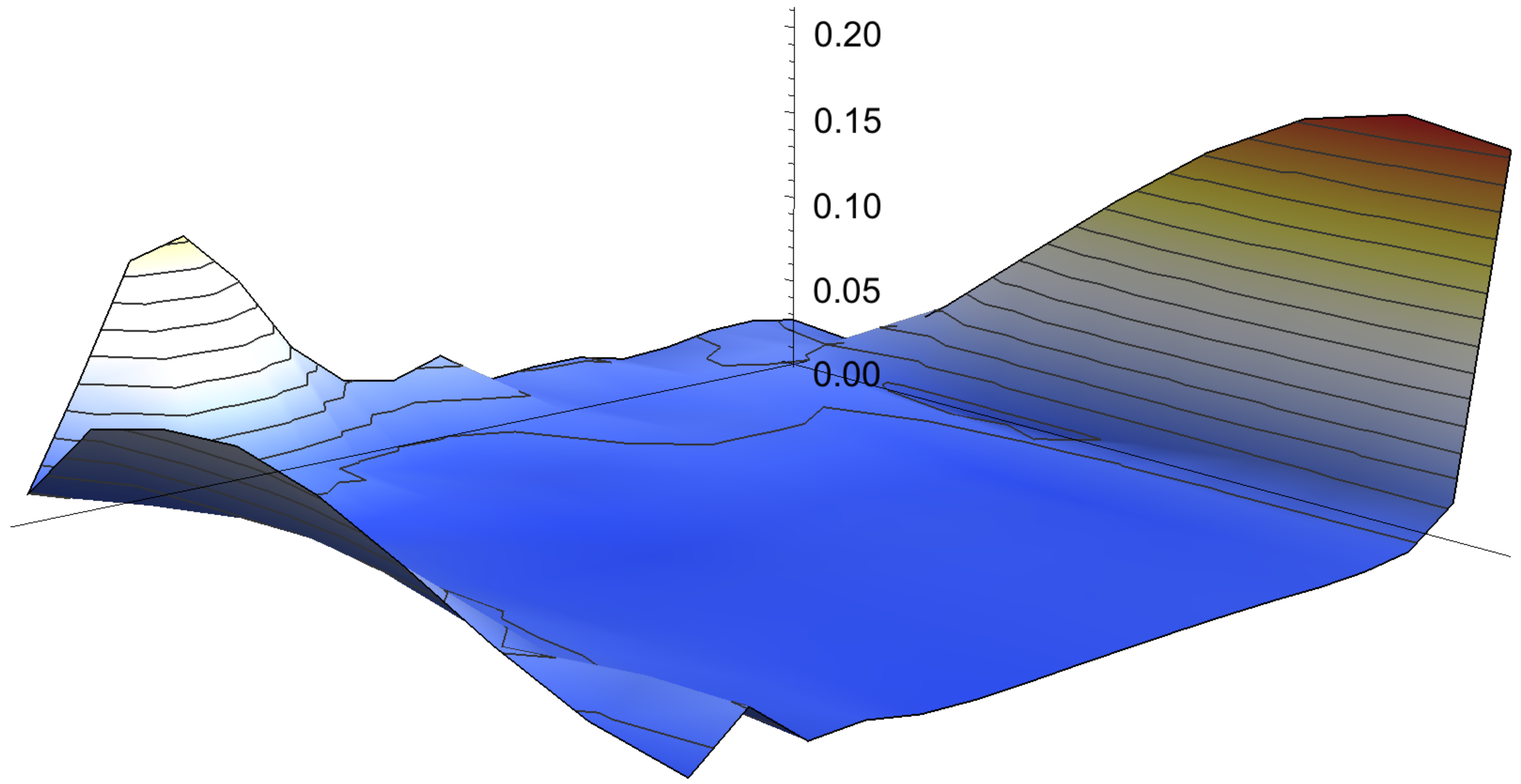

error for upper half without directional wavelets

# Divergence free wavelets



# Divergence free wavelets



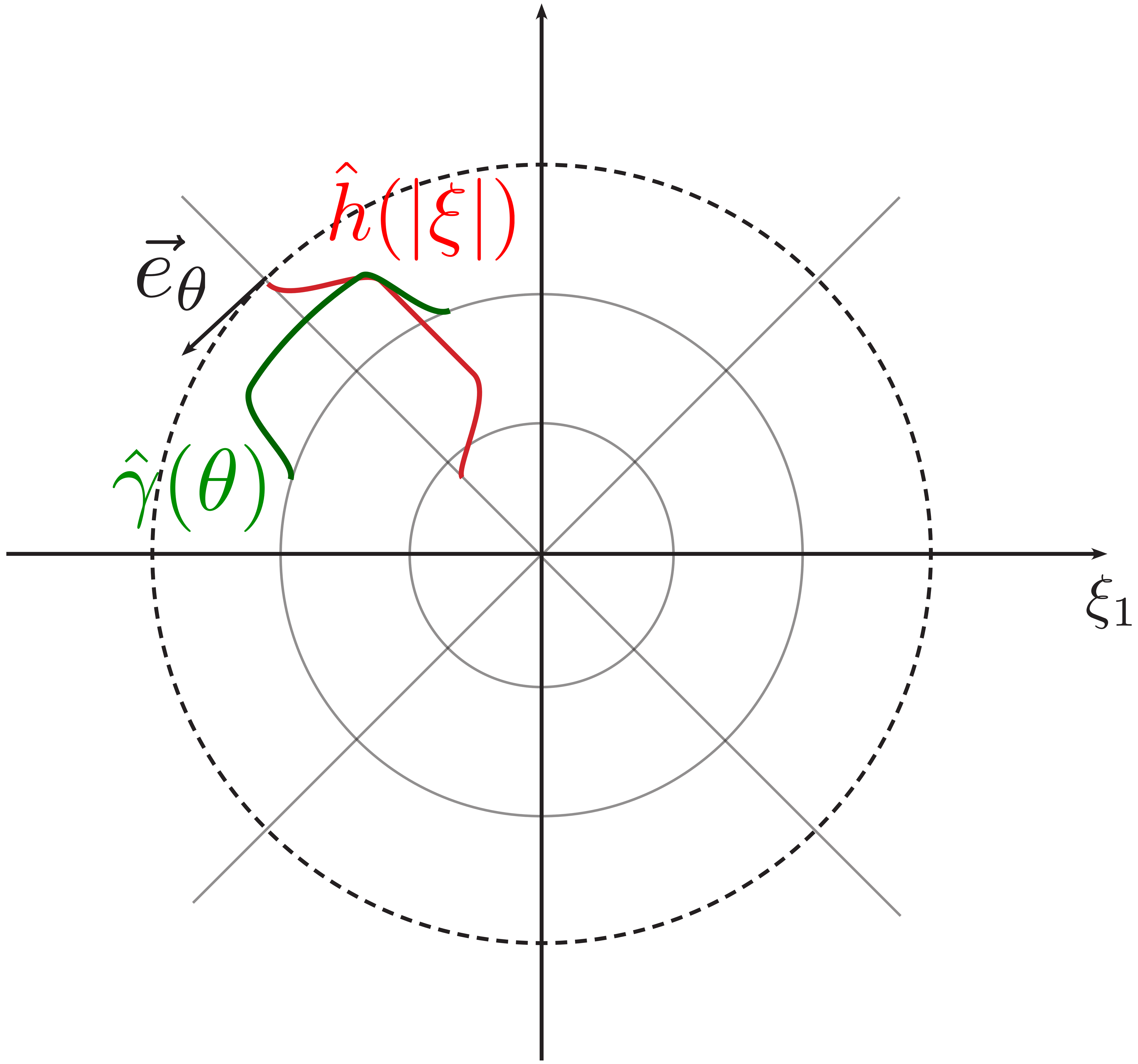




error for upper half with directional wavelets

# Divergence free wavelets in $\mathbb{R}^3$

# Divergence free wavelets in $\mathbb{R}^{\cancel{3}2}$

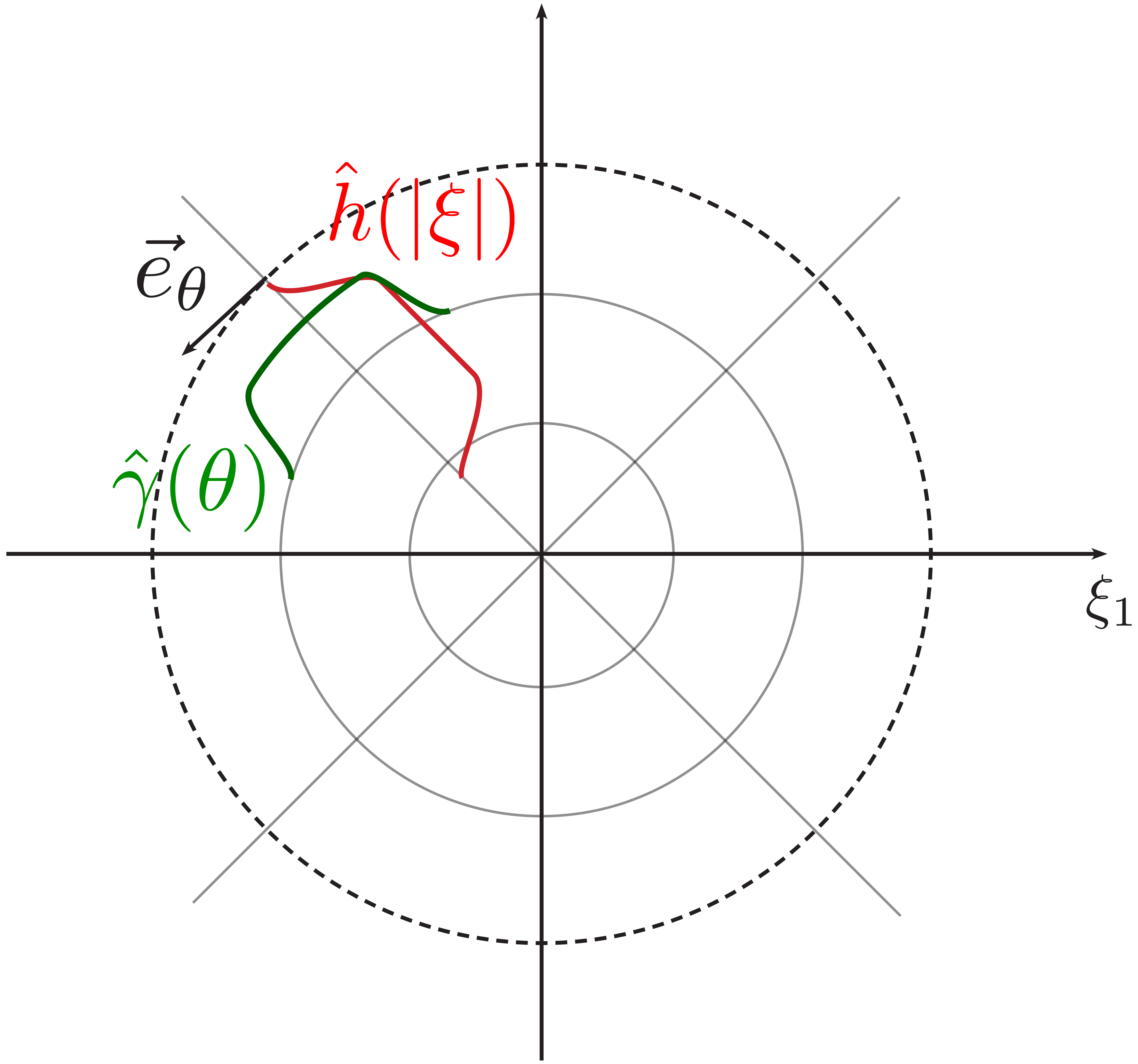
$$\hat{\vec{\psi}}(\xi) = -i\,\hat{\gamma}(\theta)\,\hat{h}(|\xi|)\,\vec{e}_\theta$$



# Divergence free wavelets in $\mathbb{R}^{\cancel{3}2}$

$$\hat{\vec{\psi}}(\xi) = -i\,\hat{\gamma}(\theta)\,\hat{h}(|\xi|)\boxed{\vec{e}_\theta}$$

ensures divergence freedom



# Divergence free wavelets in $\mathbb{R}^3$

*Proposition*: Let $\vec{\tau}_i$ be the basis vector for the sphere for the longitudinal coordinate for spherical coordinate w.r.t to the i-th axis. Then

$$\mathcal{H} = \{\vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3\}$$

Analogue of $\vec{e}_\theta$ on the sphere

forms a tight frame for $T_\omega S^2$ for all $\omega \in S^2$

# Divergence free wavelets in $\mathbb{R}^3$

- Divergence free wavelets:

$$\hat{\vec{\psi}}^i(\xi) = \hat{h}(|\xi|)\,\vec{\tau}_i$$

# Divergence free wavelets in $\mathbb{R}^3$

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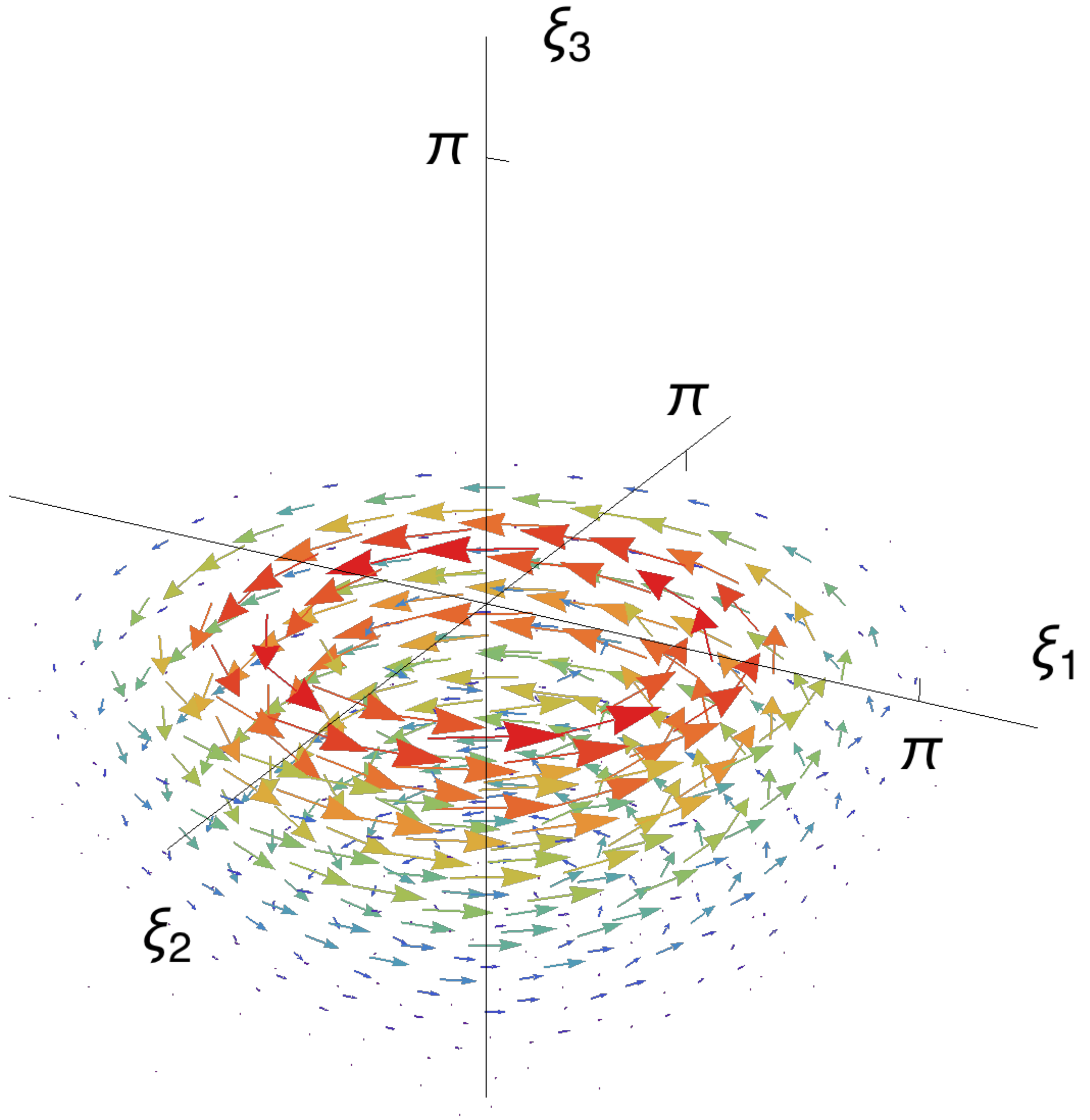
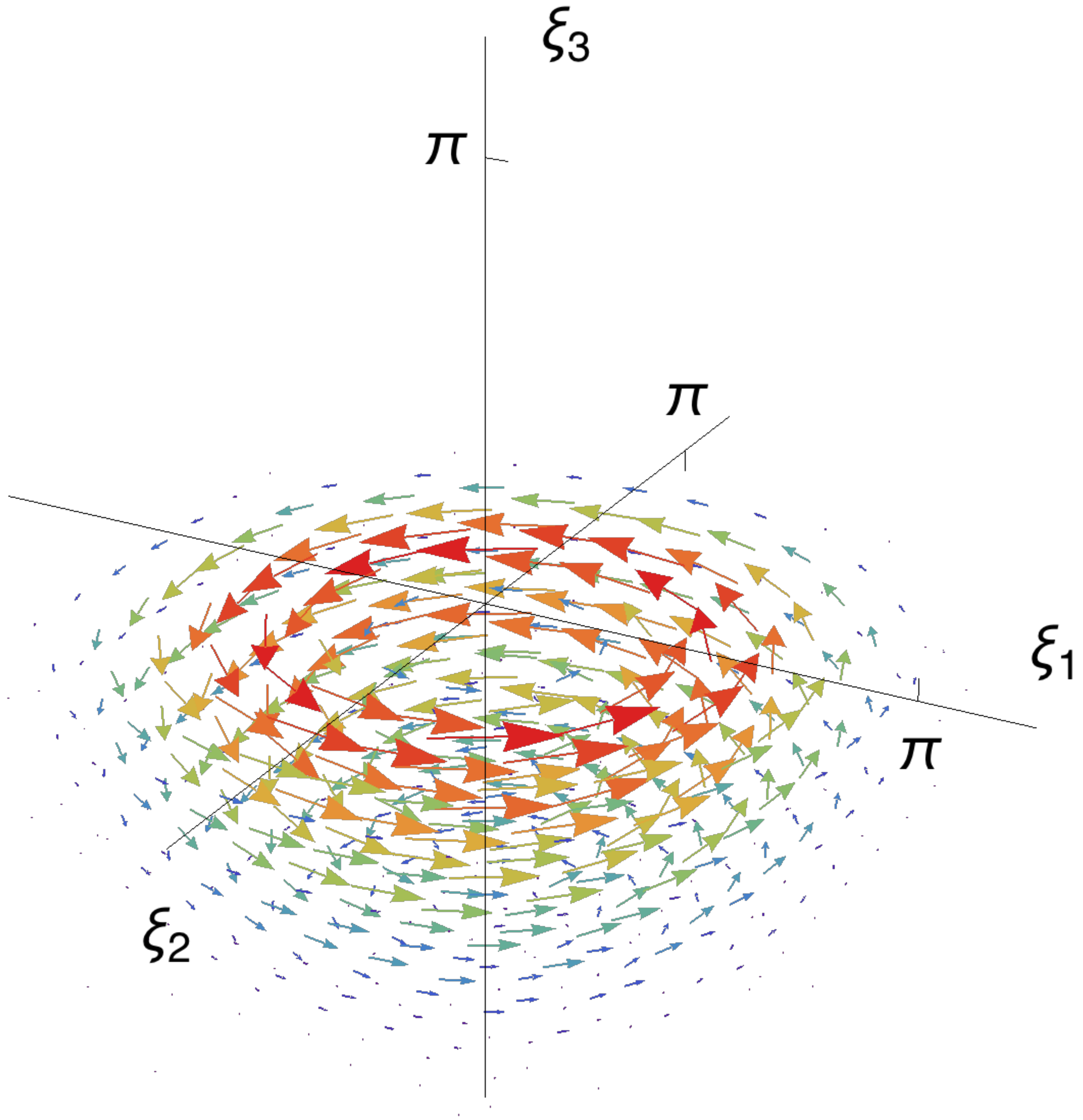
$$\hat{\vec{\psi}}^i(\xi) = \hat{h}(|\xi|)\,\vec{\tau}_i$$



# Divergence free wavelets in $\mathbb{R}^3$



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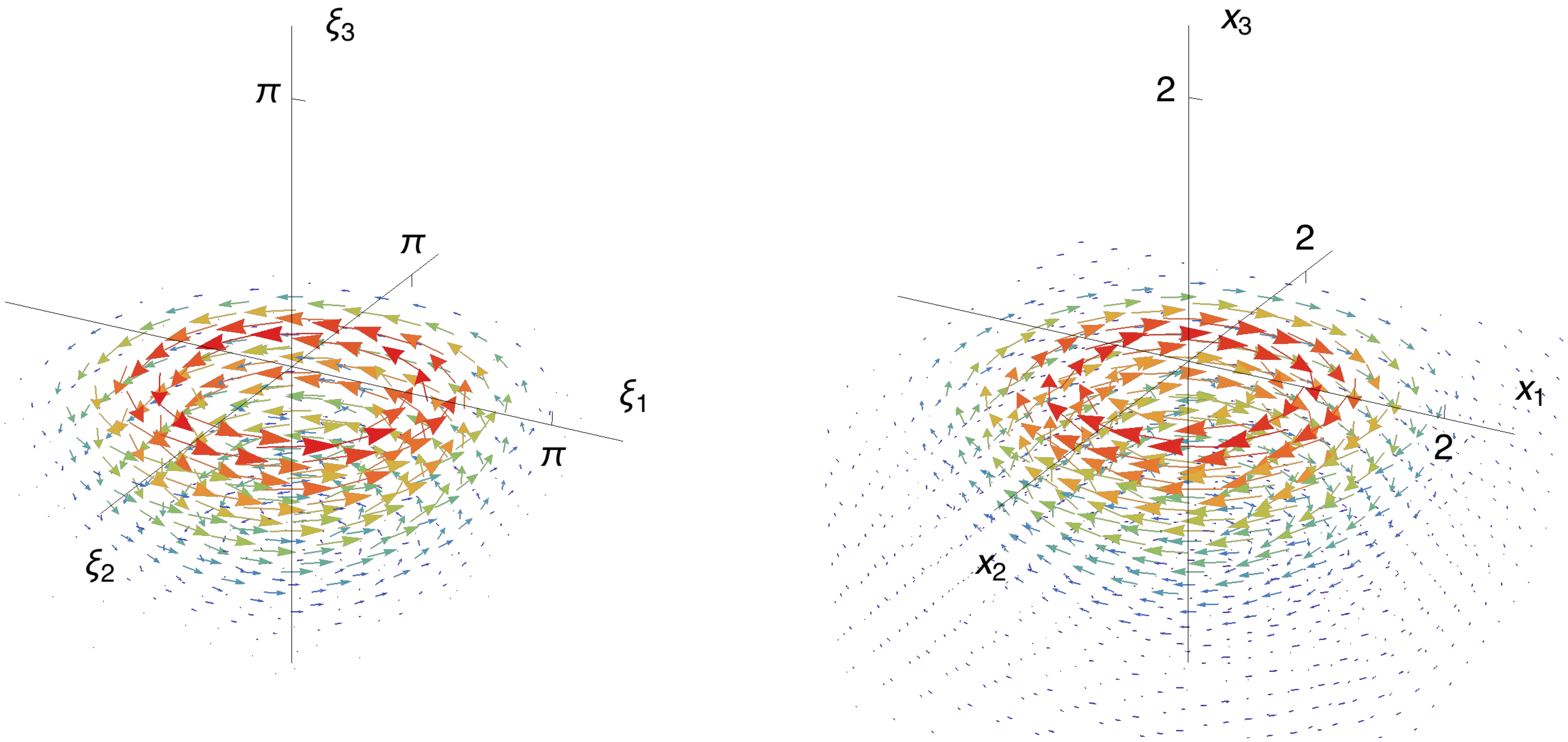


# Divergence free wavelets in $\mathbb{R}^3$

- Divergence free wavelets:

$$\hat{\vec{\psi}}^i(\xi) = \hat{h}(|\xi|)\,\vec{\tau}_i$$

# Divergence free wavelets in $\mathbb{R}^3$

- Divergence free wavelets:

$$\hat{\vec{\psi}}^i(\xi) = \hat{\gamma}(\bar{\xi})\,\hat{h}(|\xi|)\,\vec{\tau}_i$$

# Divergence free wavelets in $\mathbb{R}^3$

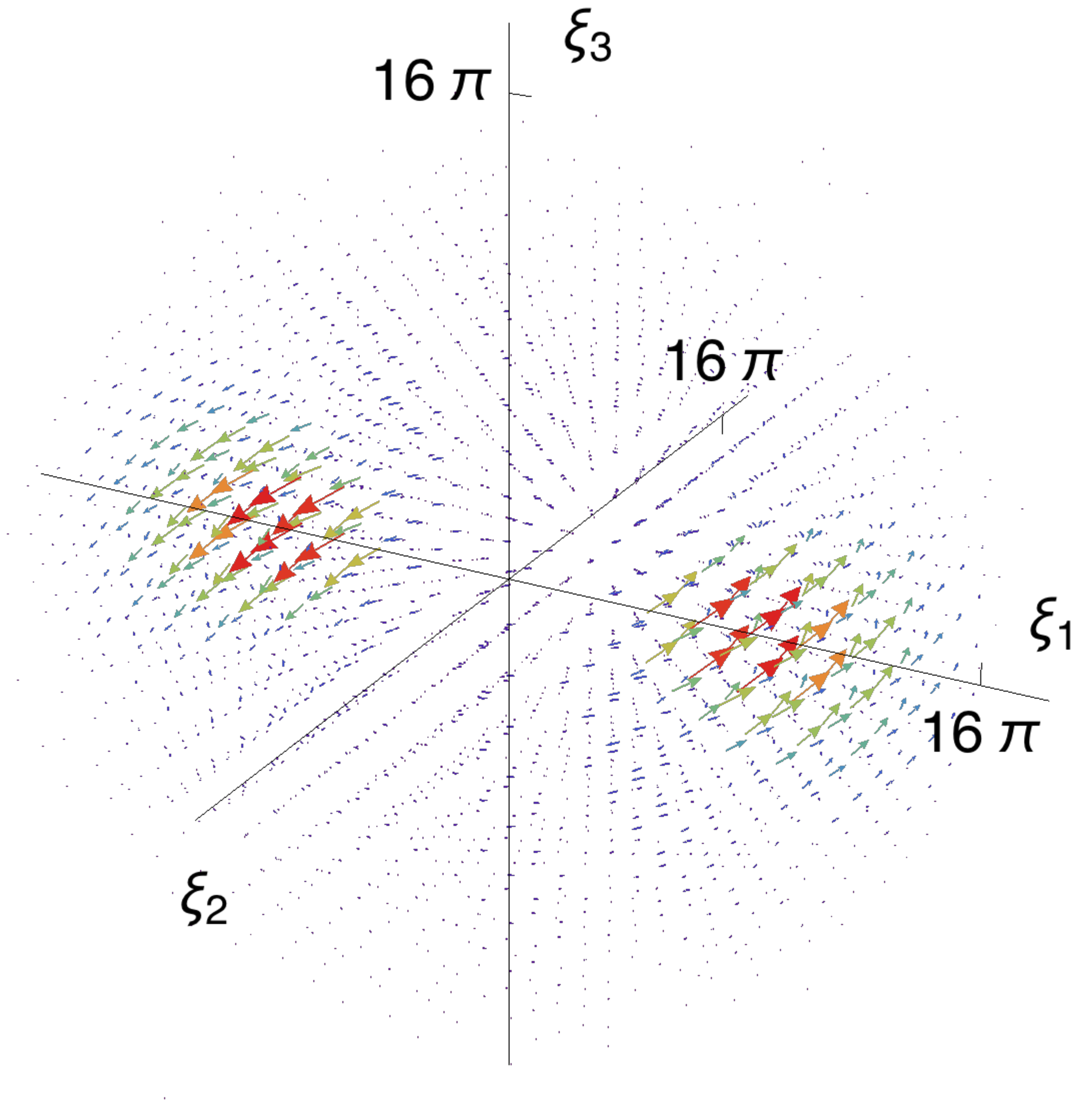
- Divergence free wavelets:

$$\hat{\vec{\psi}}^i(\xi) = \underbrace{\hat{\gamma}(\bar{\xi})}_{\hat{\gamma}(\bar{\xi}) = \sum_{l,m} \gamma_{lm}\, y_{lm}(\bar{\xi})}\, \hat{h}(|\xi|)\, \vec{\tau}_i$$

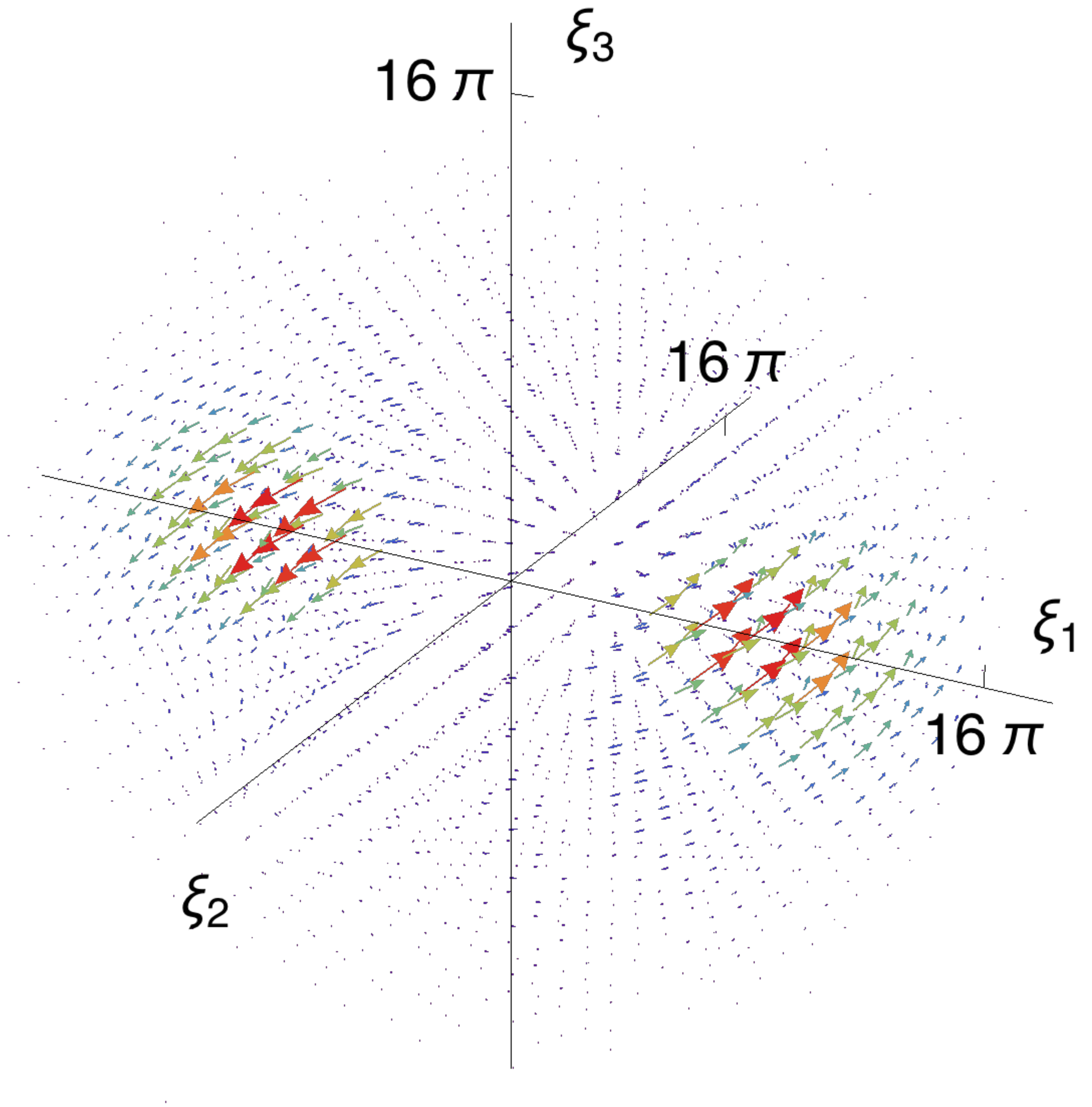
# Divergence free wavelets in $\mathbb{R}^3$

- Divergence free wavelets:

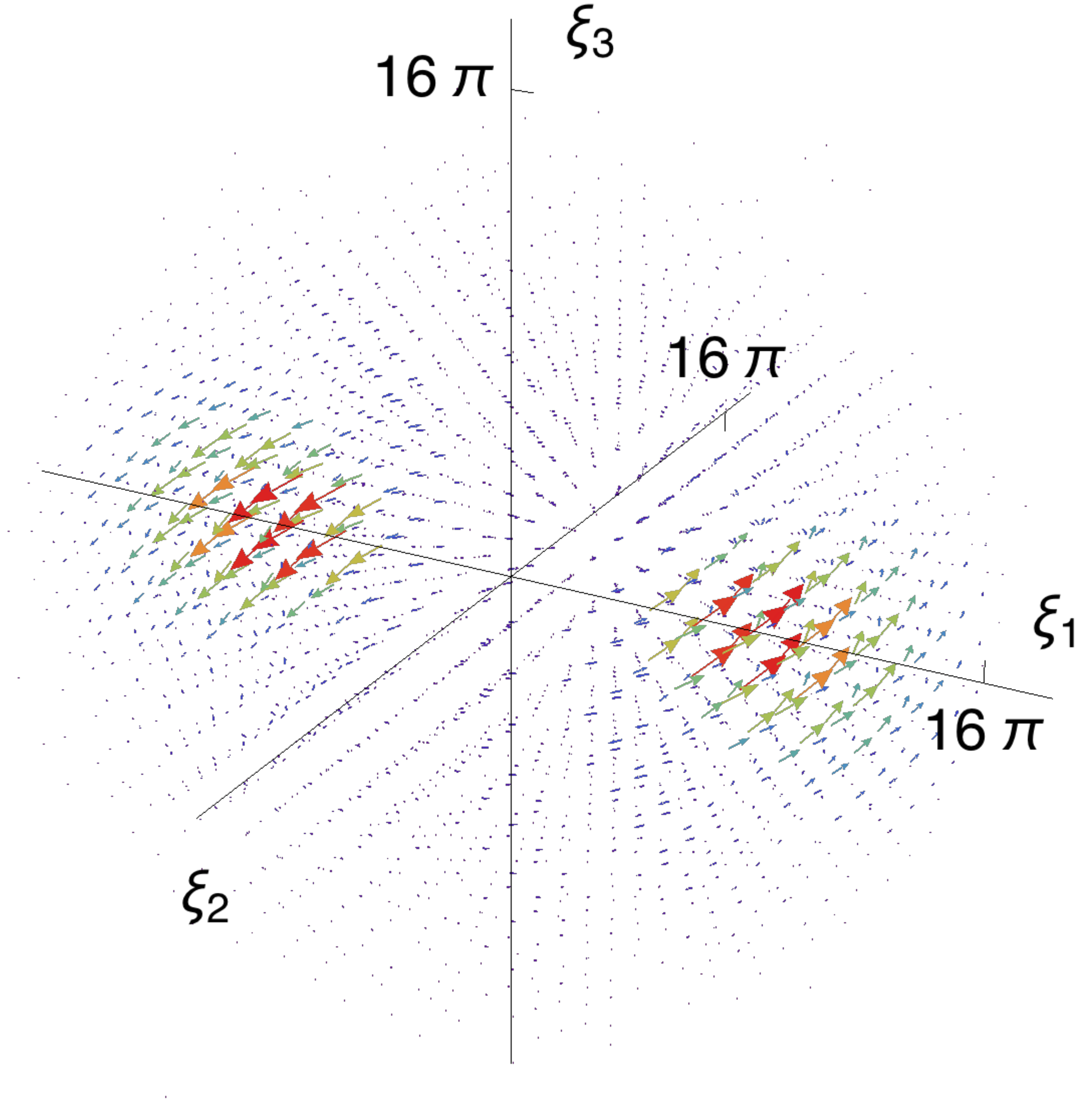
$$\hat{\vec{\psi}}^i(\xi) = \underbrace{\hat{\gamma}(\bar{\xi})}_{\hat{\gamma}(\bar{\xi}) = \sum_{l,m} \gamma_{lm}\, y_{lm}(\bar{\xi})} \hat{h}(|\xi|)\, \vec{\tau}_i$$

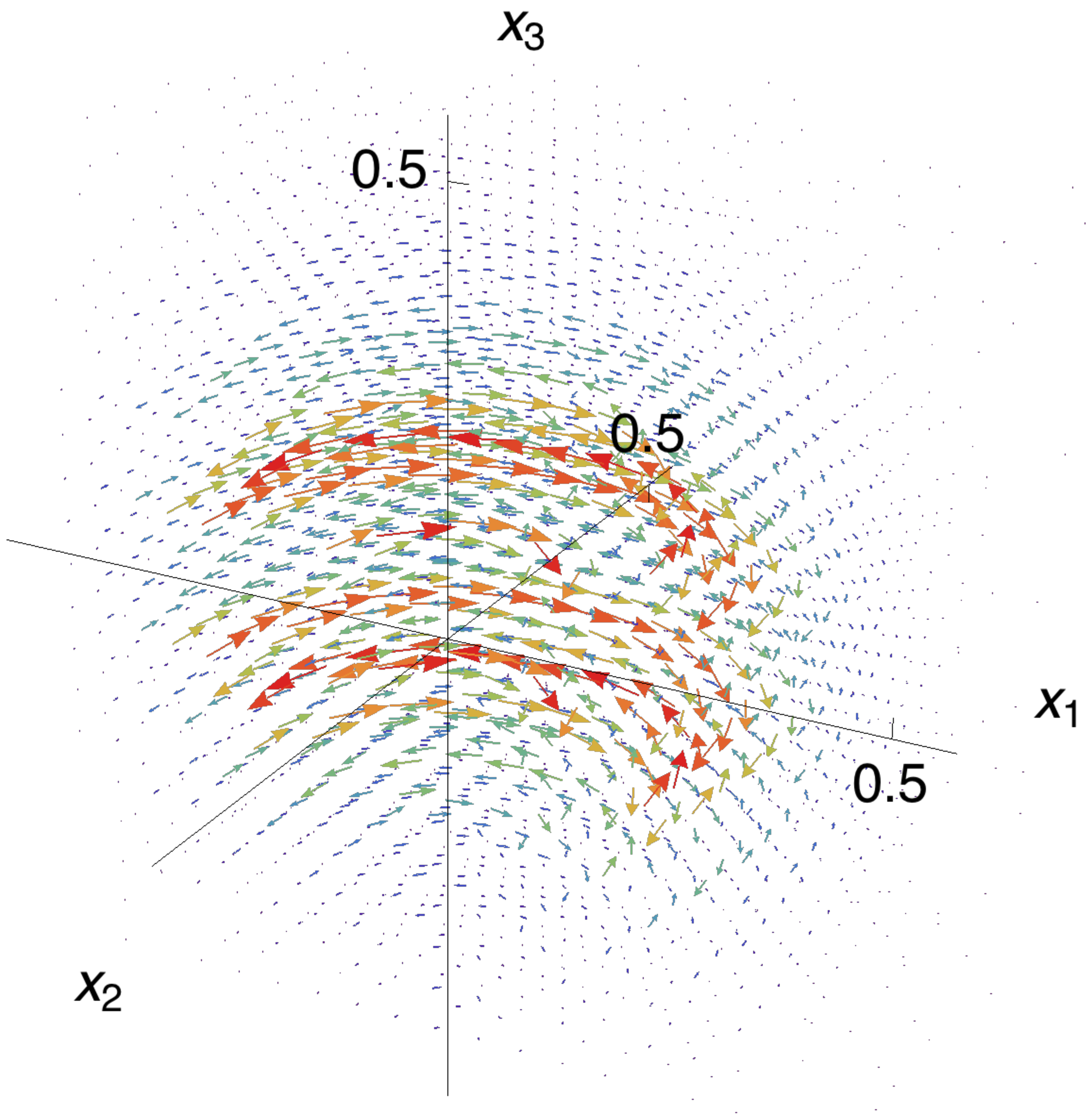


# Divergence free wavelets in $\mathbb{R}^3$

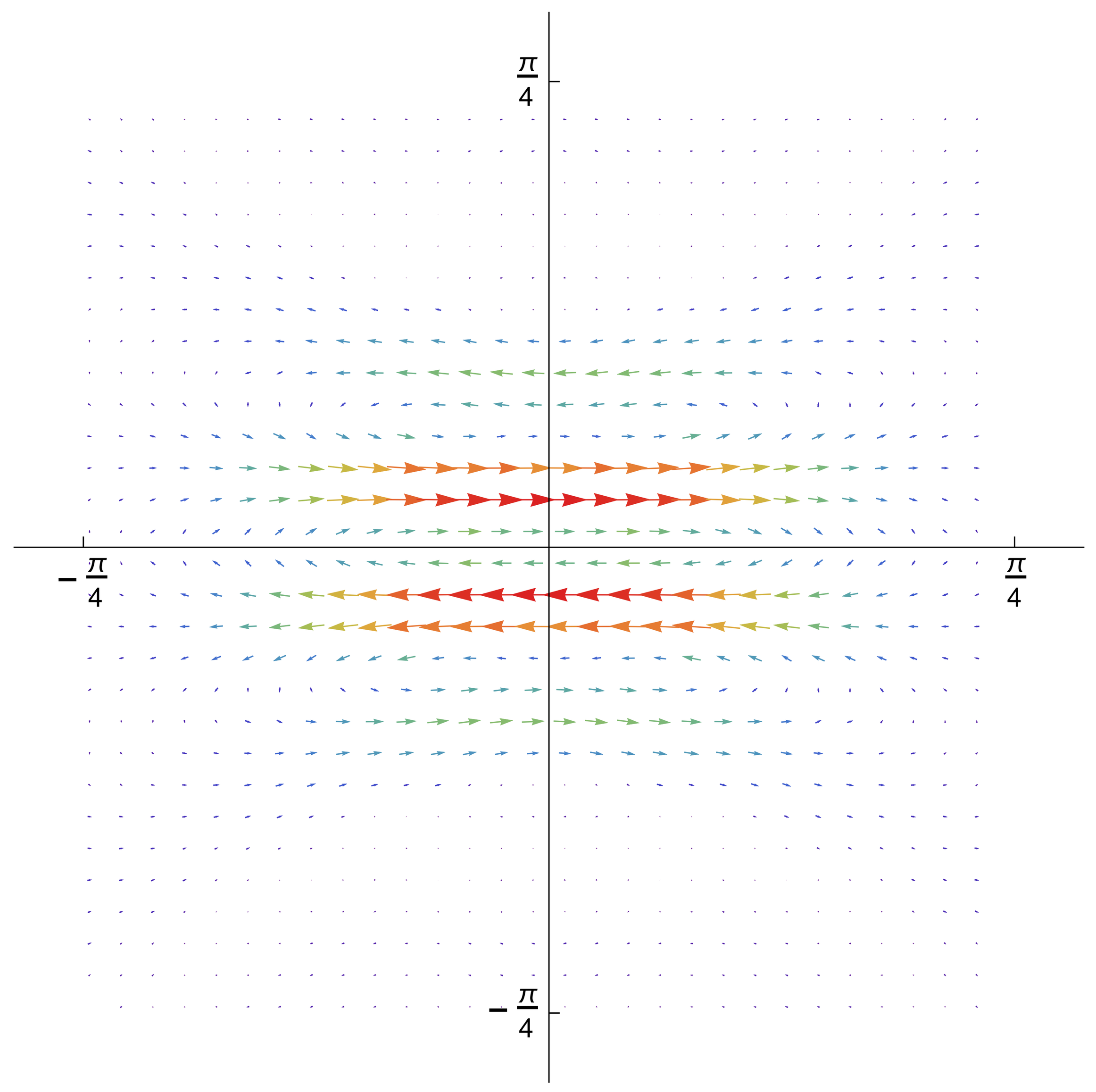


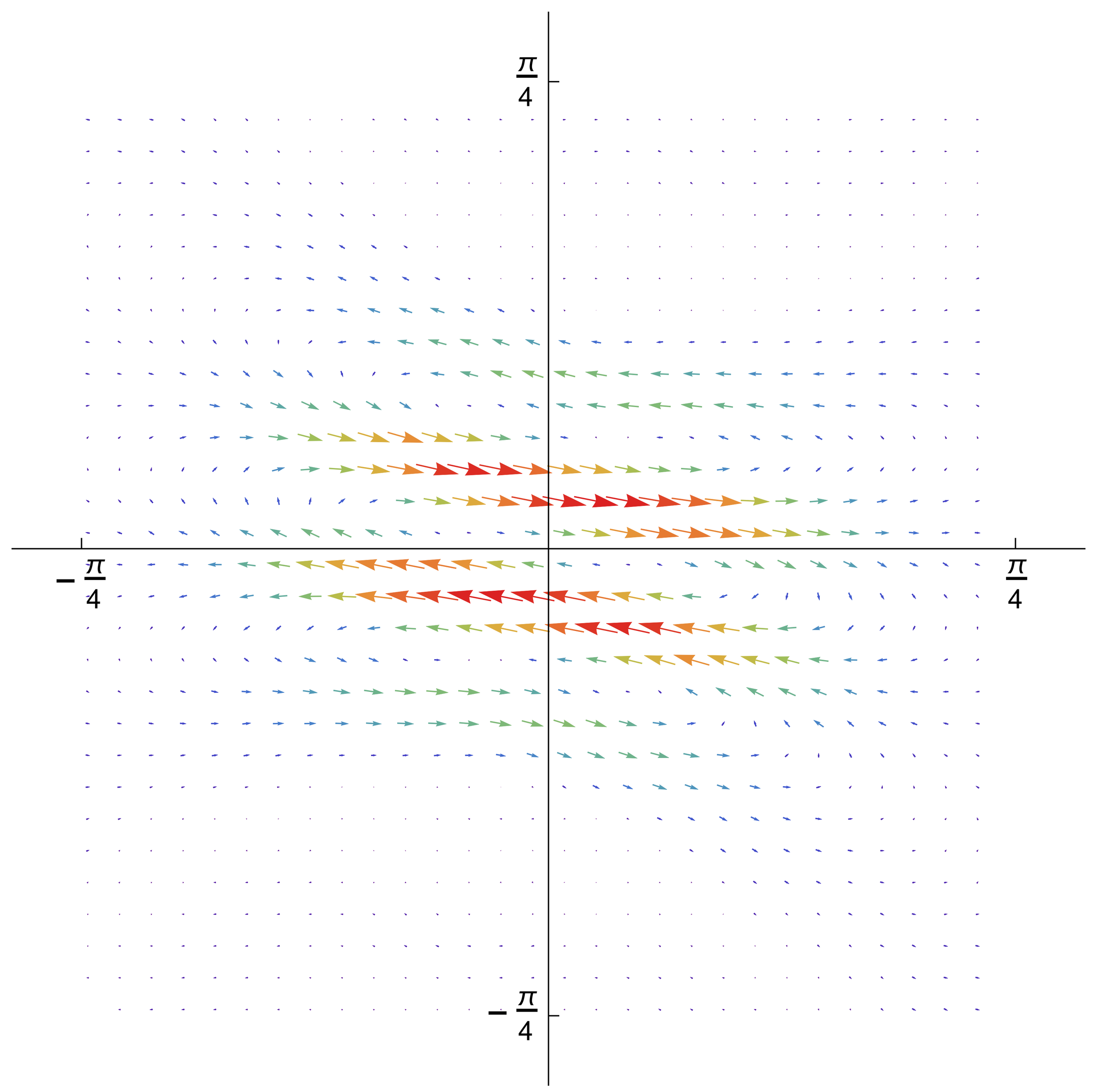
# Divergence free wavelets in $\mathbb{R}^3$

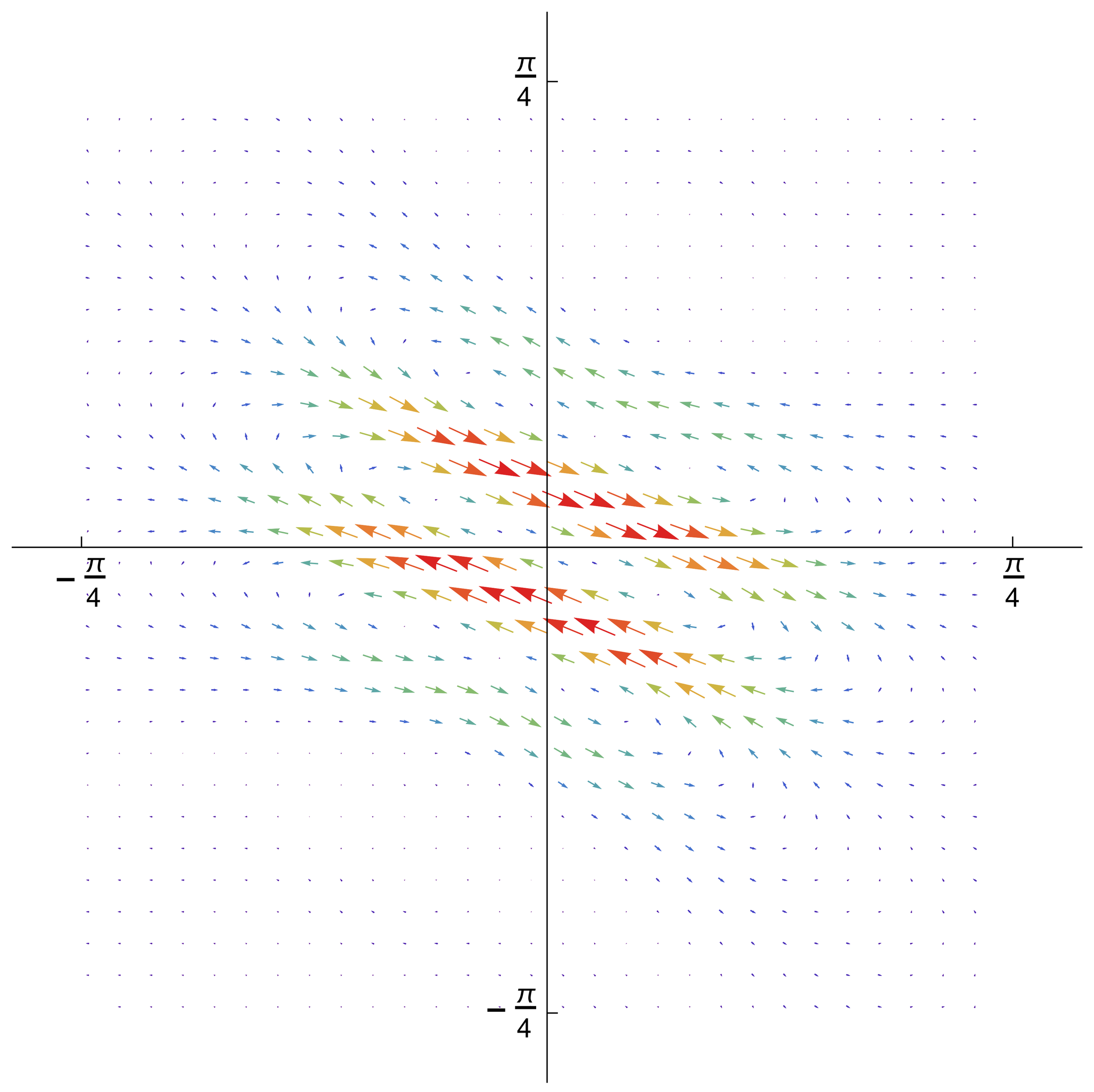


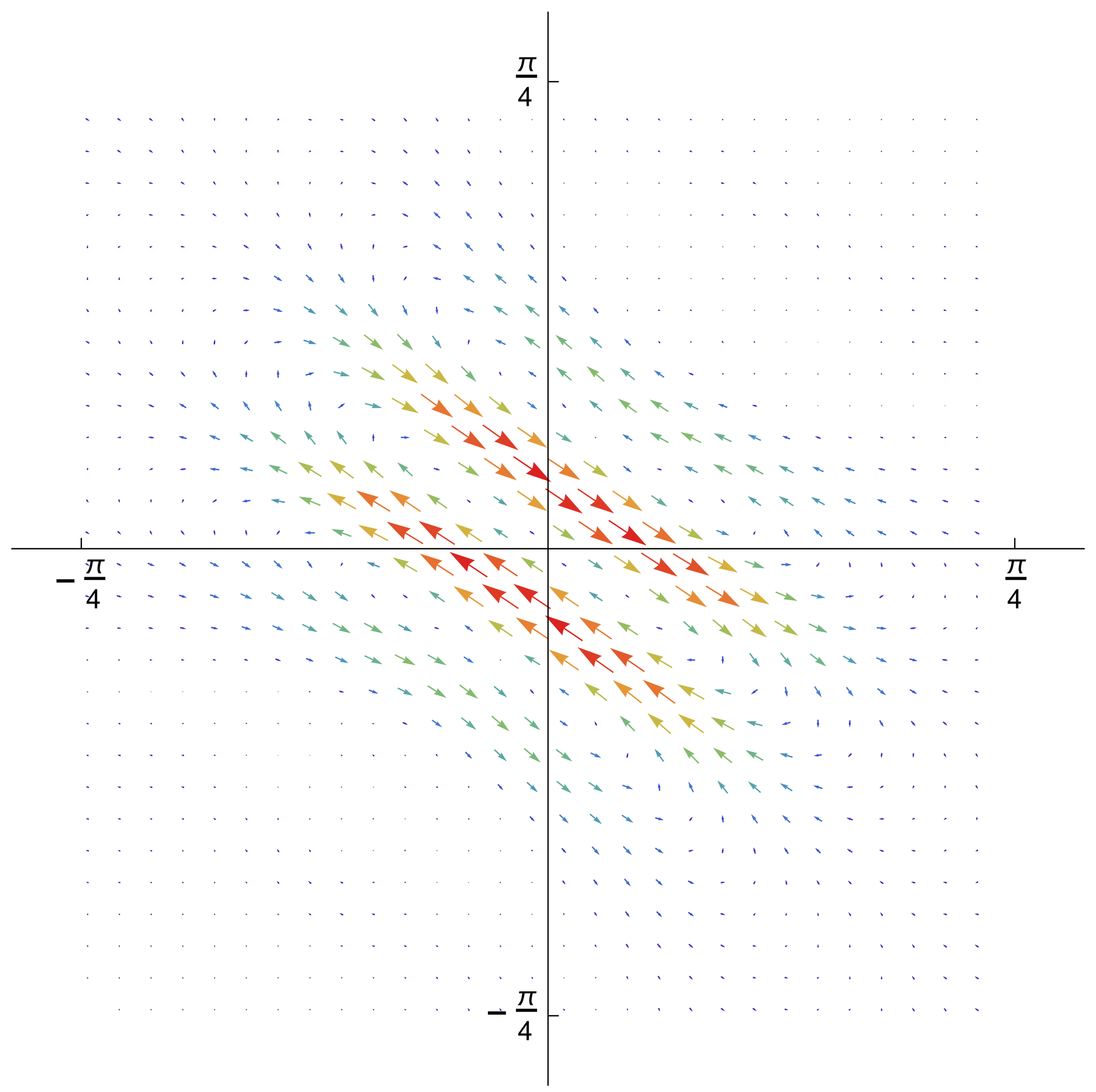



# Divergence free wavelets in $\mathbb{R}^3$



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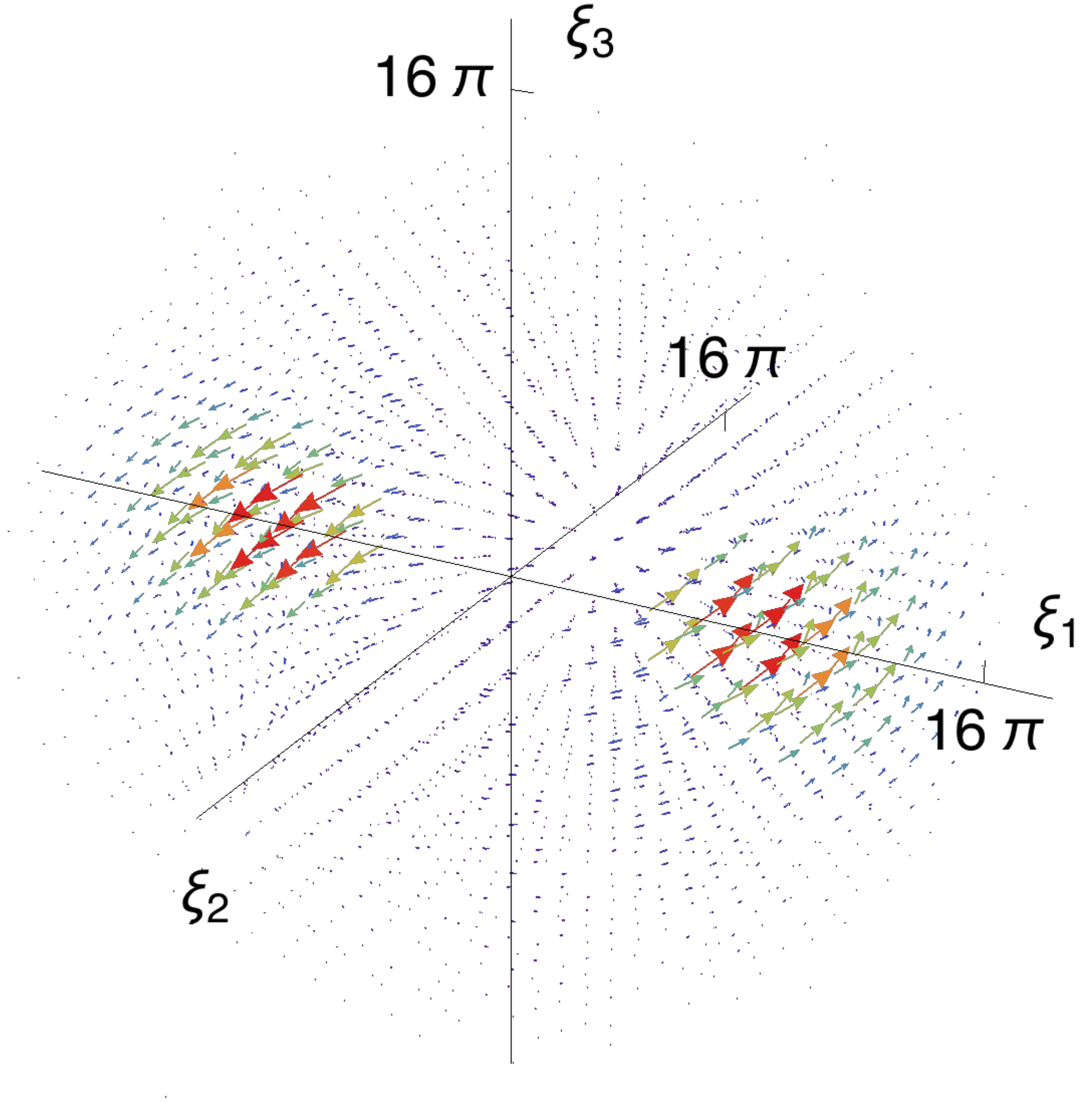
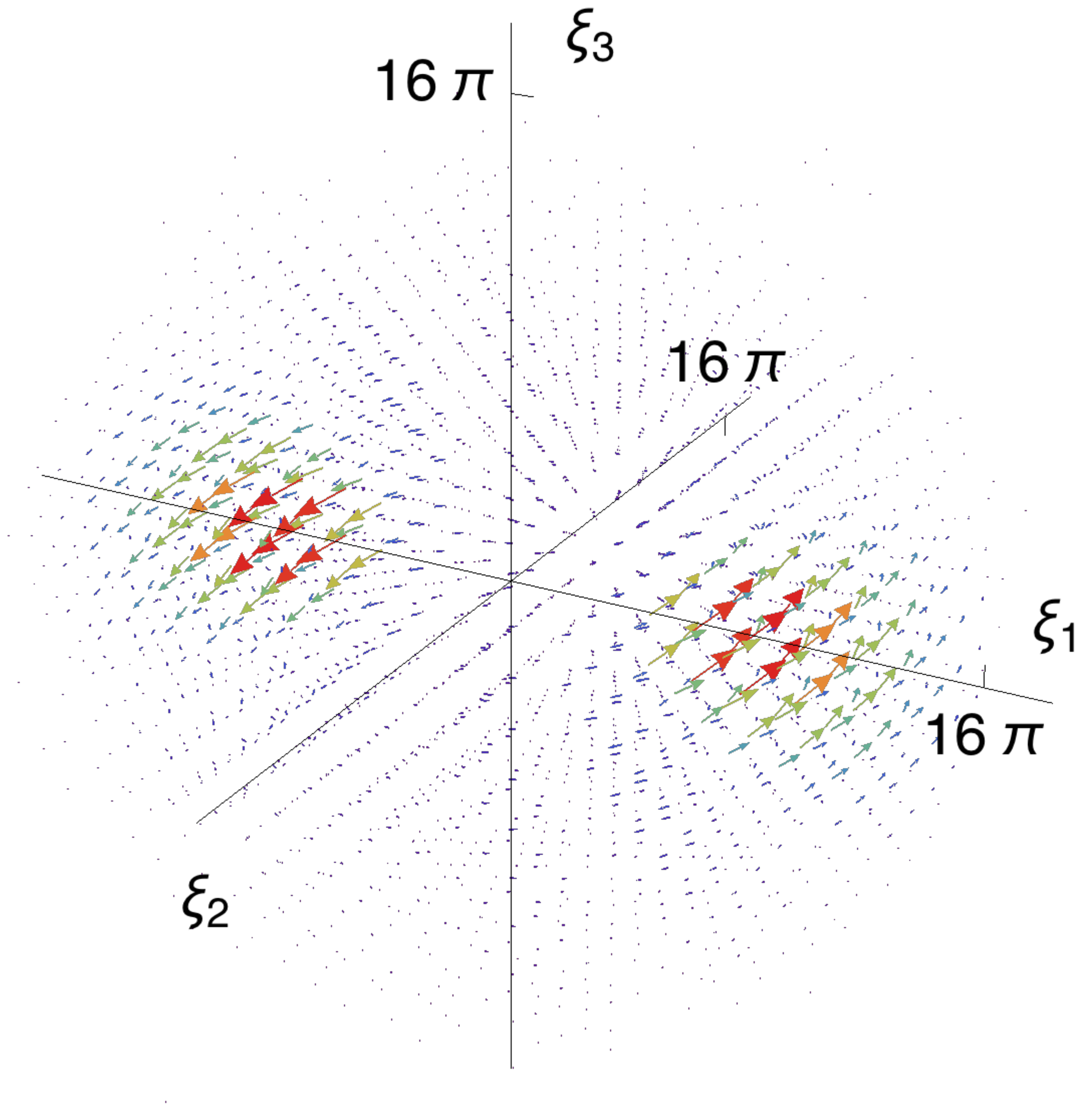
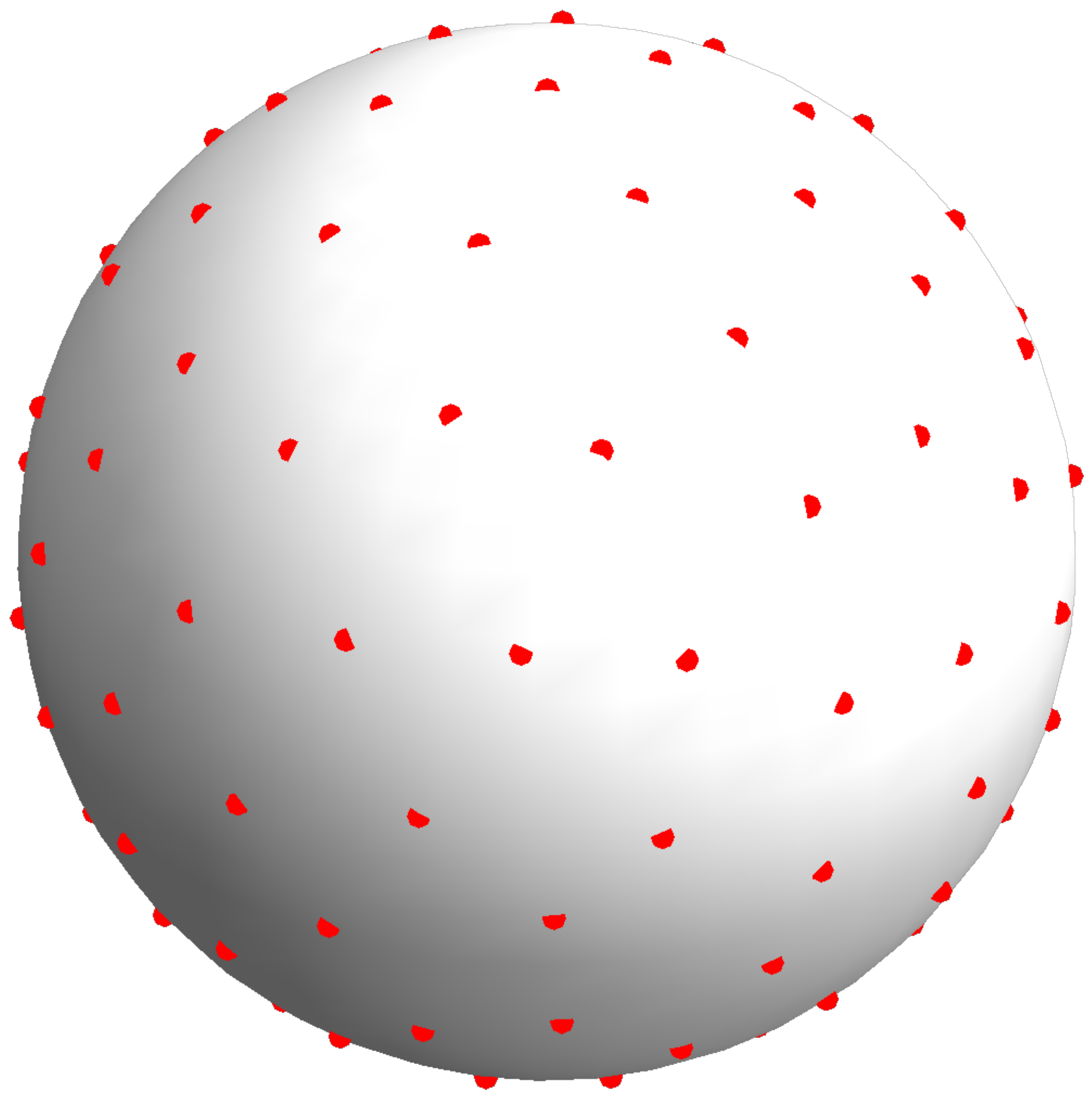
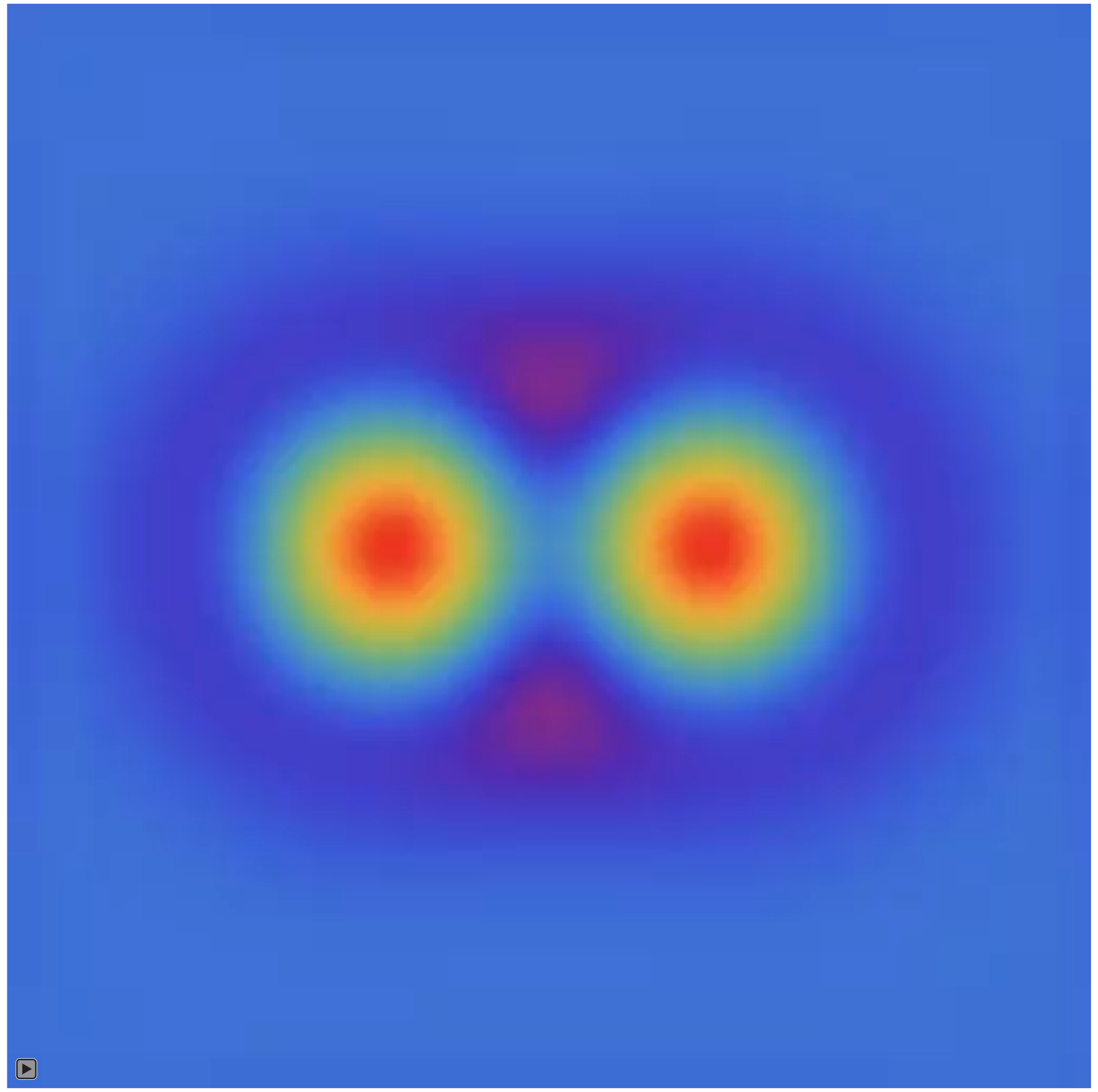


# Divergence free wavelets in $\mathbb{R}^3$

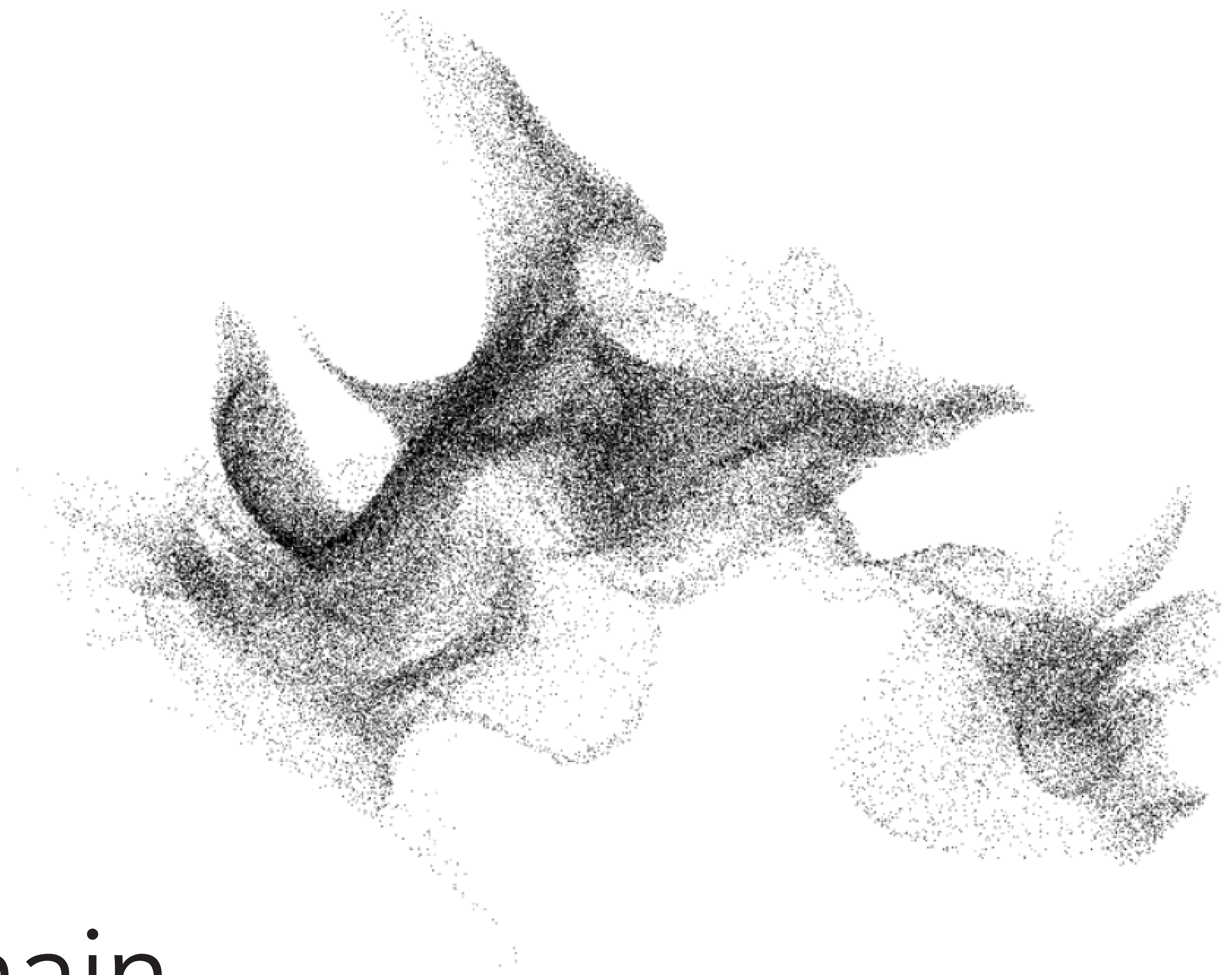


# Back to the motivation



# Divergence free wavelets



- Tight frames for $L_2^{\mathrm{div}}(\mathbb{R}^n)$
- Isotropic and directional functions
- Closed form expression in spatial domain
- Quasi-optimal approximation of boundary conditions (in $\mathbb{R}^2$, under suitable assumptions)
- More details: https://arxiv.org/abs/1805.02062