Polarlets: Supplementary Material

Abstract

In this document we provide mathematical details that have been omitted from the submitted manuscript due to space limitations.

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1. Preliminaries

1.1. The Fourier transform

The unitary Fourier transform of a function $f : \mathbb{R}^n \to \mathbb{C}$ is defined as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n_x} f(x)\, e^{-i\langle x,\xi\rangle}\, dx \tag{1a}$$

with inverse transform

$$\mathcal{F}^{-1}(f)(\xi) = f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n_\xi} \hat{f}(\xi)\, e^{i\langle \xi,x\rangle}\, d\xi. \tag{1b}$$

1.2. Spherical harmonics

The analogue of the Fourier transform in Eq. 1 on the sphere is the spherical harmonics expansion. For any $f \in L_2(S^2)$ it is given by

$$f(\omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \langle f(\eta), y_{lm}(\eta) \rangle \, y_{lm}(\omega) \tag{2a}$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm}\, y_{lm}(\omega) \tag{2b}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard L_2 inner product on S^2 given by

$$\langle f(\omega), g(\omega) \rangle = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f(\theta,\phi)\, g(\theta,\phi)\, \sin\theta\, d\theta d\phi. \tag{3}$$

We use standard (geographic) spherical coordinates with $\theta \in [0, \pi]$ being the polar angle and $\phi \in [0, 2\pi]$ the azimuthal one. The spherical harmonics basis functions in Eq. 2 are given by

$$y_{lm}(\omega) = y_{lm}(\theta, \phi) = C_{lm}\, P_l^m(\cos\theta)\, e^{im\phi} \tag{4}$$

where the $P_l^m(\cdot)$ are associated Legendre polynomials and C_{lm} is a normalization constant so that the $y_{lm}(\omega)$ are orthonormal over the sphere. The associated Legendre polynomials are defined as

$$P_l^m(\cos\theta) = (-1)^m \sum_{p=0}^{r} c_{lmp}\, \sin\theta^m (\cos\theta)^{l-m-2p} \tag{5a}$$

where $r = \lfloor (l-m)/2 \rfloor$ and

$$c_{lmp} = (-1)^p \frac{2^{-l}(2l-2p)!}{p!(l-p)!(l-m-2p)!}. \tag{5b}$$

The associated Legendre polynomials are not L_2-normalized by satisfy

$$\int_{-1}^{1} P_{l_1}^{m_1}(x)\, P_{l_2}^{m_2}(x)\, dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{l_1 l_2}. \tag{5c}$$

Spherical Harmonics Addition Theorem. The spherical harmonics addition theorem is given by

$$P_l(\bar{x}_1 \cdot \bar{x}_1) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} y_{lm}(\omega_1)\, y_{lm}^*(\omega_2) \tag{6}$$

where ω_1 are the spherical coordinates for the unit vector $\bar{x}_1 \in \mathbb{R}^3$ and ω_2 are those for $\bar{x}_2 \in \mathbb{R}^3$. It follows immediately from the above formula that $P_l(\bar{x}_1 \cdot \bar{x}_1)$ is the reproducing kernel for the space spanned by all spherical harmonics of fixed l.

Clebsch-Gordon coefficients. In contrast to the Fourier series, where the product $e^{im_1\theta}\, e^{im_2\theta}$ is given by another Fourier series basis function, $e^{i(m_1+m_2)\theta}$, for spherical harmonics the product is not diagonal and the coupling coefficients are known as Clebsch-Gordon coefficients $C_{l_1,m_1;l_2,m_2}^{l,m}$. In particular, the projection of the product of $y_{l_1,m_1}(\omega)$ and $y_{l_2,m_2}(\omega)$ onto the spherical harmonics $y_{lm}(\omega)$ is given by

$$\int_{S^2} y_{l_1 m_1}(\omega)\, y_{l_2 m_2}(\omega)\, y_{lm}^*(\omega)\, d\omega = \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}}\, C_{l_1,0;l_2,0}^{l,0}\, C_{l_1,m_1;l_2,m_2}^{l,m}$$

and we call

$$G_{l_1,m_1;l_2,m_2}^{l,m} \equiv \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}}\, C_{l_1,0;l_2,0}^{l,0}\, C_{l_1,m_1;l_2,m_2}^{l,m} \tag{7a}$$

the spherical harmonics product coefficient. The Clebsch-Gordon coefficients are sparse and non-zero only when

$$m = m_1 + m_2, \tag{7b}$$

that is the m parameter is not strictly needed but conventionally used, and

$$l_1 + l_2 - l \geq 0 \tag{7c}$$

$$l_1 - l_2 + l \geq 0 \tag{7d}$$

$$-l_1 + l_2 + l \geq 0. \tag{7e}$$

1.3. Fourier Transform in Polar and Spherical Coordinates

Jacobi-Anger formula. In the plane, the Fourier transform can also be written in polar coordinates using the Jacobi-Anger formula [1],

$$e^{i\langle \xi, x\rangle} = \sum_{m\in\mathbb{Z}} i^m\, e^{im(\phi_x - \phi_\xi)} J_m(|\xi|\,|x|), \tag{8}$$

that relates the complex exponential in Euclidean and polar coordinates. In Eq. 8, $J_m(z)$ is the Bessel function of the first kind and $(\phi_x, |x|)$ and $(\phi_\xi, |\xi|)$ are polar coordinates for the spatial and frequency domains, respectively. The ordering of the ϕ_x and ϕ_ξ on the right hand side is arbitrary and when the left hand side is conjugated i^m becomes i^{-m}.

The Fourier transform in two dimensions. The Jacobi-Anger formula allows one to compute the Fourier transform in polar coordinates. Let $f(\phi, |x|) \equiv$

$f(\bar{x}|x|) = f(x)$ with $\bar{x} = (\cos\phi, \sin\phi)$, for fixed radius $|x|$. The function $f(x)$ can then be written as the Fourier series expansion

$$f(x) = f(\phi_x, |x|) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} f_n(|x|)\, e^{in\phi_x}. \tag{9a}$$

Inserting this together with the Jacobi-Anger formula into Eq. 1a and performing a change of variables to polar coordinates we obtain

$$\begin{aligned} \hat{f}(\xi) =& \frac{1}{2\pi} \int_{\mathbb{R}^+_{|x|}} \int_{\phi_x=0}^{2\pi} \left(\frac{1}{2\pi} \sum_{n\in\mathbb{Z}} f_n(|x|)\, e^{2\pi i n\phi_x} \right) \\ & \times \left(\sum_{m\in\mathbb{Z}} i^m e^{-im(\phi_x - \phi_\xi)} J_m(|\xi|\,|x|) \right) |x|\, d\phi_x\, d|x|. \end{aligned} \tag{9b}$$

The integral over ϕ_x is trivial since it only involves the complex exponentials $e^{in\phi_x}$ and $e^{-im\phi_x}$, giving $2\pi\,\delta_{nm}$ and also collapsing the product of sums into a single sum. The Fourier transform $\hat{f}(\xi) = \hat{f}(\phi_\xi, |\xi|)$ is thus

$$\hat{f}(\xi) = \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} i^m\, e^{im\phi_\xi} \int_{\mathbb{R}^+_{|x|}} f_m(|x|)\, J_m(|\xi|\,|x|)|x|\, d|x| \tag{9c}$$

$$= \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} i^m\, \hat{f}_m(|\xi|)\, e^{im\phi_\xi}, \tag{9d}$$

with an analogous expression for the inverse transform. Note that the Fourier transform preserves the polar structure: $f(\phi_x, |x|)$ described in polar coordinates is mapped to $\hat{f}(\phi_\xi, |\xi|)$ in polar coordinates in the frequency domain.

Rayleigh formula. The analogue of the Jacobi-Anger formula in three dimensions is the Rayleigh formula,

$$e^{i\langle \xi, x\rangle} = \sum_{l=0}^{\infty} i^l\, (2l+1)\, P_l\big(\bar{\xi}\cdot\bar{x}\big)\, j_l(|\xi|\,|x|) \tag{10a}$$

$$= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l\, y_{lm}\big(\bar{\xi}\big)\, y_{lm}\big(\bar{x}\big)\, j_l(|\xi|\,|x|) \tag{10b}$$

where $j_l(\cdot)$ is the spherical Bessel function and the second line follows by the spherical harmonics addition theorem in Eq. 6. Using a calculation analogous to those in Eq. 9, the Rayleigh formula enables the calculation of the Fourier transform in spherical coordinates.

2. Admissibility Conditions for Polar Wavelets

2.1. Admissibility Conditions in Two Dimensions

In two dimensions, polarlets are defined by [2]

$$\hat{\psi}(\xi) = \hat{\gamma}(\theta_\xi)\, \hat{h}(|\xi|) = \left(\sum_{n=-N_j}^{N_j} \beta_{j,n}^t \, e^{in\theta_\xi} \right) \hat{h}(|\xi|). \tag{11a}$$

where $\beta_{j,n}^t = e^{int2\pi/M_j}\, \beta_{j,n}$ for some suitable coefficient sequence β_n and M_j is the number of different orientations on level j.[1] In the spatial domain, Eq. 11a becomes

$$\psi(x) = \frac{1}{2\pi} \sum_{n=-N_j}^{N_j} i^n \beta_{j,n}^t \, e^{in\theta_x} \underbrace{\int_{\mathbb{R}_{|\xi|}^+} \hat{h}(|\xi|)\, J_m(|\xi|\,|x|)\, d|\xi|}_{h_m(|x|)}, \tag{11b}$$

see the main text or the accompanying code for the closed form expression for $h_m(|x|)$. Eq. 11 is the mother wavelet for

$$\psi_{jkt}(x) \equiv \frac{2^j}{2\pi} \psi\big(R_{jt}\, 2^j x - k\big) \tag{12}$$

where $j \in \mathbb{Z}$, $k \in \mathbb{Z}^2$, and R_{jt} is a rotation by $2\pi/M_j$.

The following lemma provides the conditions that the functions defined by Eq. 12 form a tight frame.

Lemma 1. *Let U_j be the $(M_j \times 2N_j + 1)$-dimensional matrix formed by the $\beta_{j,n}^t$ coefficients for all M_j orientations. Then the wavelets in Eq. 12 generate a Parseval tight frame for $L_2(\mathbb{R}^2)$ when the radial window satisfies the Caldèron admissibility condition*

$$\sum_{j \in \mathbb{Z}} \left| \hat{h}\big(2^{-j}|\xi|\big) \right|^2 = 1 \quad , \quad \forall \xi \in \mathbb{R}_\xi^2 \tag{13a}$$

[1] In general, the radial and angular part can have independent level variables but to simplify the exposition we assume that these are coupled.

and

$$U_j^H U_j = D_j \quad \text{with} \quad \mathrm{tr}(D_j) = 1, \tag{13b}$$

where D_j is diagonal, for all levels j.

The above lemma was proved by Unser and Chenouard [2] using the higher-order Riesz transform. The proof below uses a direct argument. We will establish the result for a semi-continuous frame with continuous translation parameter. The lemma then follows from standard arguments using the Shannon sampling theorem, see e.g. [3].

Proof. We want to show

$$f(x) = \sum_j \sum_{t=0}^{M_j-1} \big(f * \psi_{j,t}\big) * \psi_{j,t} \tag{14a}$$

for $f \in L_2(\mathbb{R}^2)$. Taking the Fourier transform of both sides we obtain

$$\hat{f}(\xi) = \sum_j \sum_{t=0}^{M_j-1} (\hat{f} \cdot \hat{\psi}_{j,t}^*)\, \hat{\psi}_{j,t}. \tag{14b}$$

Since $\hat{f}$ does not depend on j we can write

$$\hat{f} = \hat{f} \sum_j \sum_{t=0}^{M_j-1} \hat{\psi}_{j,t}^*\, \hat{\psi}_{j,t}. \tag{14c}$$

For the lemma to hold we thus have to show

$$\sum_j \sum_{t=0}^{M_j-1} |\hat{\psi}_{j,e}(\xi)|^2 = 1 \quad , \quad \forall \xi \in \mathbb{R}_\xi^2. \tag{14d}$$

With the definition of the window function in Eq. 11a, the equation becomes

$$\begin{aligned} &\sum_j \sum_{t=0}^{M_j-1} |\hat{\psi}_{j,t}|^2 \qquad (14e) \\ &\quad = \sum_j \sum_{t=0}^{M_j-1} \left(\sum_{n=-N_j}^{N_j} \beta_{j,n}^t\, e^{in\theta}\, \hat{h}(2^{-j}|\xi|) \right) \left(\sum_{m=-N_j}^{N_j} \beta_{j,m}^t\, e^{im\theta}\, \hat{h}(2^{-j}|\xi|) \right)^* . \end{aligned}$$

Using linearity we have

$$\sum_j \sum_{t=0}^{M_j-1} |\hat{\psi}_{j,t}|^2 = \sum_j \sum_{t=0}^{M_j-1} \sum_{n=-N_j}^{N_j} \sum_{m=-N_j}^{N_j} \beta_{j,n}^t \, \beta_{j,m}^{t*} \, e^{in\theta} \, e^{-im\theta} \left|\hat{h}(2^{-j}|\xi|)\right|^2$$

Assuming the radial window functions satisfy the Caldèron admissibility condition in Eq. 13a, the lemma holds when

$$1 = \sum_{n=-N_j}^{N_j} \sum_{m=-N_j}^{N_j} \left(\sum_{t=0}^{M_j-1} \beta_{j,n}^t \, \beta_{j,m}^{t*} \right) e^{i(n-m)\theta}. \tag{14f}$$

For the right hand side to be unity for all θ we have to have that the Fourier series in $(m-n)$ has $\delta_{m-n,0}$ as coefficients, that is

$$c_{m-n} = \sum_{t=0}^{M_j-1} \beta_{j,n}^t \, \beta_{j,m}^{t*} = \delta_{m-n,0}. \tag{14g}$$

This is the first condition in Eq. 13b, i.e. $U^H U = D$. The Fourier series coefficient is unity in magnitude with the trace condition in Eq. 13b. □

2.2. Admissibility Conditions in Three Dimensions

In three dimensions, polarlets are defined by [4, 5]

$$\hat{\psi}(\xi) = \hat{\gamma}(\bar{\xi}) \, \hat{h}(|\xi|) = \left(\sum_{l=0}^{L_j} \sum_{m=-l}^{l} \kappa_{lm}^{j,t} \, y_{lm}(\bar{\xi}) \right) \hat{h}(|\xi|). \tag{15a}$$

where $\kappa_{lm}^t = W_{lm}^{m'}(\lambda_t)\kappa_{lm'}^t$ for some suitable coefficient sequence $\kappa_{lm'}$ and where $W_{lm}^{m'}(\lambda_t)$ is the Wigner-D matrix that rotates the functions to the location λ_t. In the spatial domain Eq. 15a becomes

$$\psi(x) = \sum_{l=0}^{L_j} \sum_{m=-l}^{l} i^l \kappa_{lm}^{j,t} \, y_{lm}(\bar{x}) \underbrace{\int_{\mathbb{R}^+_{|\xi|}} \hat{h}(|\xi|) \, j_l(|\xi| \, |x|) |\xi|^2 d|\xi|}_{h_l(|x|)} \tag{15b}$$

see the accompanying code for the closed form expression for $h_l(|x|)$. Eq. 15 is the mother wavelet for

$$\psi_{jkt}(x) \equiv \frac{2^{2j}}{2\pi} \psi\big(R_{jt} \, 2^j x - k\big) \tag{16}$$

where $j \in \mathbb{Z}$, $k \in \mathbb{Z}^2$, and R_{jt} the rotation to λ_j. The following lemma provides the conditions that the functions defined by Eq. 16 to form a tight frame.

Lemma 2. *Let $u_{j,t}$ be the $(L_j+1)^2)$-dimensional vector formed by the $\kappa_{lm}^{j,t}$ for fixed j, t. Then the wavelets in Eq. 16 form a Parseval tight frame when*

$$\sum_{j\in\mathbb{Z}} \left|\hat{h}\big(2^{-j}|\xi|\big)\right|^2 = 1 \quad , \quad \forall \xi \in \mathbb{R}^2_\xi \tag{17a}$$

and

$$\delta_{l,0}\delta_{m,0} = \sum_{t=0}^{M_j} u_{j,t}\, G^{lm}\, u_{j,t} \tag{17b}$$

where G^{lm} is the matrix formed by the spherical harmonics product coefficients in Eq. 7 for fixed (l,m).

Proof. With an argument fully analogous to those in the proof of Lemma 1 it suffices to show that

$$\sum_j \sum_{t=0}^{M_j} |\hat{\psi}_{j,t}|^2 = 1 \quad , \quad \forall \xi \in \mathbb{R}^3. \tag{18}$$

With the definition of the window functions and after re-arranging terms one obtains

$$\sum_j \sum_{t=0}^{M_j} |\hat{\psi}_j|^2 = \sum_j \sum_{t=0}^{M_j} \sum_{l_1,m_1} \sum_{l_2 m_2} \kappa_{l_1 m_1}^{j,t} y_{l_1 m_1}\big(\bar{\xi}\big)\, \kappa_{l_2 m_2}^{j,t} y_{l_2 m_2}^*\big(\bar{\xi}\big) \left|\hat{h}(2^{-j}|\xi|)\right|^2$$

Assuming the Caldèron condition in Eq. 17a is satisfied, the lemma holds when the product of the angular part evaluates to the identity for every band j and every direction $\bar{\xi}$. This means that for every j the projection of the angular part in the above equation onto spherical harmonics has to satisfy

$$\delta_{l,0}\,\delta_{m,0} = \sum_{t=0}^{M_j} \left\langle \sum_{l_1,m_1} \kappa_{l_1 m_1}^{j,t} y_{l_1 m_1}\big(\bar{\xi}\big) \sum_{l_2,m_2} \kappa_{l_2 m_2}^{j,t} y_{l_2 m_2}^*\big(\bar{\xi}\big)\,,\, y_{lm}(\bar{\xi}) \right\rangle. \tag{19a}$$

Rearranging terms we obtain

$$\delta_{l,0}\,\delta_{m,0} = \sum_{t=0}^{M_j} \sum_{l_1,m_1} \sum_{l_2,m_2} \kappa_{l_1 m_1}^{j,t}\, \kappa_{l_2 m_2}^{j,t*} \left\langle y_{l_1 m_1}\big(\bar{\xi}\big) y_{l_2 m_2}^*\big(\bar{\xi}\big)\,,\, y_{lm}^*(\bar{\xi})\right\rangle. \tag{19b}$$

The product of two spherical harmonics projected into another spherical harmonic is given by the product coefficnets in Eq. 7. We can hence write

$$\delta_{l,0}\,\delta_{m,0} = \sum_{t=0}^{M_j} \sum_{l_1,m_1} \sum_{l_2,m_2} \kappa_{l_1 m_1}^{j,t}\, \kappa_{l_2 m_2}^{j,t}\, G_{l_1,m_1;l_2,-m_2}^{l,m_1-m_2}. \tag{19c}$$

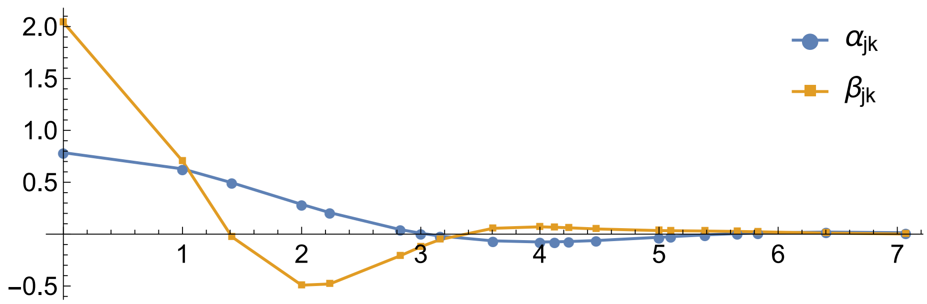

Figure 1: Spatial decay of filter tap coefficients for the fast transform in $2D$, see Eq. 21, for isotropic functions.

Collecting the $\kappa_{l_i m_i}^{j,t}$ into vectors we obtain the condition in the lemma. □

3. Fast Transform

The fast transform splits the signal f_{j+1} at scale $j+1$ into a low frequency part, represented by scaling function coefficients, and a high frequency part, represented by the wavelet coefficients. By linearity it suffices to determine the projection for the basis functions, yielding filter taps

$$\alpha_{j,k} = \langle \phi_{j,0}(x), \phi_{j+1,k}(x) \rangle \tag{20a}$$

$$\beta_{j,k,t} = \langle \psi_{j,0,t}(x), \phi_{j+1,k}(x) \rangle \, . \tag{20b}$$

By the Parseval identity $\alpha_{j,k}$ is given by

$$\beta_{j,k,t} = \left\langle \hat{\psi}_{j,0,t}(\xi) \, , \, \overline{\hat{\phi}_{j+1,k}(\xi)} \right\rangle . \tag{21a}$$

In polar coordinates this equals

$$\beta_{j,k,t} = \int_{\mathbb{R}^+_{|\xi|}} \int_{S^1_\theta} \left(\sum_n \beta_n \, e^{in\theta} \, \hat{h}\big(|2^{-j}\xi|\big) \right) \left(\hat{g}\big(|2^{-j-1}\xi|\big) \, e^{i\langle \xi, 2^{-j-1}k \rangle} \right) |\xi| \, d\theta \, d|\xi|.$$

where $\hat{g}(|\xi|)$ is the window for the scaling function. The translation term $e^{i\langle \xi, 2^{-j-1}k \rangle}$ can be expanded using the Jacobi-Anger formula. After using linearity, this yields

$$\beta_{j,k,t} = \sum_n \sum_{m=0}^{\infty} i^m \, \beta_n \, e^{im\theta_k} \int_{S^1_\theta} e^{-im\theta} \, e^{in\theta} \, d\theta \tag{21b}$$

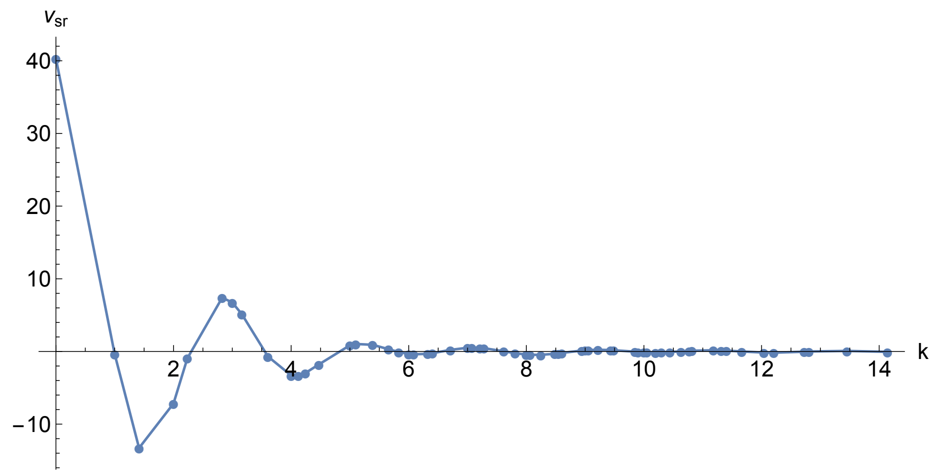

Figure 2: Spatial decay of v_{sr} coefficients in Eq. 22h.

$$\times \int_{\mathbb{R}^+_{|\xi|}} \hat{h}\big(|2^{-j}\xi|\big)\hat{g}\big(|2^{-j-1}\xi|\big)\, J_m\big(|\xi|\,|2^{-j-1}k|\big)\, |\xi|\, d|\xi|.$$

The angular integral is $2\pi\,\delta_{mn}$. The radial one has a closed form solution as well, see the accompanying code for the expression. We will refer to it as $B_j^m(|k|)$ in the following. Thus, we have

$$\beta_{j,k,t} = 2\pi \sum_n i^n\, \beta_n\, B_j^m(|k|) \tag{21c}$$

which is a finite sum with the same cardinality as those in the definition of $\hat{\psi}_{j-1,0}(\xi)$. Hence, the filter taps can be computed in closed form. For the α-coefficients, which couple scaling functions on adjacent levels, an analogous derivation applies. Since the scaling functions are isotropic we have

$$\alpha_{jk} = A_j(|k|) = 2\pi \int_{\mathbb{R}^+_{|\xi|}} \hat{g}\big(|2^{-j}\xi|\big)\hat{g}\big(|2^{-j-1}\xi|\big)\, J_0\big(|\xi|\,|2^{-j-1}k|\big)\, |\xi|\, d\theta\, d|\xi|. \tag{21d}$$

When the Rayleigh formula is used instead of the Jacobi-Anger one, an analogous derivation can be used to derive the filter taps in three dimensions.

4. Galerkin Projection of the Laplace Operator

The Galerkin projection of the Laplace operator is given by

$$v_{sr} = \big\langle \Delta\psi_s\,,\,\psi_r \big\rangle. \tag{22a}$$

where $s = (j_s, k_s, t_s)$, $r = (j_r, k_r, t_r)$ are multi-indices. Using the Parseval identity we can compute the coefficients v_{sr} in the Fourier domain,

$$v_{sr} = \left\langle \widehat{\Delta} \hat{\psi}_s \, , \, \overline{\hat{\psi}_r} \right\rangle. \tag{22b}$$

The Fourier multiplier $\widehat{\Delta}$ of the Laplacian is $\widehat{\Delta} = -|\xi|^2$ so that

$$v_{sr} = -\left\langle |\xi|^2 \, \hat{\psi}_s \, , \, \overline{\hat{\psi}_r} \right\rangle. \tag{22c}$$

Without loss of generality we can assume that ψ_s is located at the origin. Then

$$v_{sr} = -\left\langle |\xi|^2 \, \hat{\psi}(2^{-j_s}|\xi|) \, , \, \hat{\psi}(2^{-j_r}|\xi|) \, e^{i\langle \xi, 2^{-j_r} k_r \rangle} \right\rangle \tag{22d}$$

and using the definition of the frame functions we obtain

$$= -\left\langle |\xi|^2 \, \hat{h}(2^{-j_s}|\xi|) \, , \, \hat{h}(2^{-j_r}|\xi|) \, e^{i\langle \xi, 2^{-j_r} k_r \rangle} \right\rangle \tag{22e}$$

where for simplicity we assumed isotropic functions.

In two dimensions, the inner product can be computed in polar coordinates by expanding the translation term $e^{i\langle \xi, 2^{j_r} k_r \rangle}$ using the Jacobi-Anger formula. We thus have

$$\begin{aligned} v_{sr} = -\int_{\mathbb{R}^+_{|\xi|}} \int_{S^2_{\theta_\xi}} & \hat{h}(2^{-j_s}|\xi|) \, \hat{h}(2^{-j_r}|\xi|) \\ & \times \left(\sum_{m \in \mathbb{Z}} i^m \, e^{im(\theta_\xi - \theta_{k_r})} \, J_m(|\xi| \, |2^{-j_r} k_r|) \right) |\xi|^3 \, d\theta_\xi \, d|\xi| \end{aligned} \tag{22f}$$

$$\begin{aligned} = -\sum_{m \in \mathbb{Z}} i^m \, e^{-im\theta_{k_r}} & \int_{S^2_{\theta_\xi}} e^{im\theta_\xi} \, d\theta_\xi \\ & \times \int_{\mathbb{R}^+_{|\xi|}} \hat{h}(2^{-j_s}|\xi|) \, \hat{h}(2^{-j_r}|\xi|) \, J_m(|\xi| \, |2^{-j_r} k_r|) \, |\xi|^3 \, d|\xi| \end{aligned} \tag{22g}$$

The angular integral is non-zero only when $m = 0$. We hence have

$$v_{sr} = -2\pi \int_{\mathbb{R}^+_{|\xi|}} \hat{h}(2^{-j_s}|\xi|) \, \hat{h}(2^{-j_r}|\xi|) \, J_m(2^{-j_r}|\xi| \, |k_r|) \, |\xi|^3 \, d|\xi|. \tag{22h}$$

The remaining radial integral can be evaluated in closed form, see the accompanying code for the expression. An analogous argument applies in the non-isotropic case. In three dimensions a similar calculation can be performed when the Jacobi-Anger formula is replaced by the Rayleigh one.

As is apparent from Eq. 22h the v_{sr} are non-zero only when j_s and j_r adjacent, since otherwise the windows $\hat{h}(2^{-j_s}|\xi|)$ and $\hat{h}(2^{-j_r}|\xi|)$ have no common support. The matrix representing the Laplace operator in polarlets is hence sparse. The spatial decay of the v_{sr} is shown in Fig. 2.

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