Between visual fidelity and mathematical wizardary

Christian Lessig

Otto-von-Guericke-Universität Magdeburg

About me

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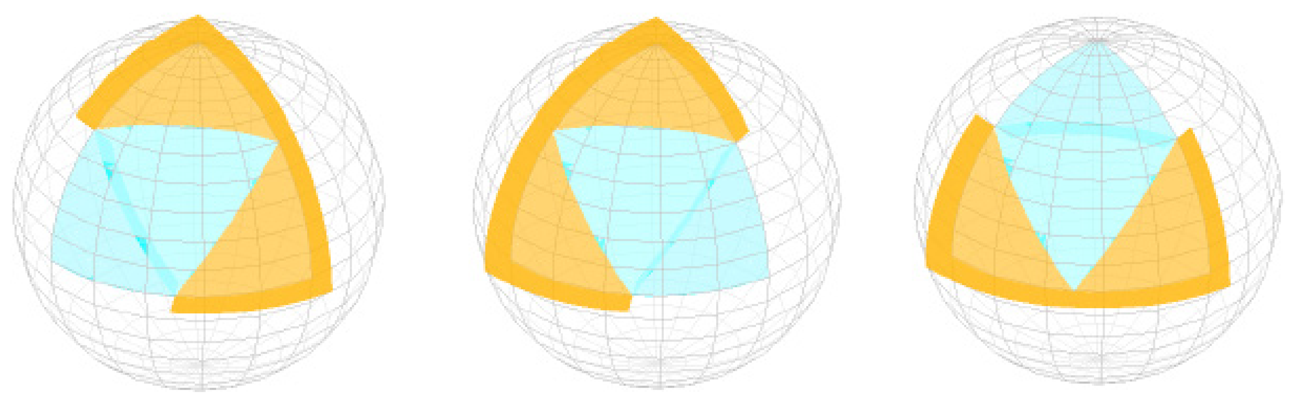

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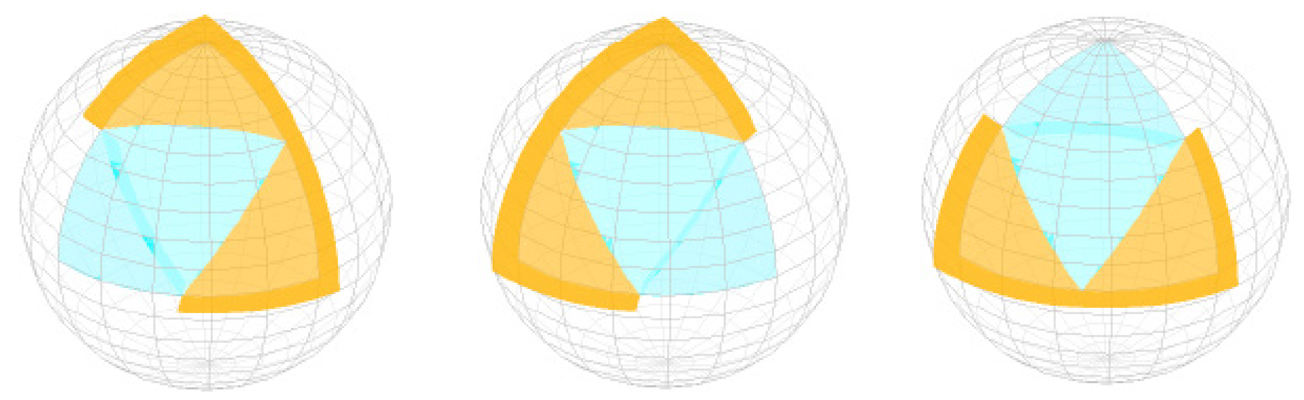

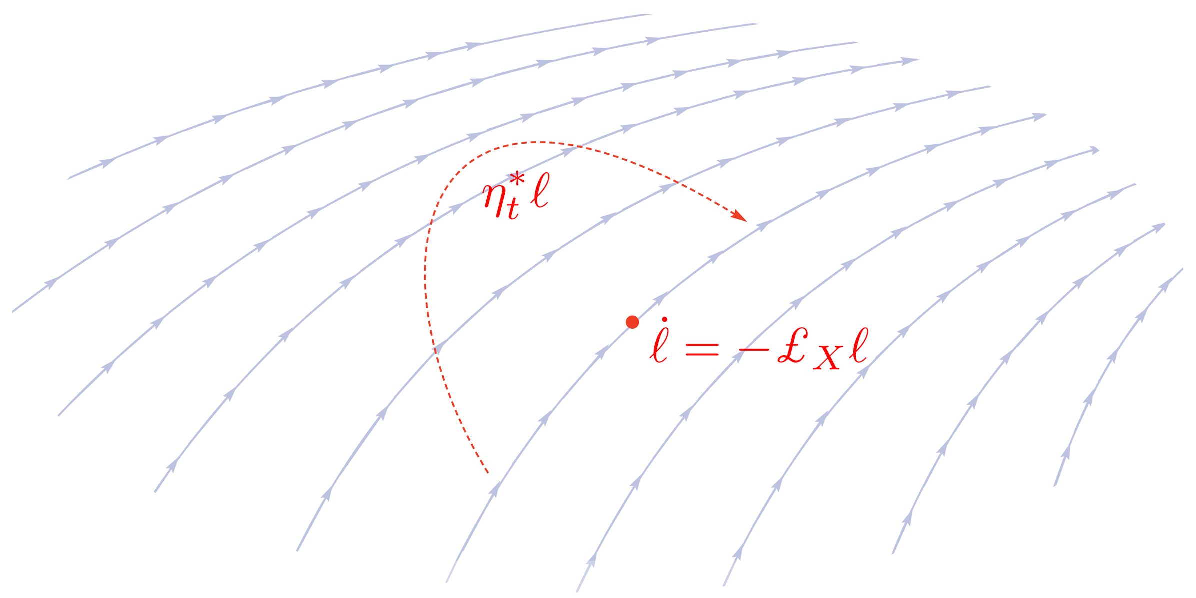

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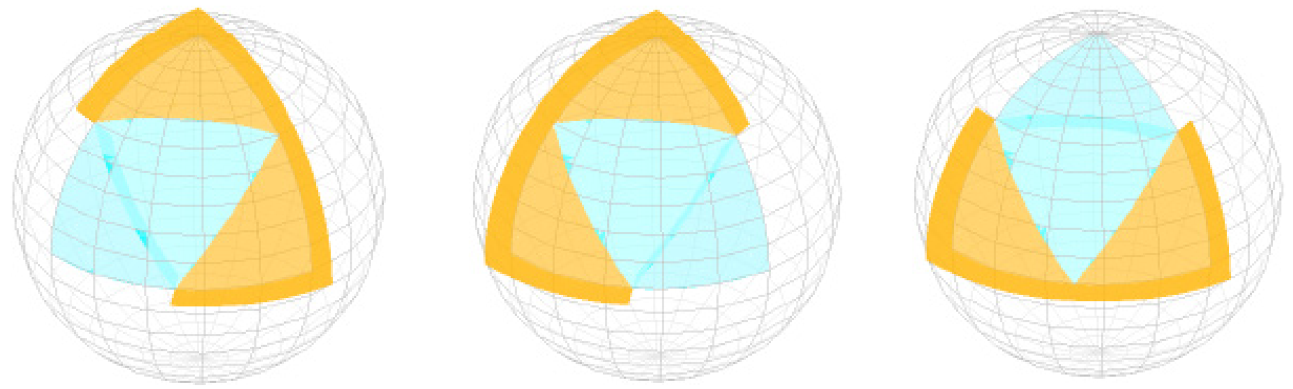

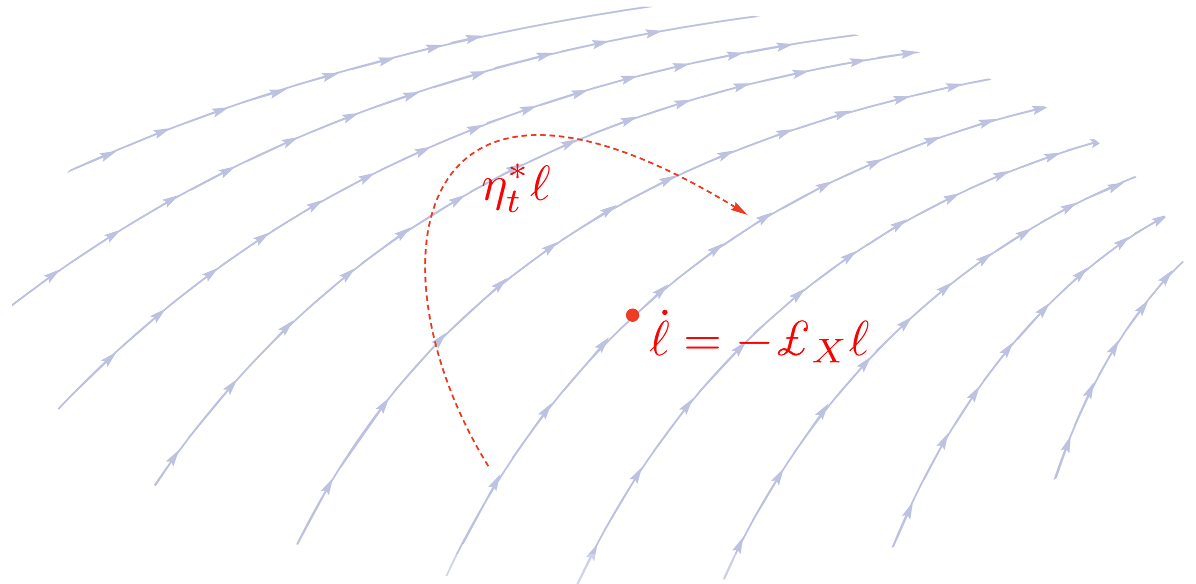

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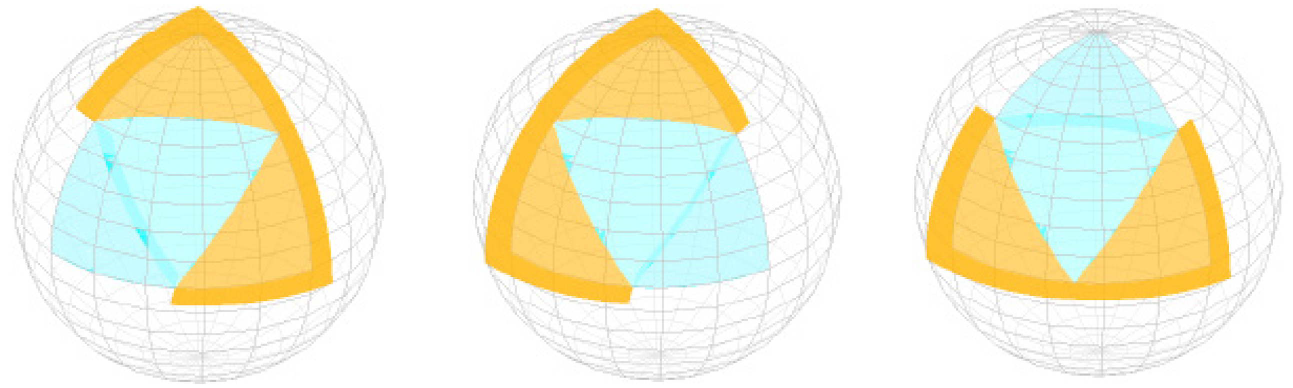

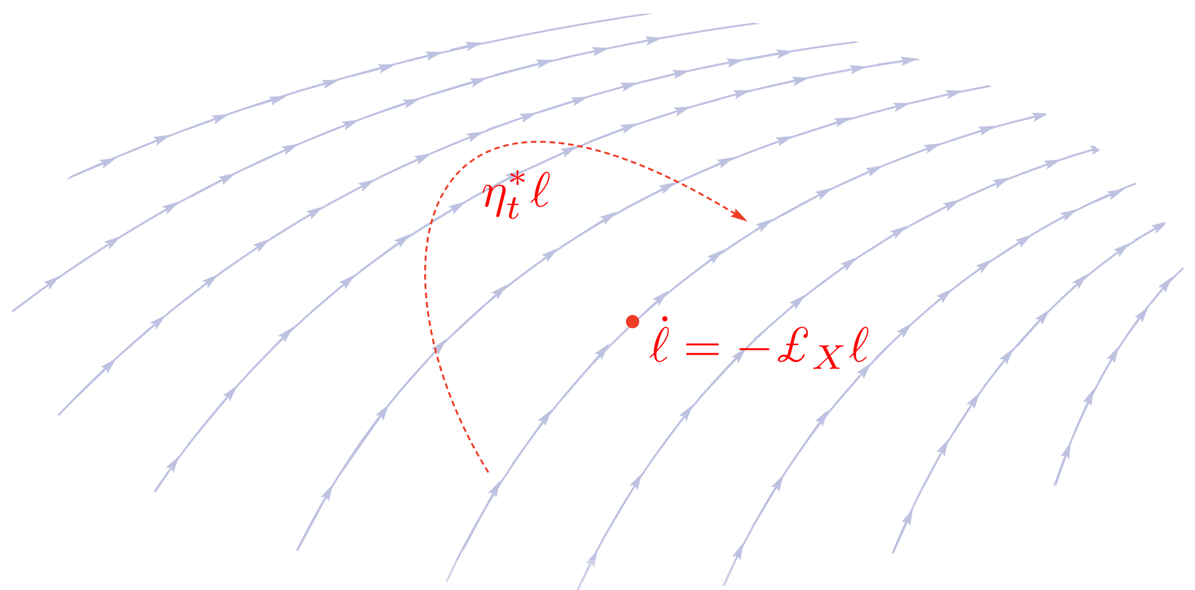

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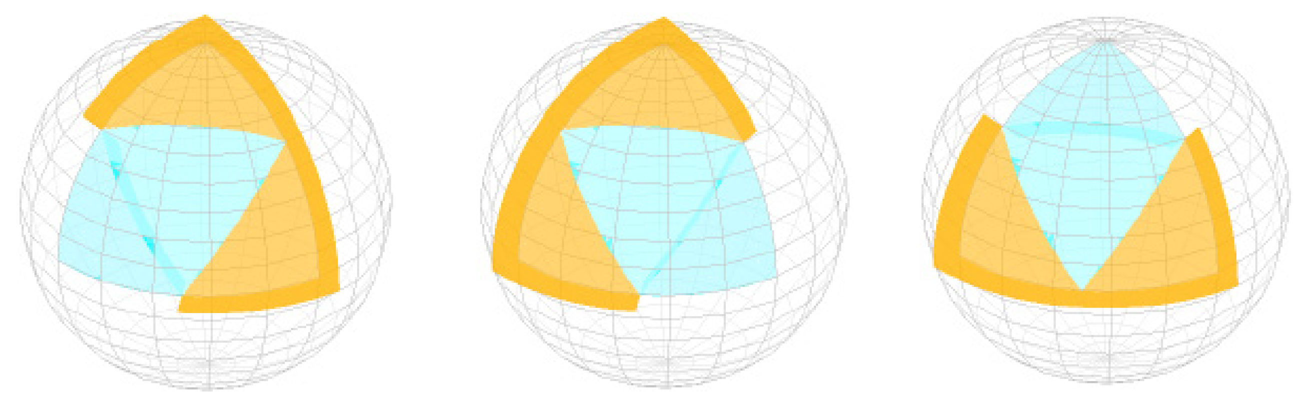
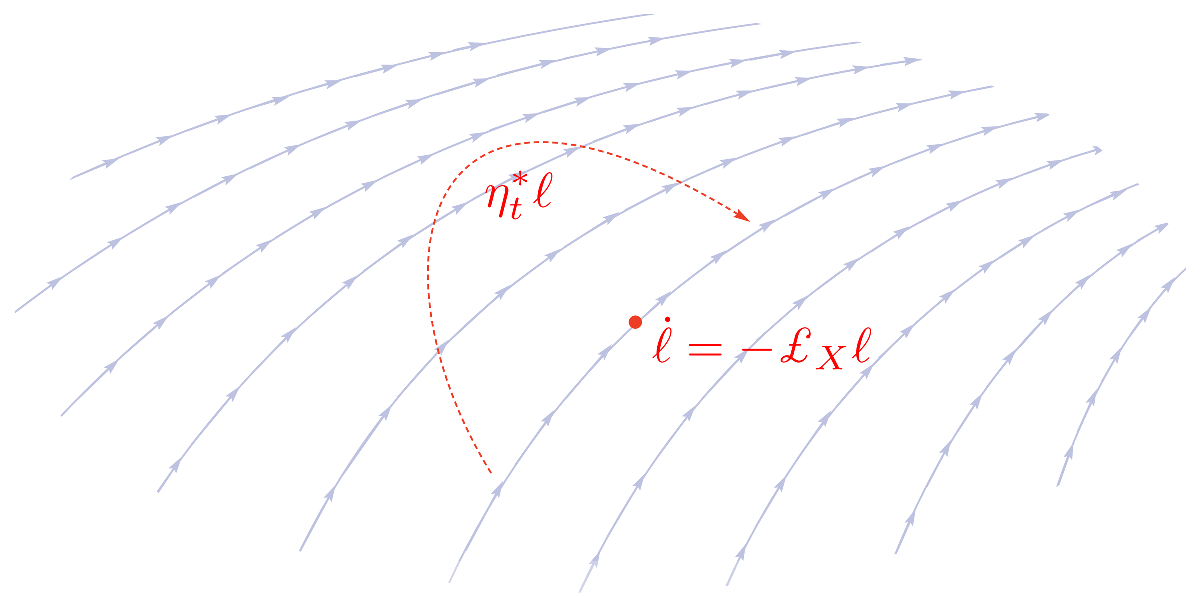
University Magdeburg





What is computer graphics?

[video]

What is computer graphics?

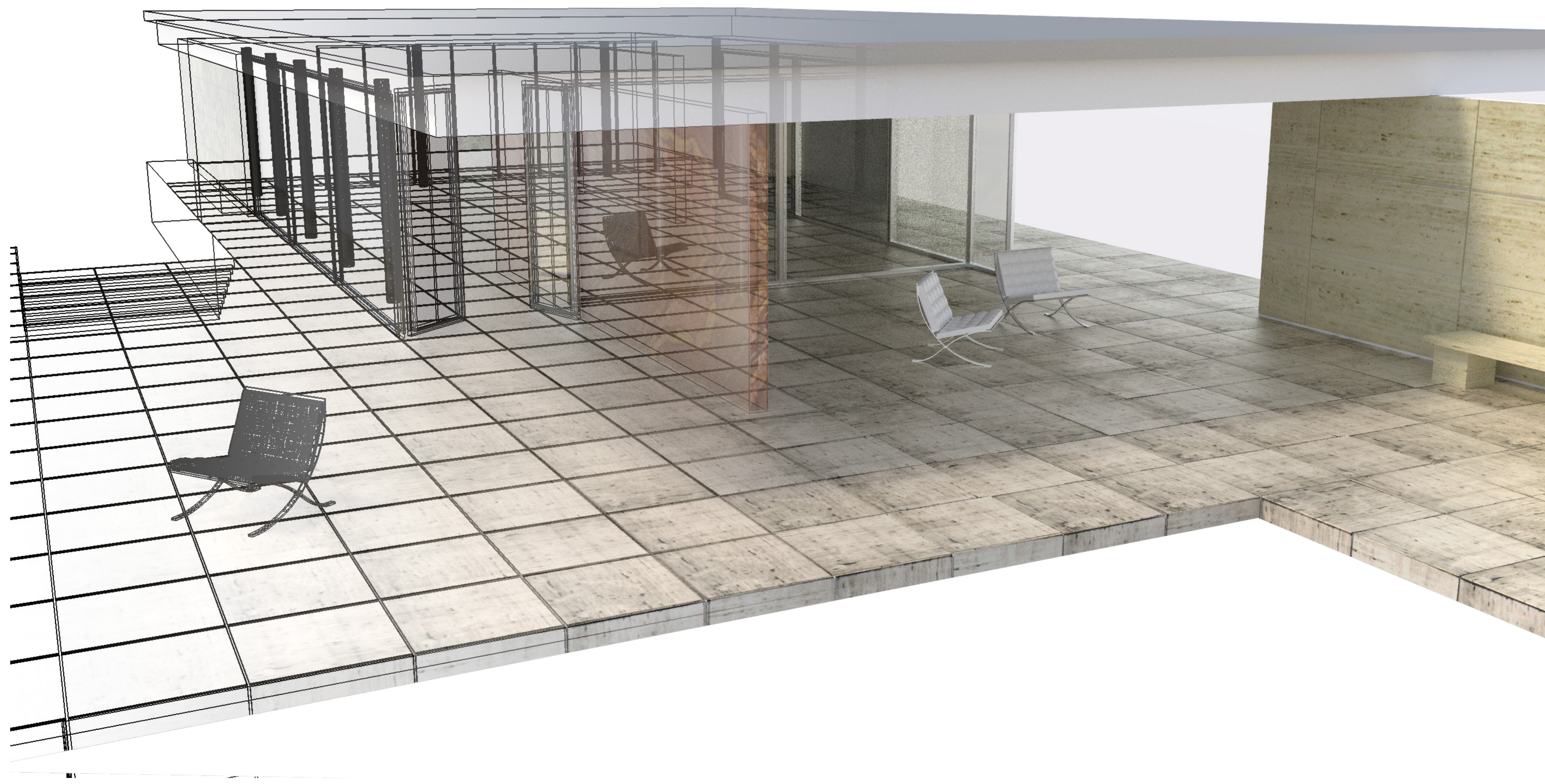
- Very diverse field between art, applied mathematics, optics, image processing, machine learning, physics, material science, …
- Driven by the movie industry, computer games, inter-active applications
- More engineering than science (usually)

What is computer graphics?

- Classical and recent topics are:
  - Image generation / rendering
  - Physically-based simulation of fluids, elastic objects, ...
  - Mesh processing and modelling
  - 3d printing
  - Computational cameras and imaging

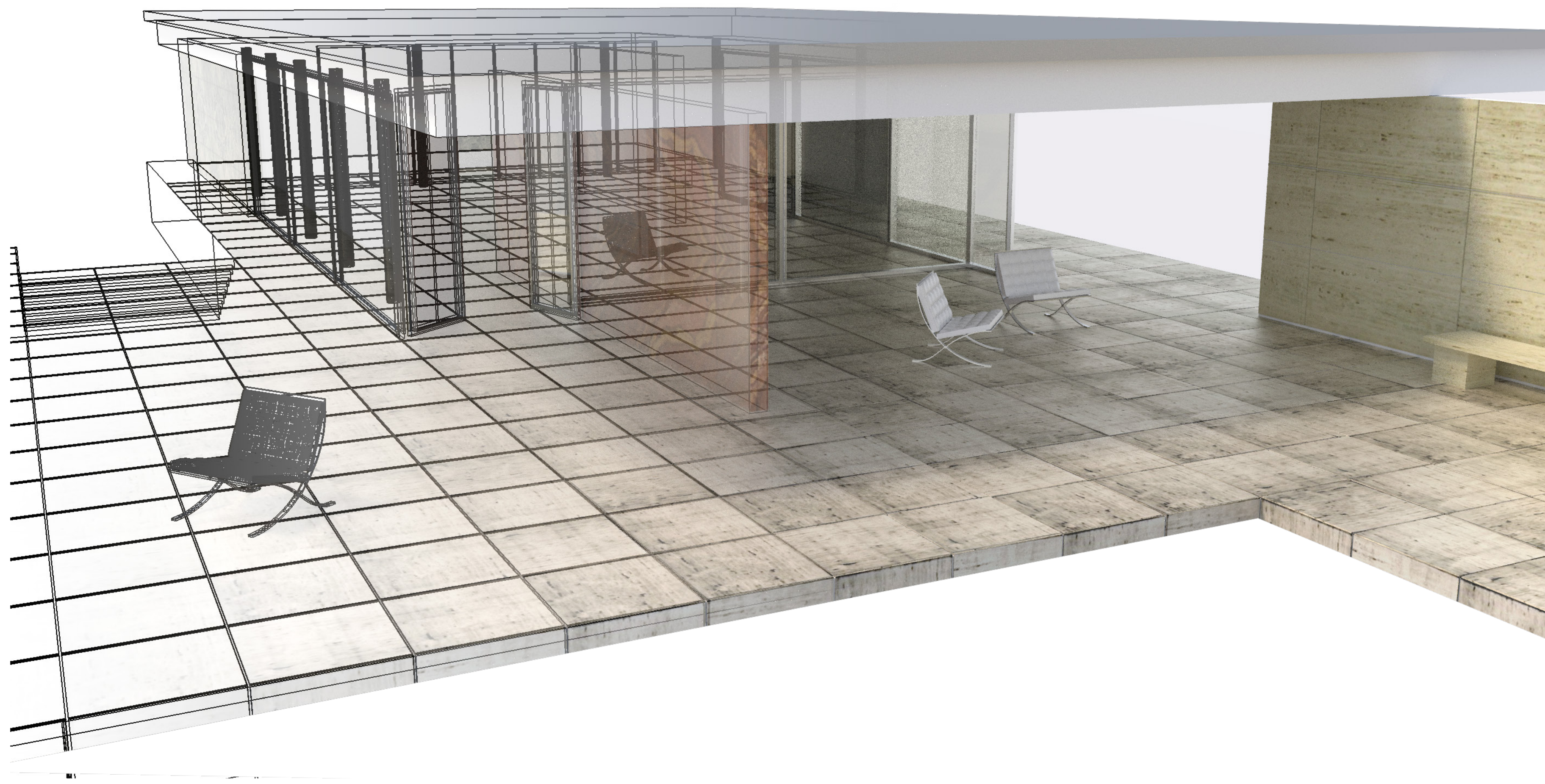
My research

My research



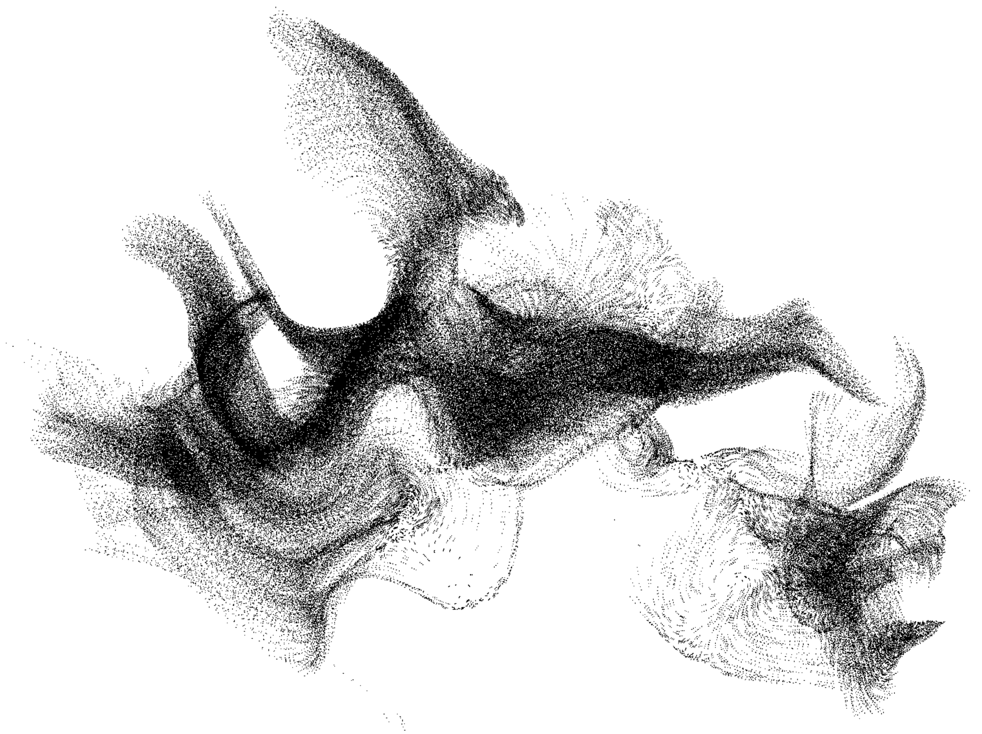
- Image generation:
  - Theoretical foundations
  - Rigorous, adaptive computational techniques

My research



- Image generation:
  - Theoretical foundations
  - Rigorous, adaptive computational techniques

- Fluid simulation
  - Geometric integrators
  - Construction of “good“ representations



Fluid simulation

[video]

Fluid simulation

- Incompressible Navier-Stokes equations:

$$\frac{\partial \vec{u}}{\partial t} + \mathcal{L}_{\vec{u}} \vec{u} = -\nabla p + \nu \Delta \vec{u}$$

$$\nabla \cdot \vec{u} = 0$$

Fluid simulation

- Incompressible Navier-Stokes equations:

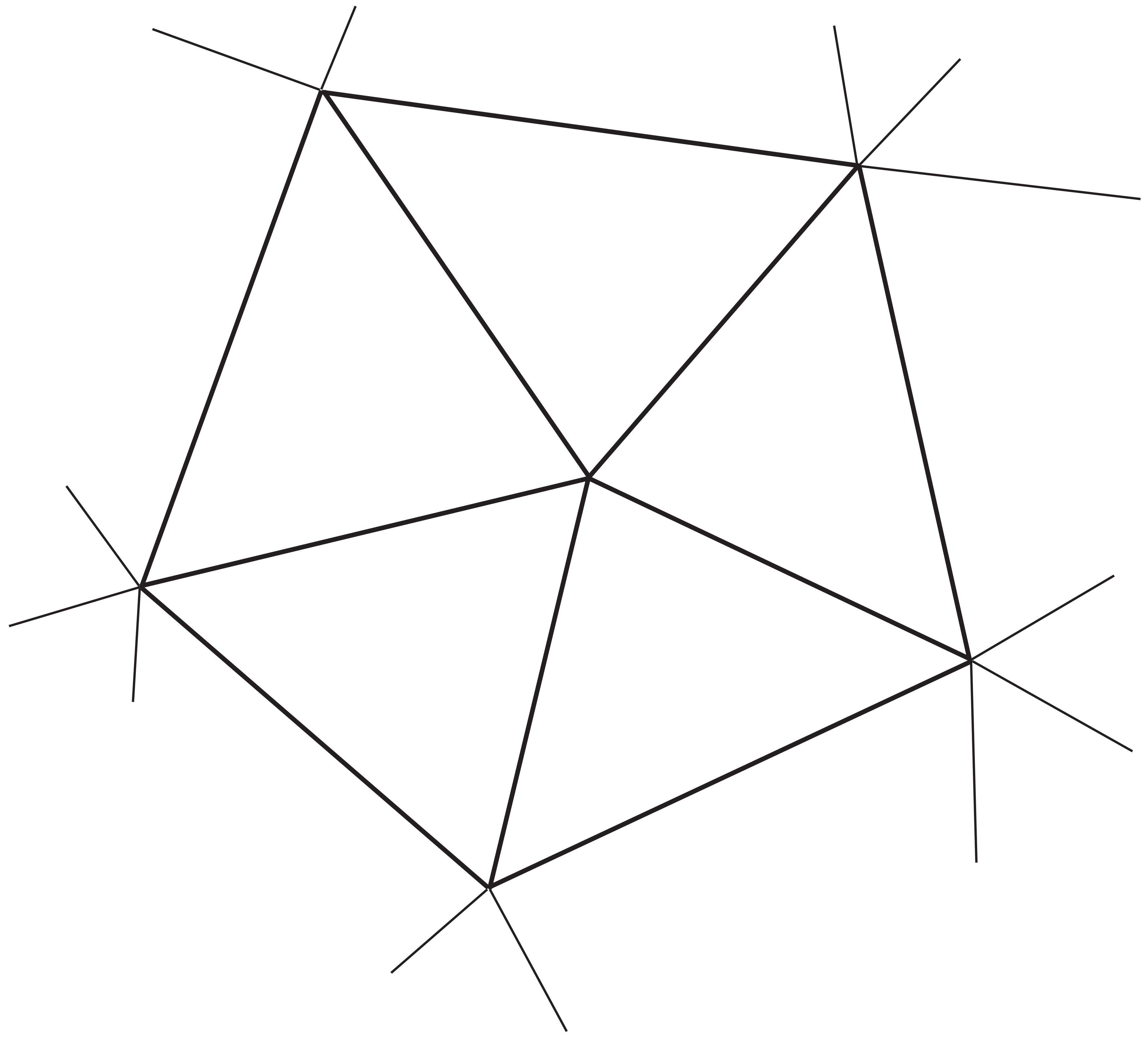
$$\frac{\partial \vec{u}}{\partial t} + \mathcal{L}_{\vec{u}}\vec{u} = -\nabla p + \nu \Delta \vec{u}$$

$$\boxed{\nabla \cdot \vec{u} = 0}$$

often difficult to
enforce numerically

Divergence freedom

- Finite element approach:



Divergence freedom

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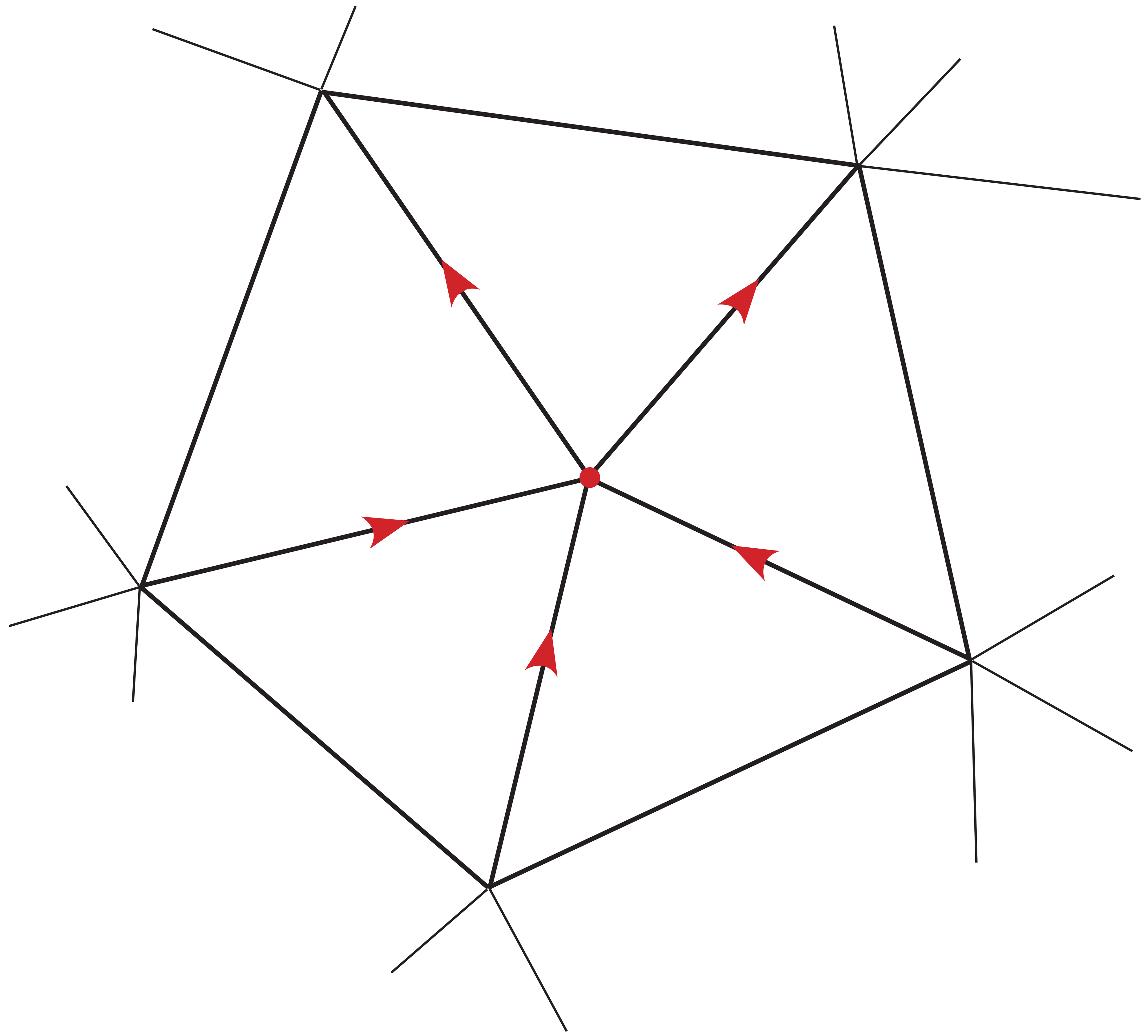

Divergence freedom

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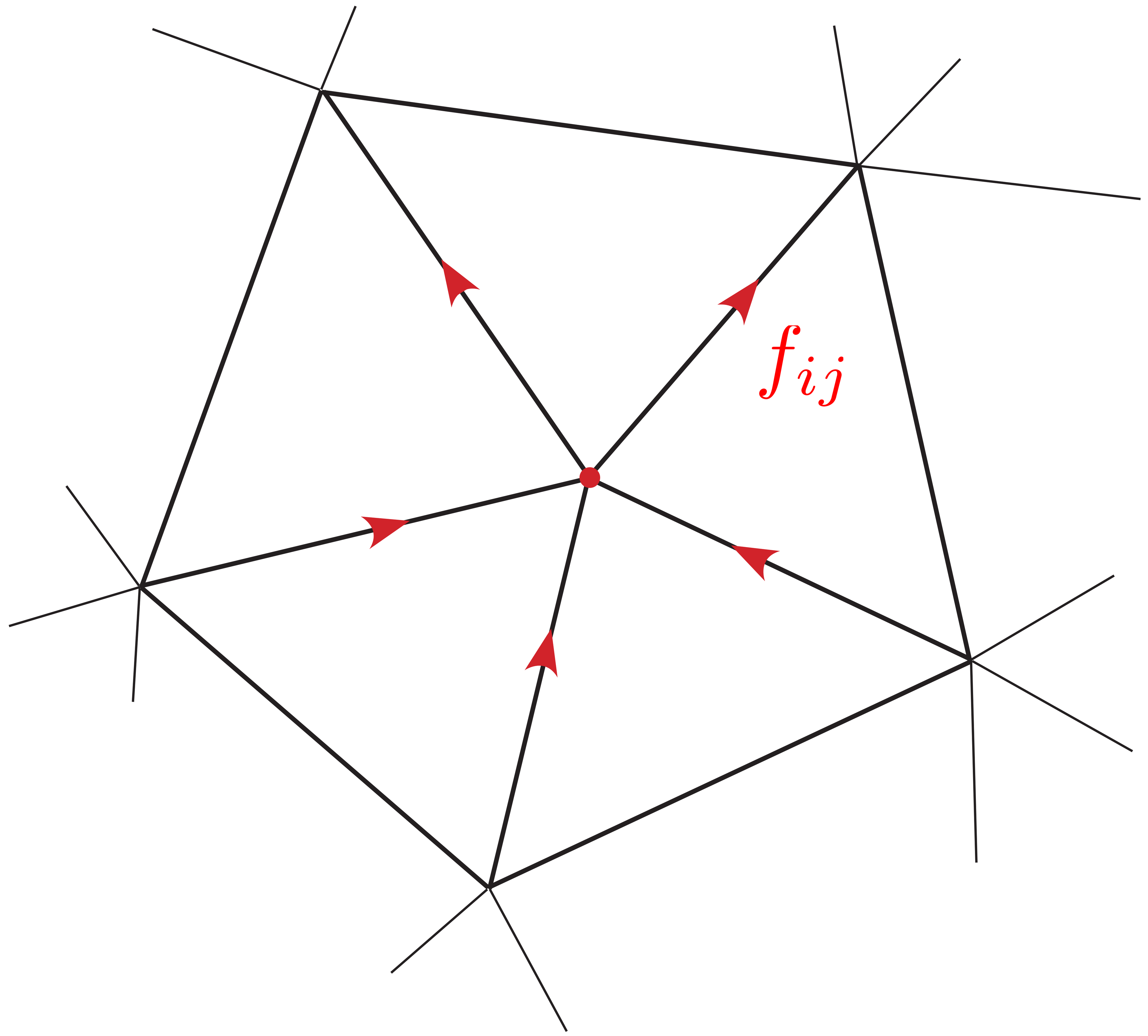

Divergence freedom

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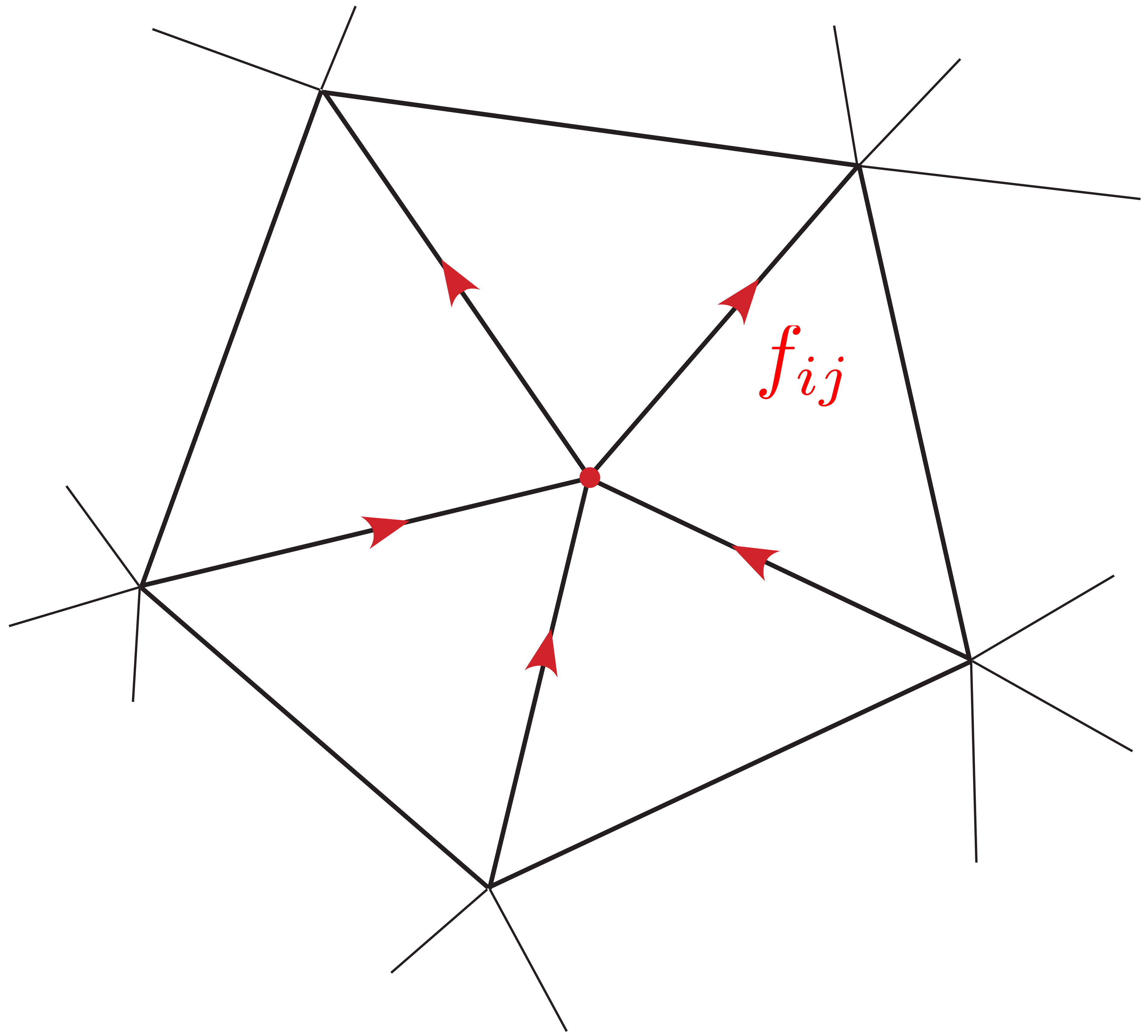


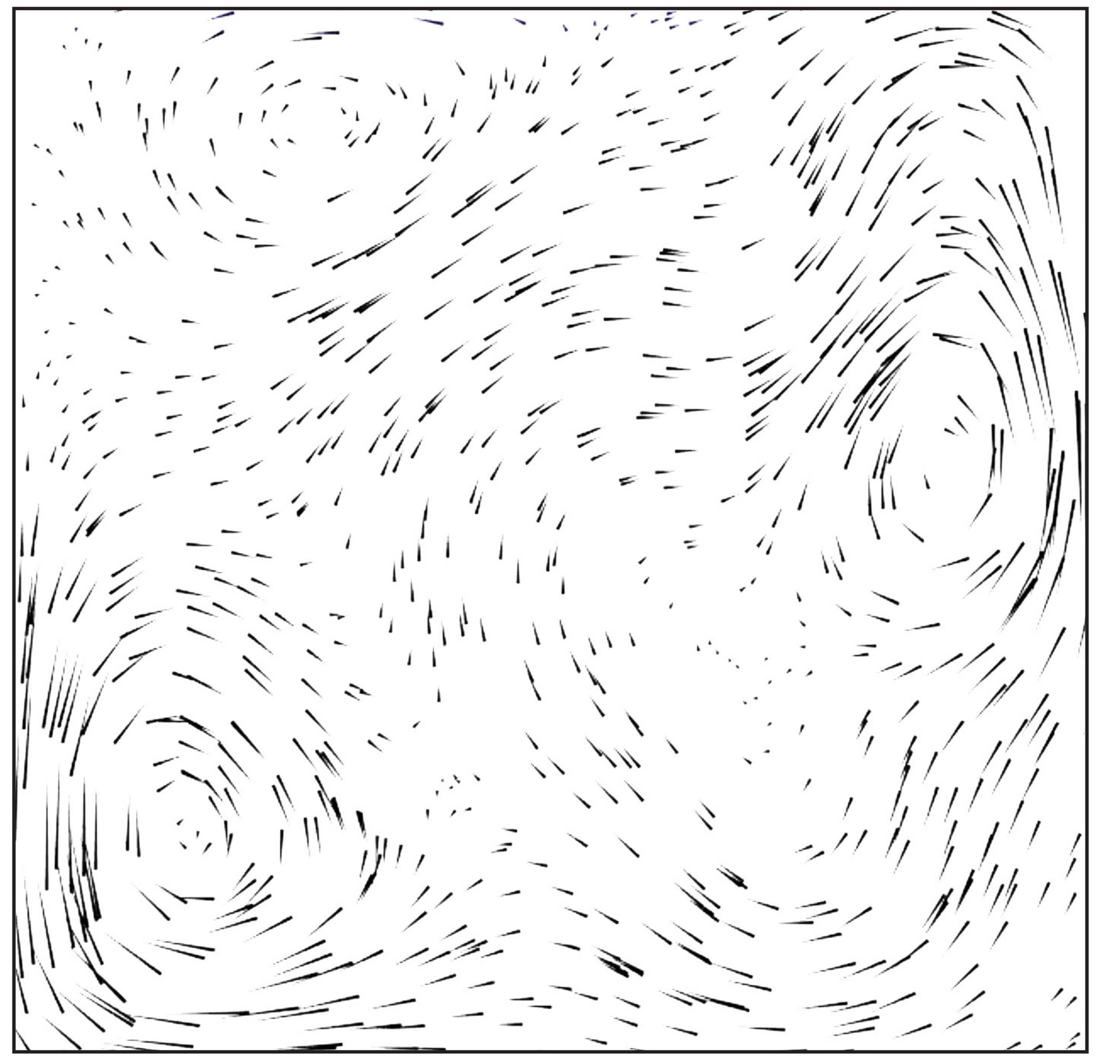
$$\sum_j f_{ij} = 0 \ , \ \forall i$$

Divergence freedom

$$\nabla \cdot \vec{u} = 0$$

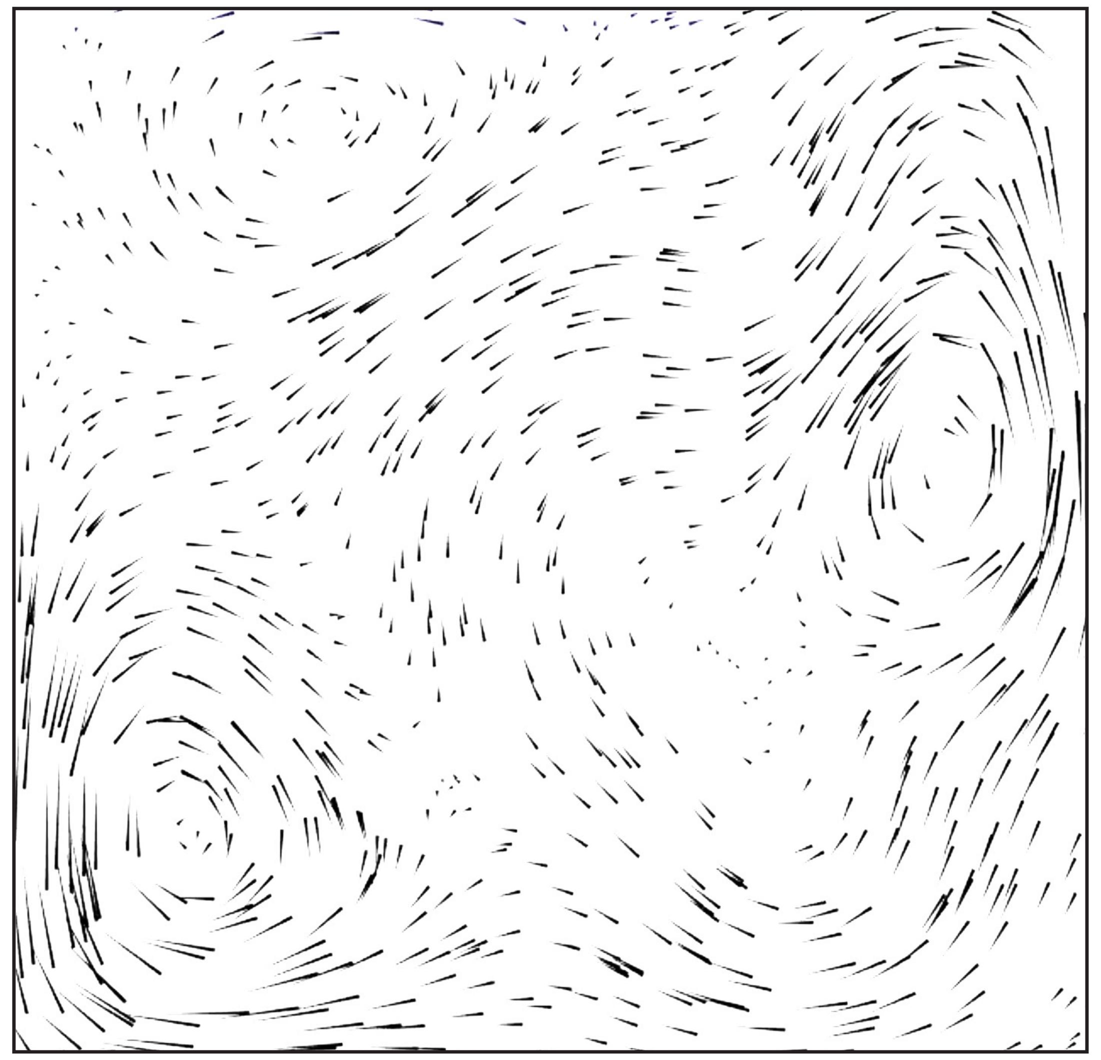
Divergence freedom

$$\nabla \cdot \vec{u} = 0$$



Divergence freedom

$$\nabla \cdot \vec{u} = 0 \xrightarrow[\text{transform}]{\text{Fourier}} \vec{\xi} \cdot \hat{\vec{u}} = 0$$



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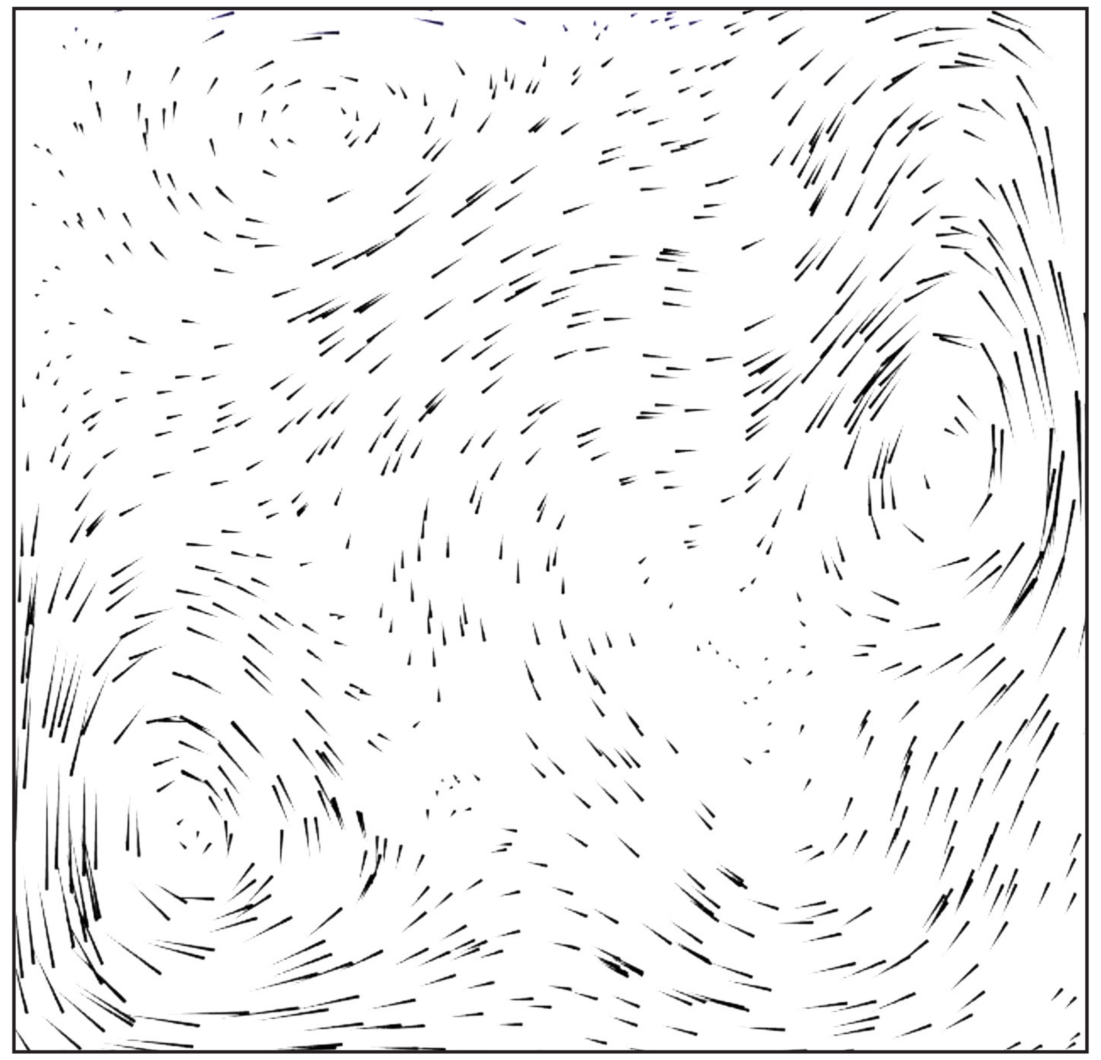
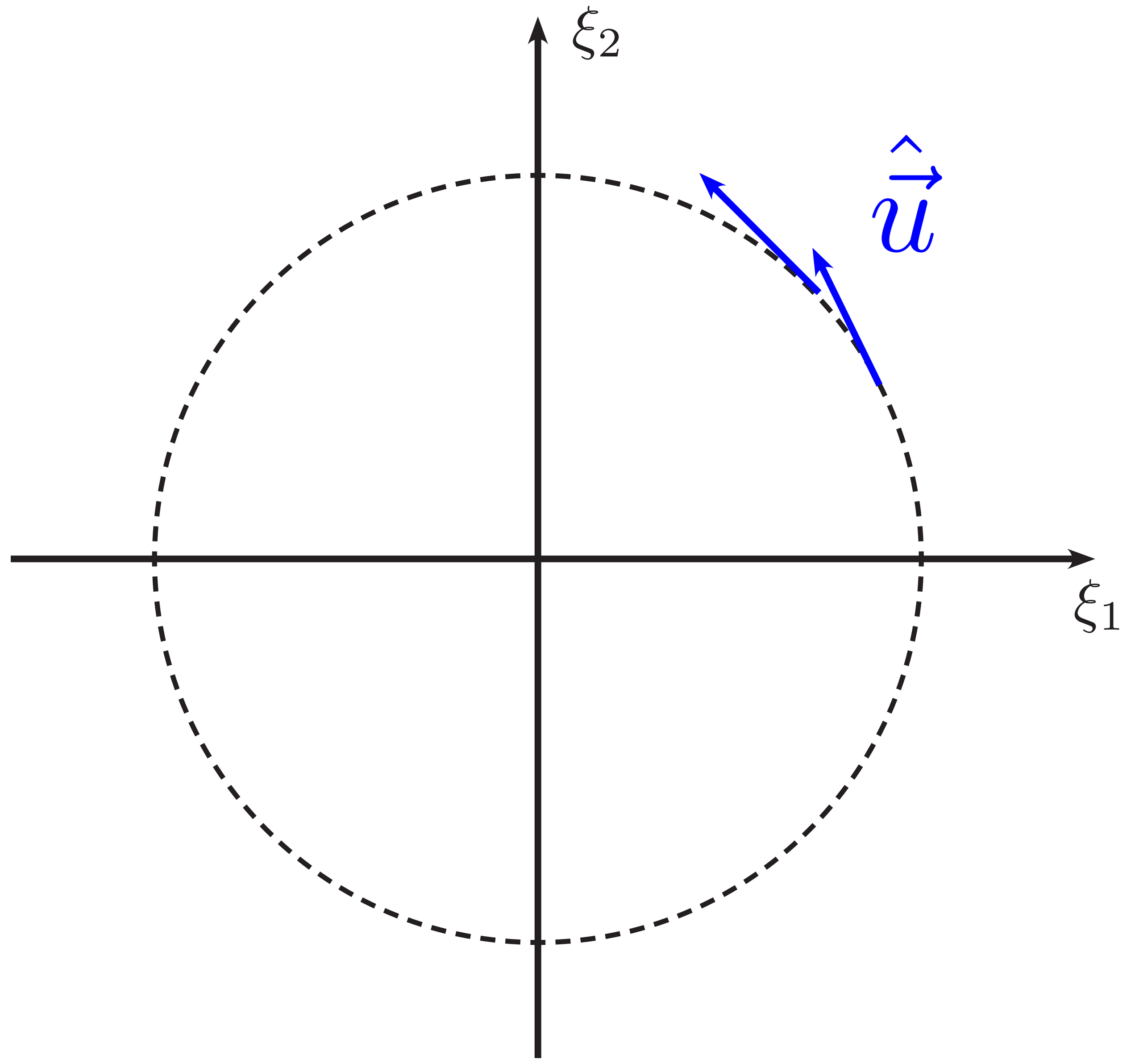
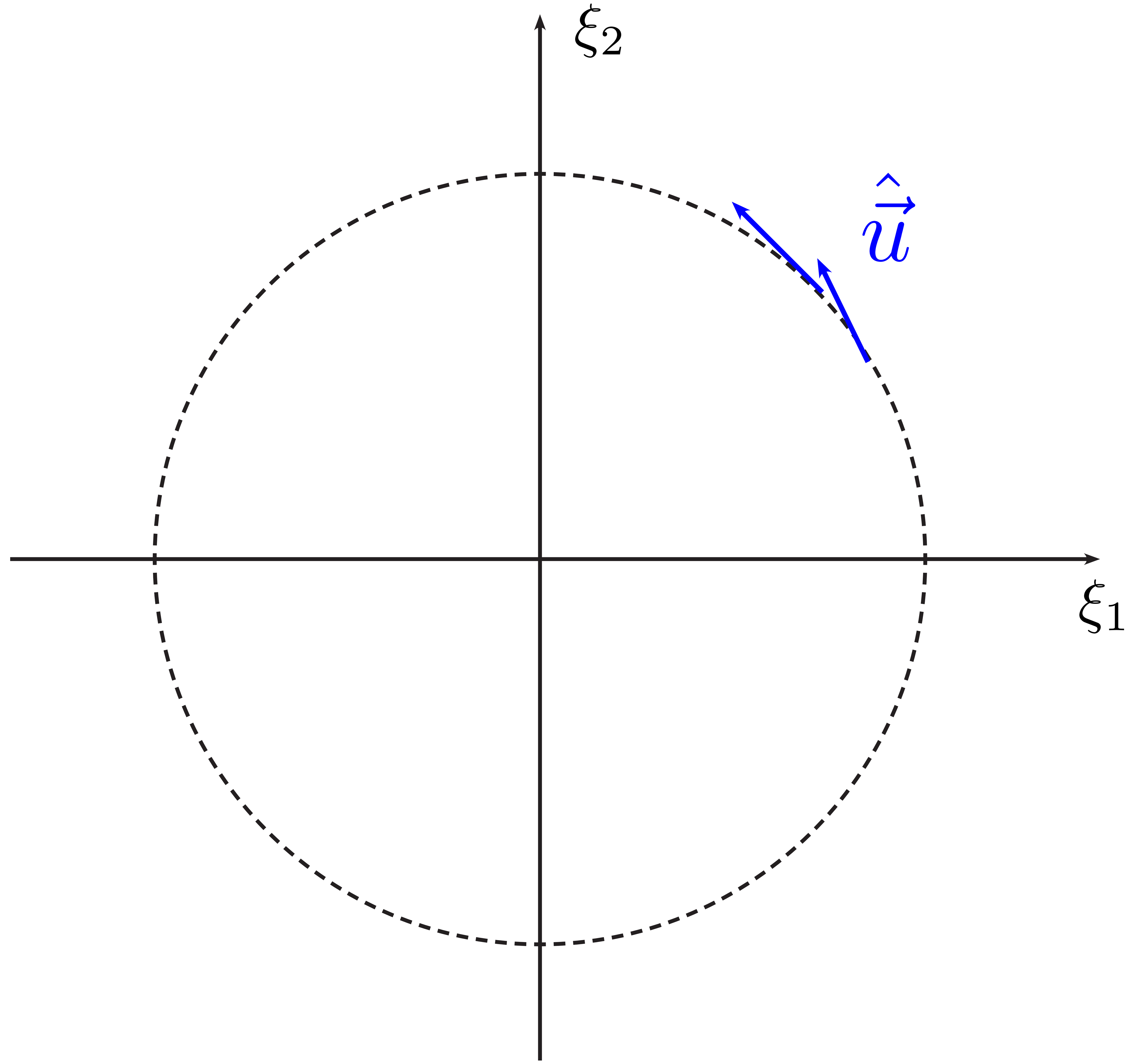




Divergence freedom

- Divergence free basis:



Divergence freedom

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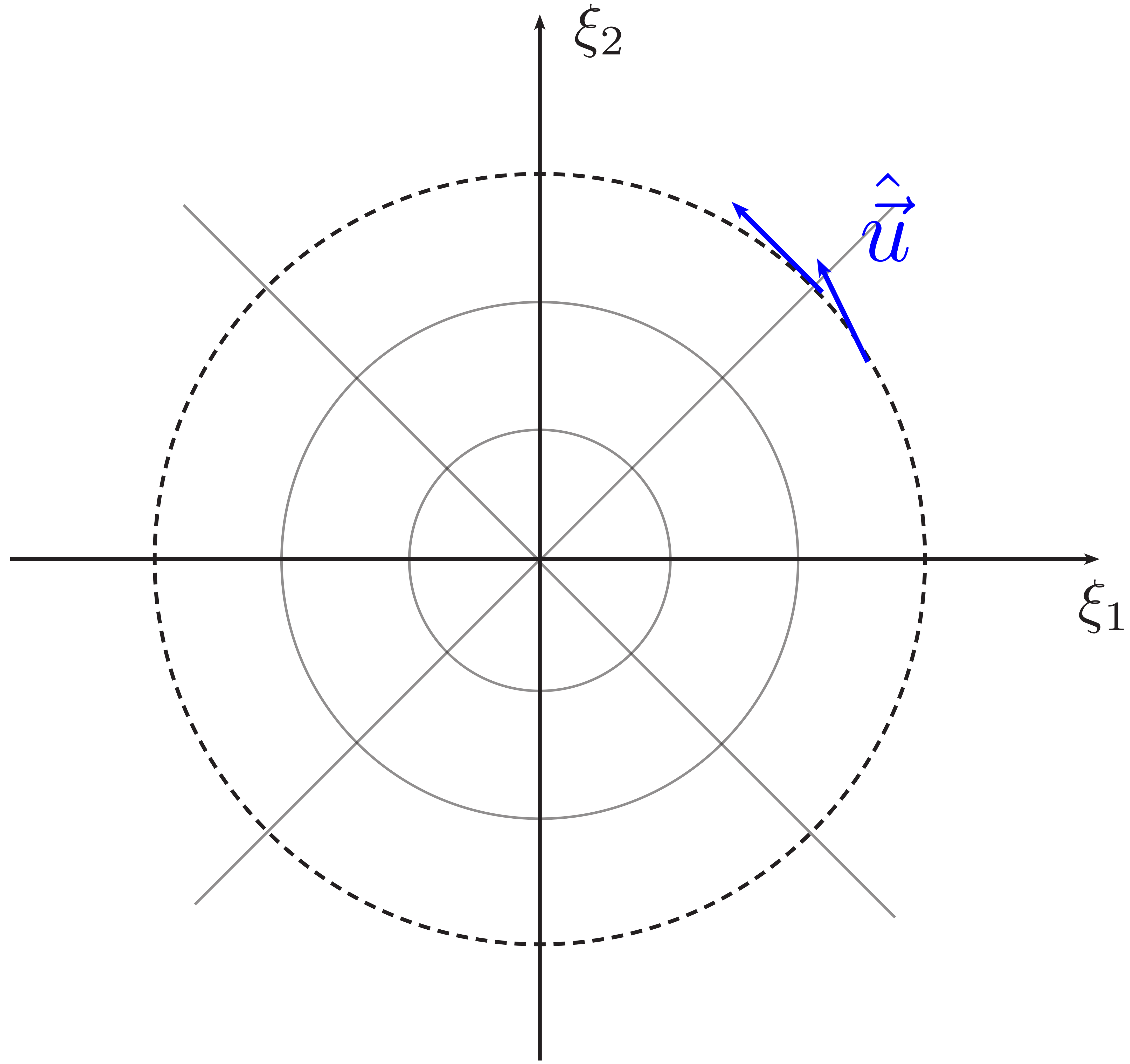

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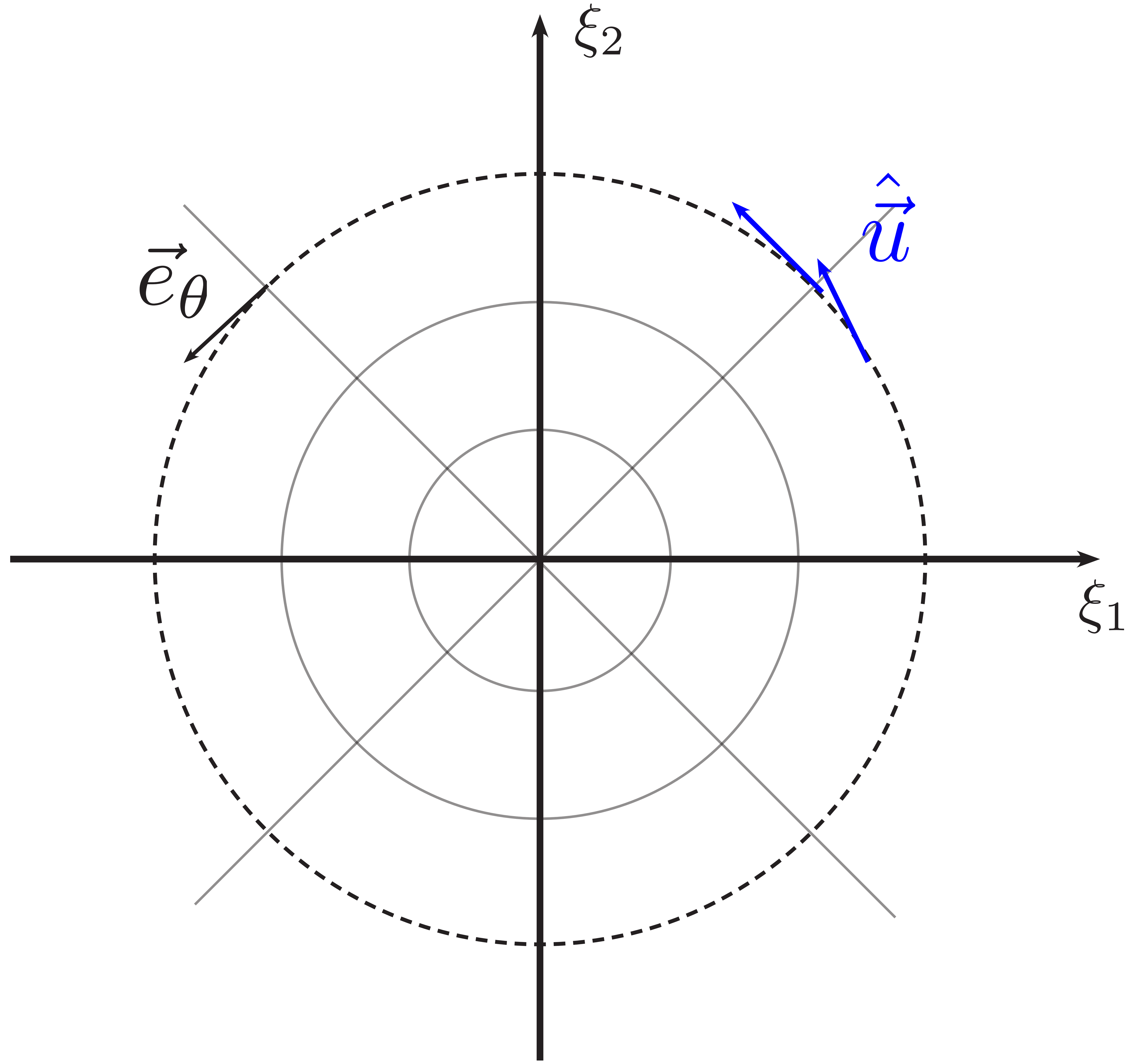
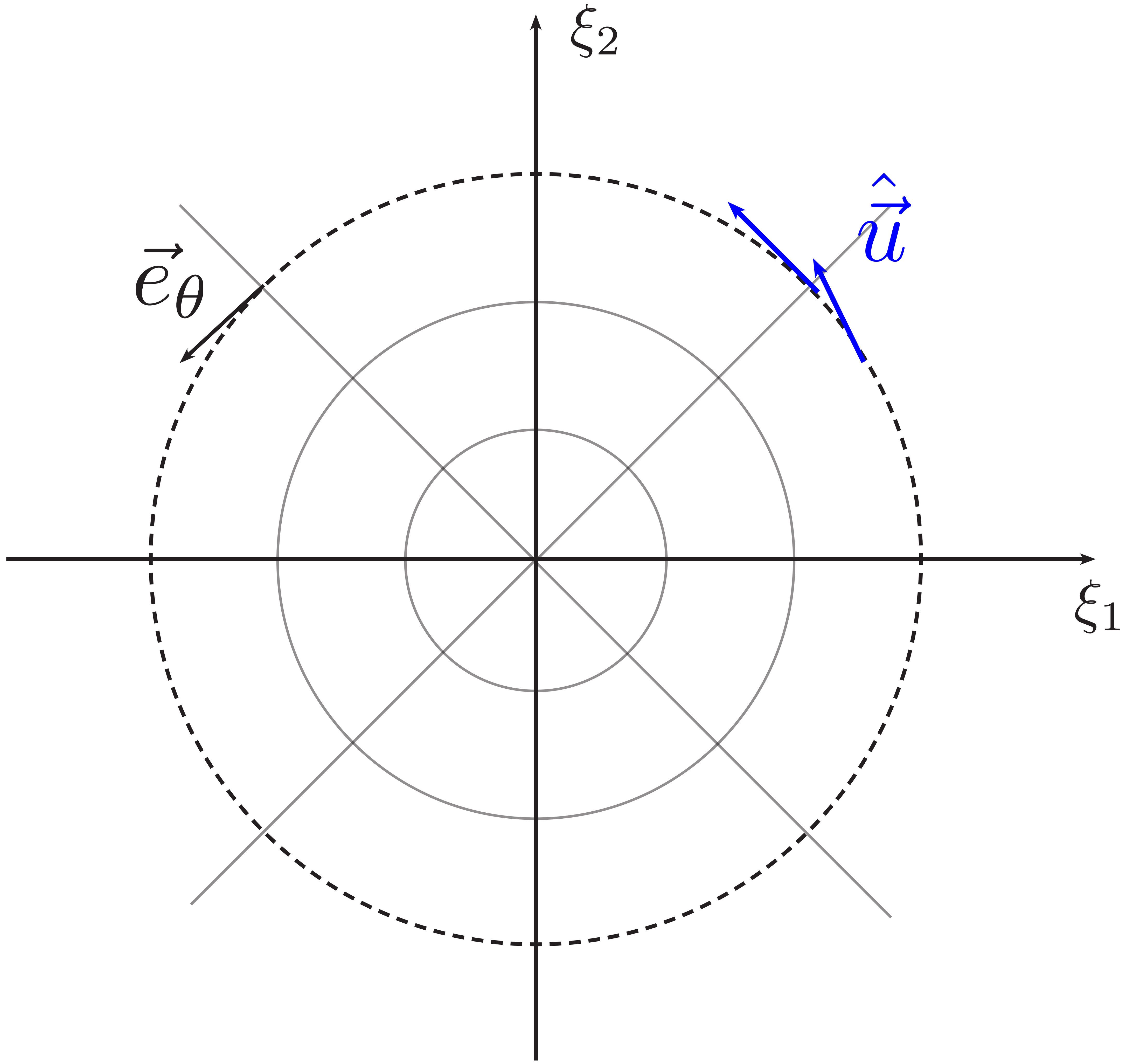


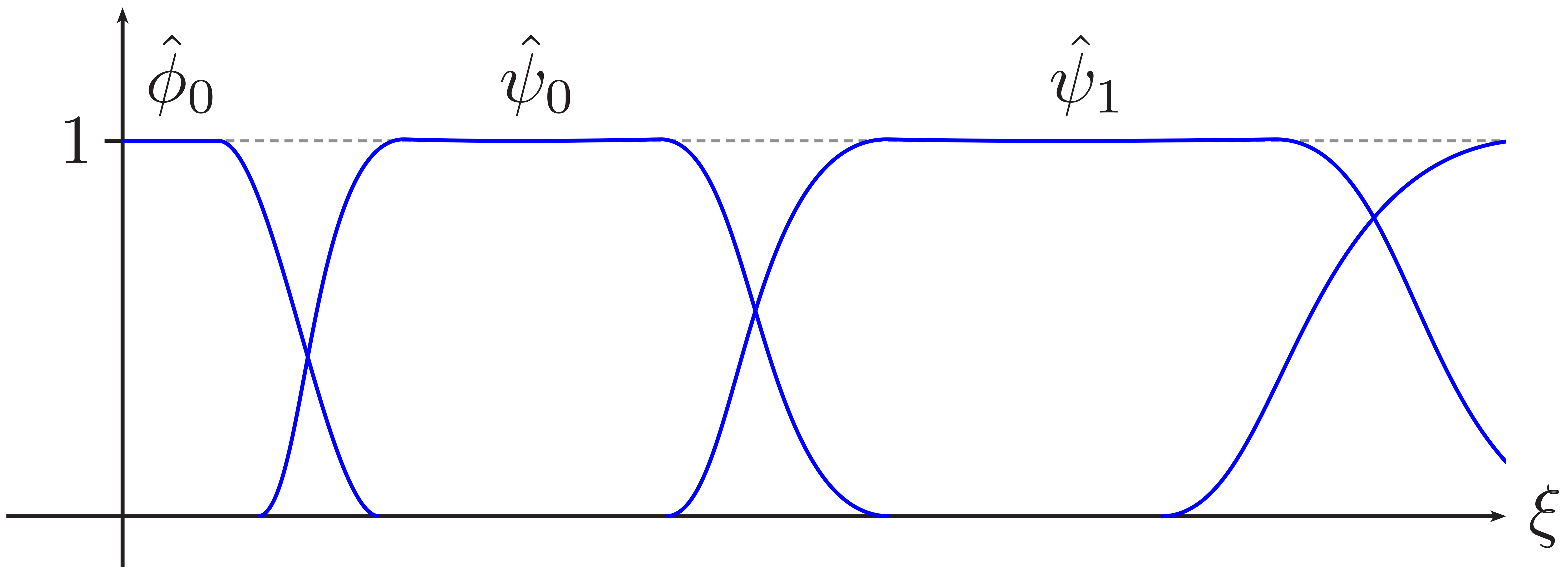
Divergence freedom

- Divergence free basis:

$$\hat{\vec{\psi}}(\xi) = \quad \ldots \quad \vec{e}_\theta$$



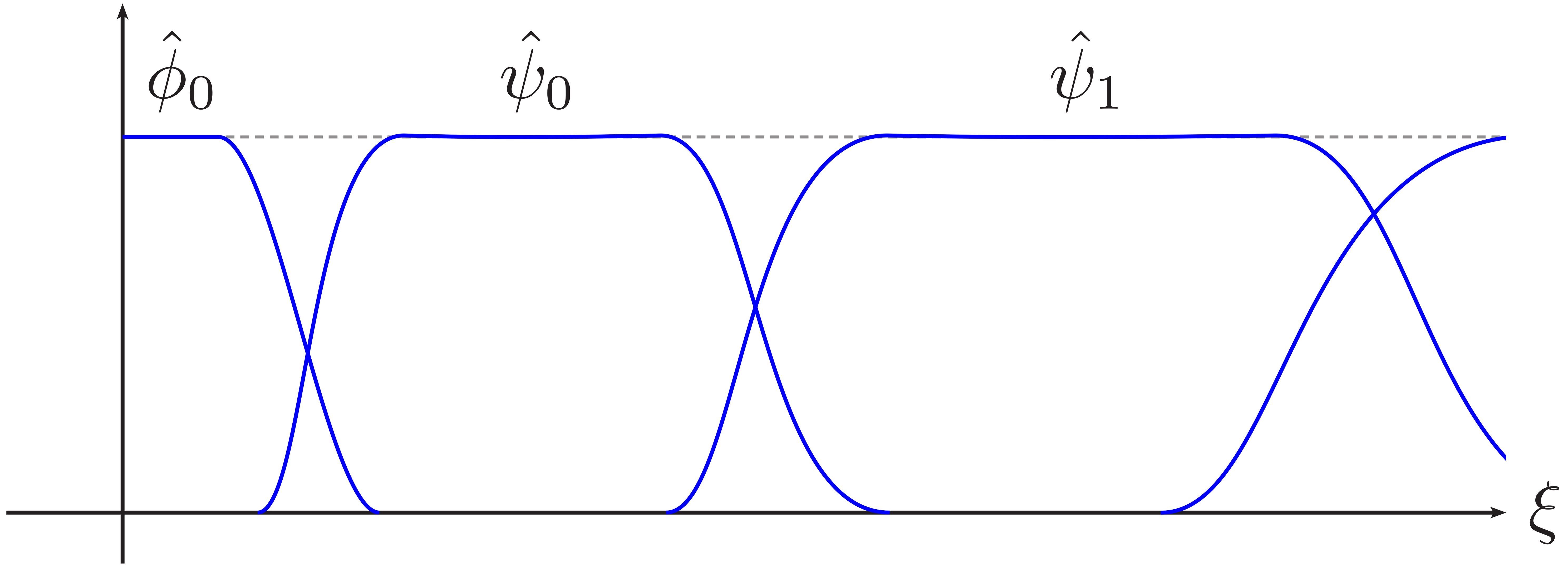
Wavelets



Wavelets

$$\sum_{j=-1}^{\infty} \left|\hat{\psi}_j(\xi)\right|^2 = 1$$

$$\forall \xi \in \mathbb{R}$$



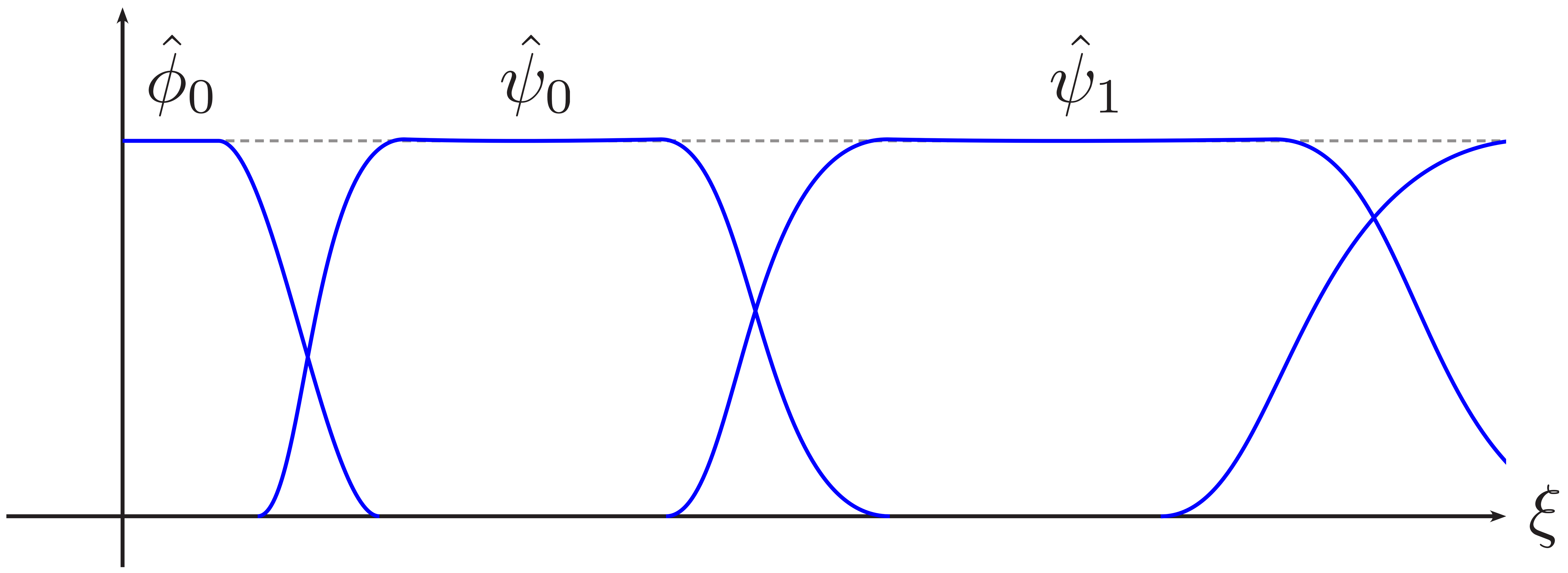
Wavelets

$$\sum_{j=-1}^{\infty} \left| \hat{\psi}_j(\xi) \right|^2 = 1$$

$$\forall \xi \in \mathbb{R}$$

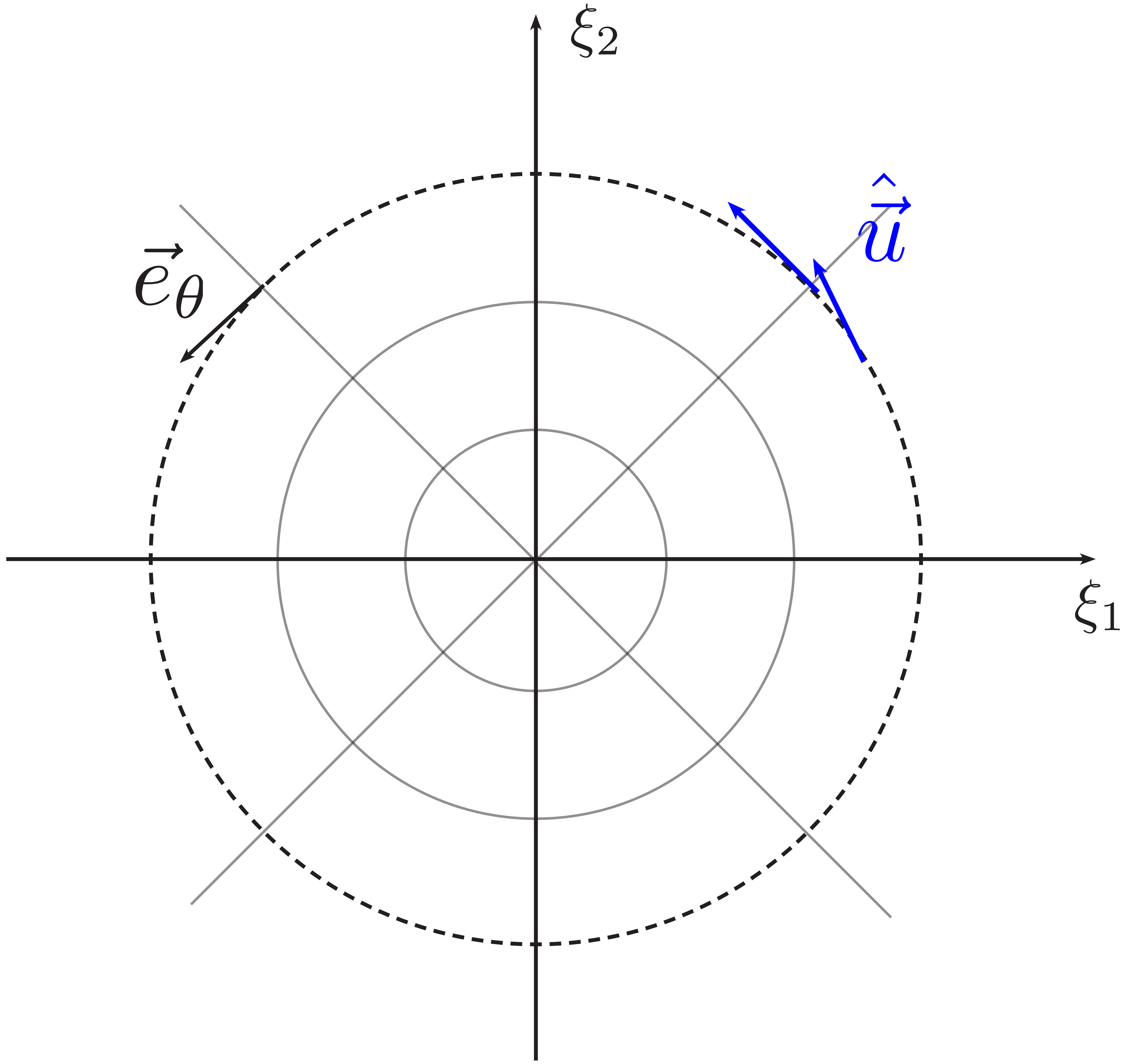
$$\Downarrow$$

forms a tight frame
in the spatial domain



Divergence free wavelets

$$\hat{\vec{\psi}}(\xi) = \quad \ldots \quad \vec{e}_\theta$$



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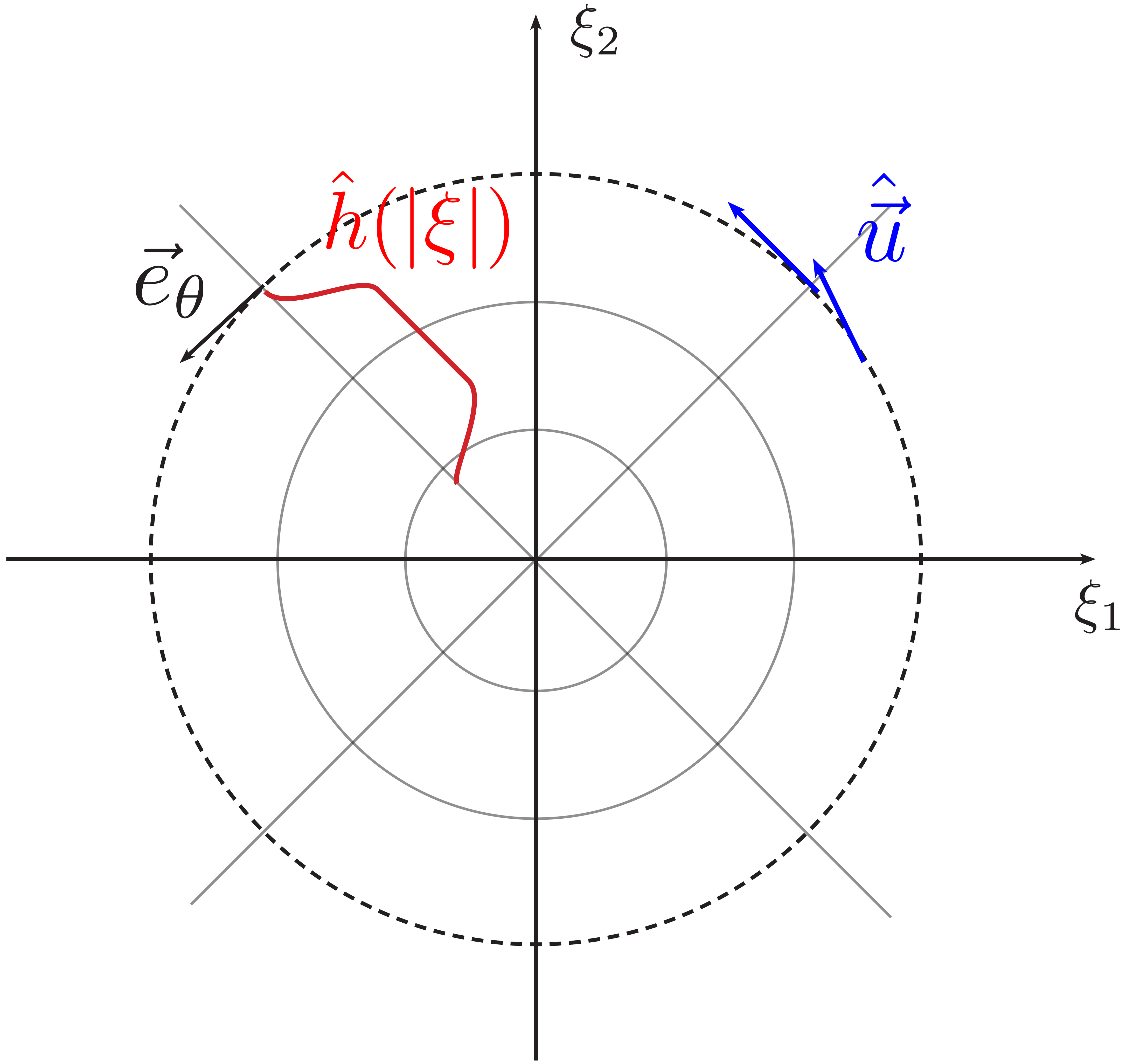
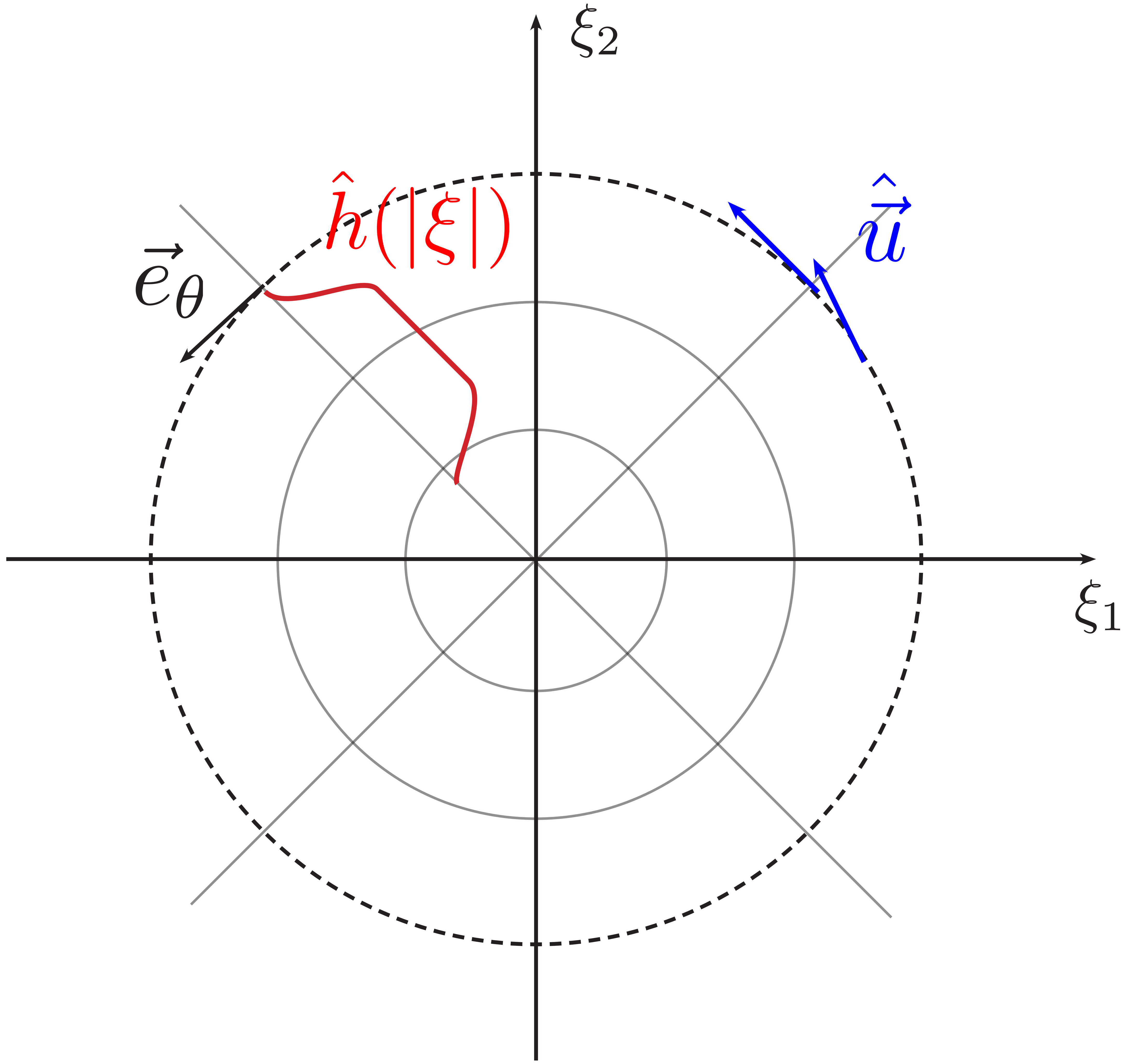


Divergence free wavelets

$$\hat{\vec{\psi}}(\xi) = -i\, h(|\xi|)\, \vec{e}_\theta$$



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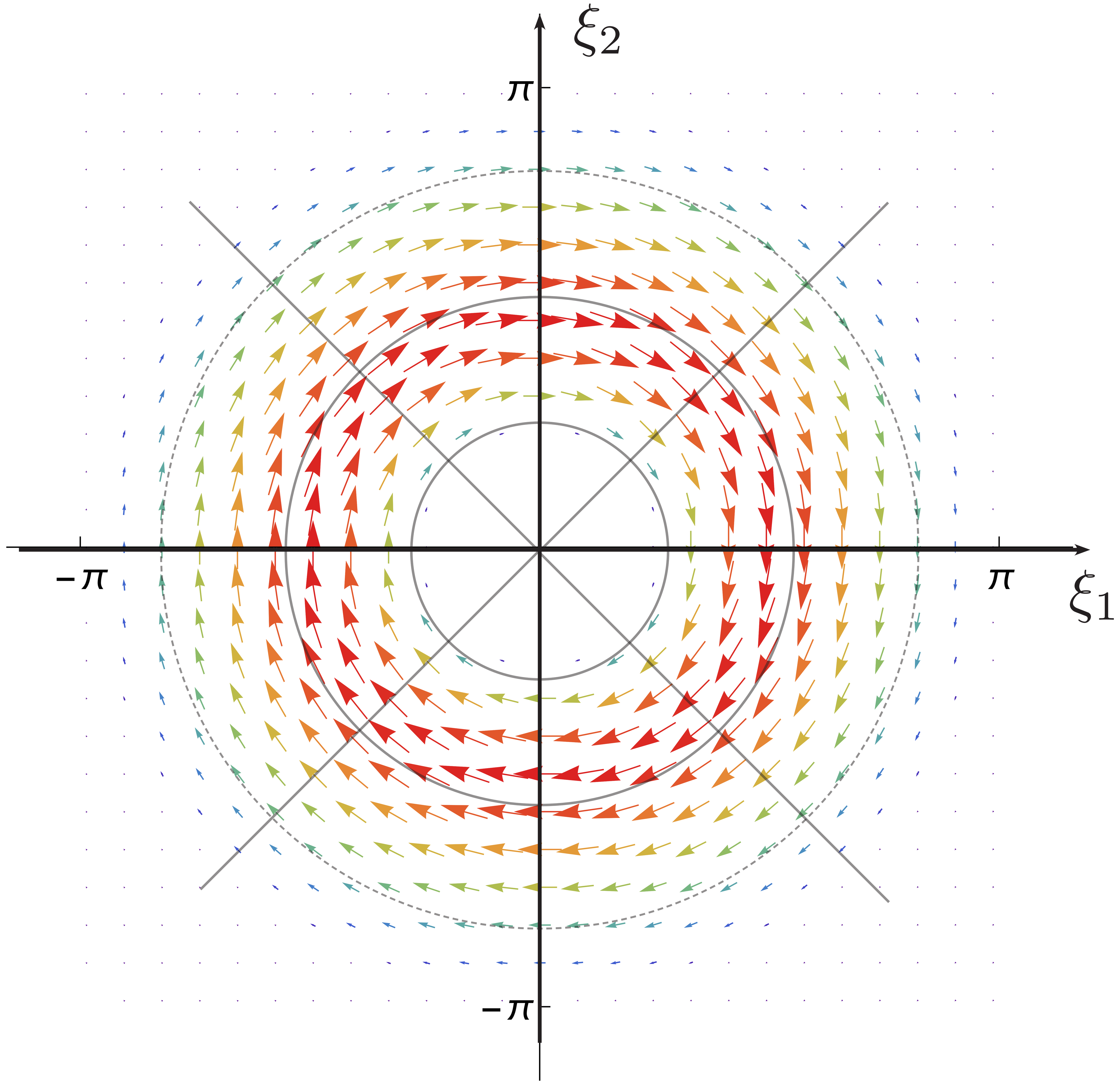

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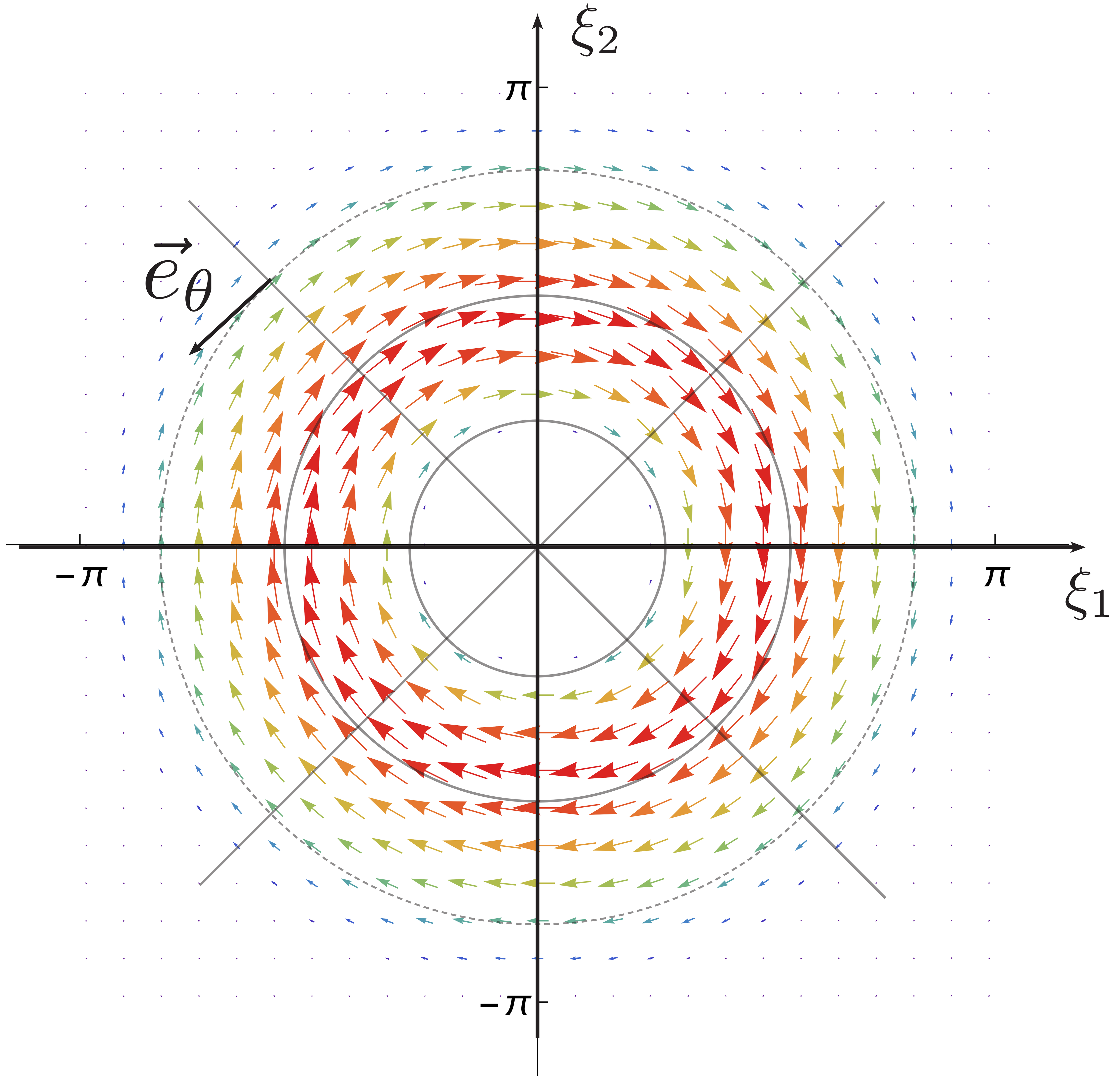

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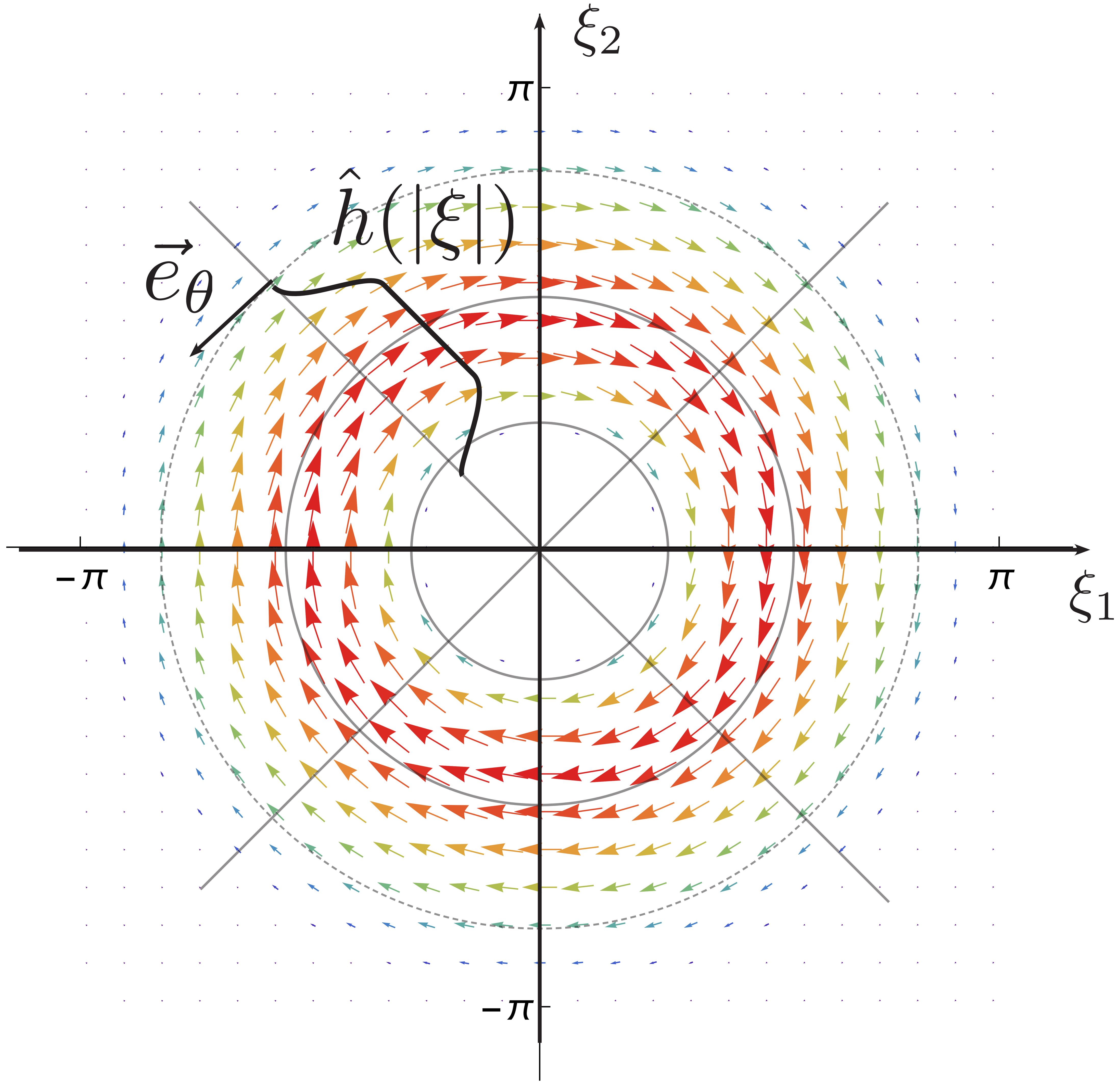


Divergence free wavelets

- Spatial representation:

$$\vec{\psi}(x) = -\frac{i}{2\pi} \int_{\mathbb{R}^2} h(|\xi|)\, \vec{e}_\theta\, e^{i\langle \xi , x \rangle}\, d\xi$$

Divergence free wavelets

- Spatial representation:

$$\vec{\psi}(x) = -\frac{i}{2\pi} \int_{\mathbb{R}^2} h(|\xi|) \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} e^{i\langle \xi , x \rangle} \, d\xi$$

Divergence free wavelets

- Spatial representation:

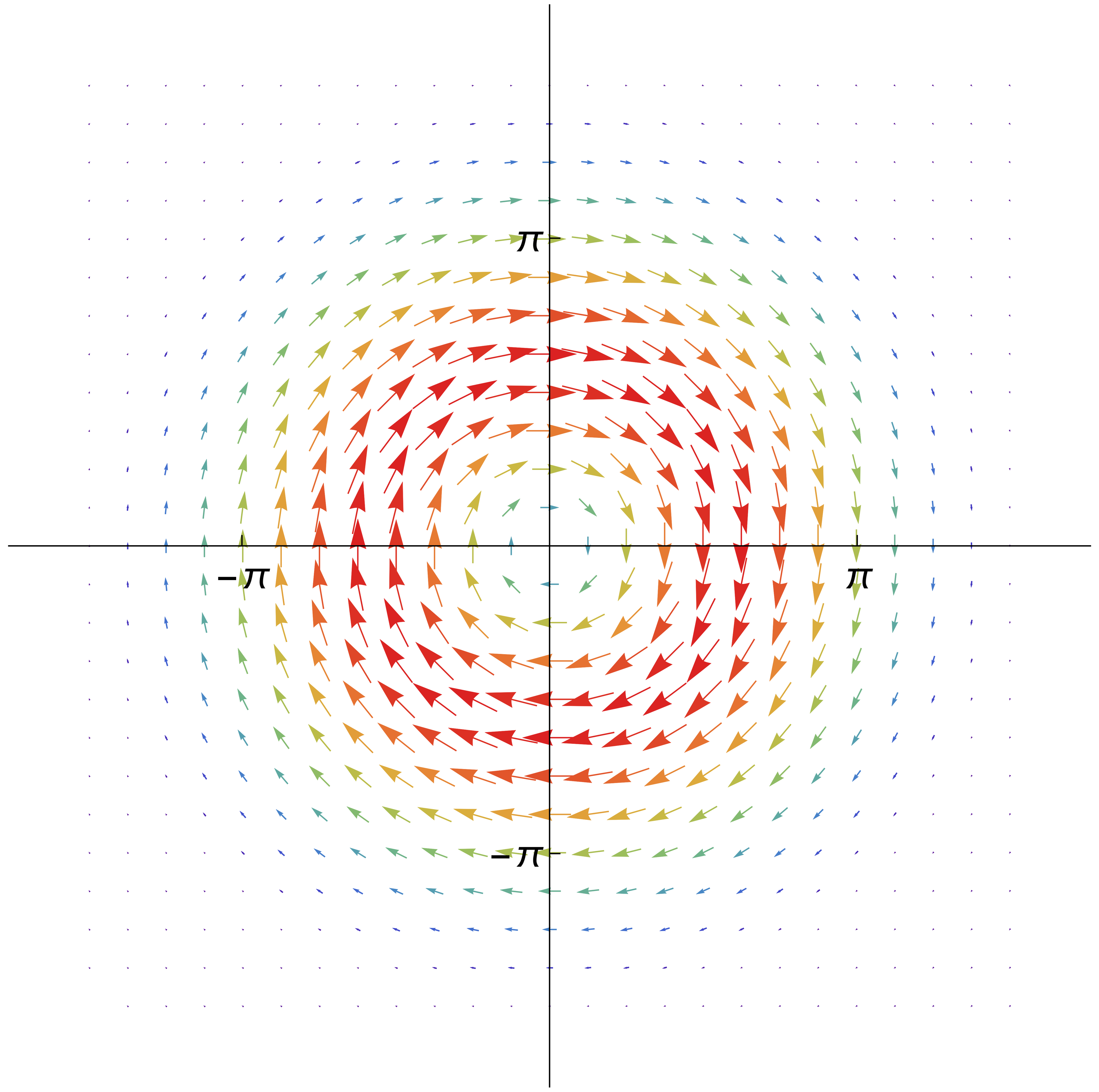
$$\vec{\psi}(x) = -\frac{i}{2\pi} \int_{\mathbb{R}^2} h(|\xi|) \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \underbrace{e^{i\langle \xi , x \rangle}} \, d\xi$$

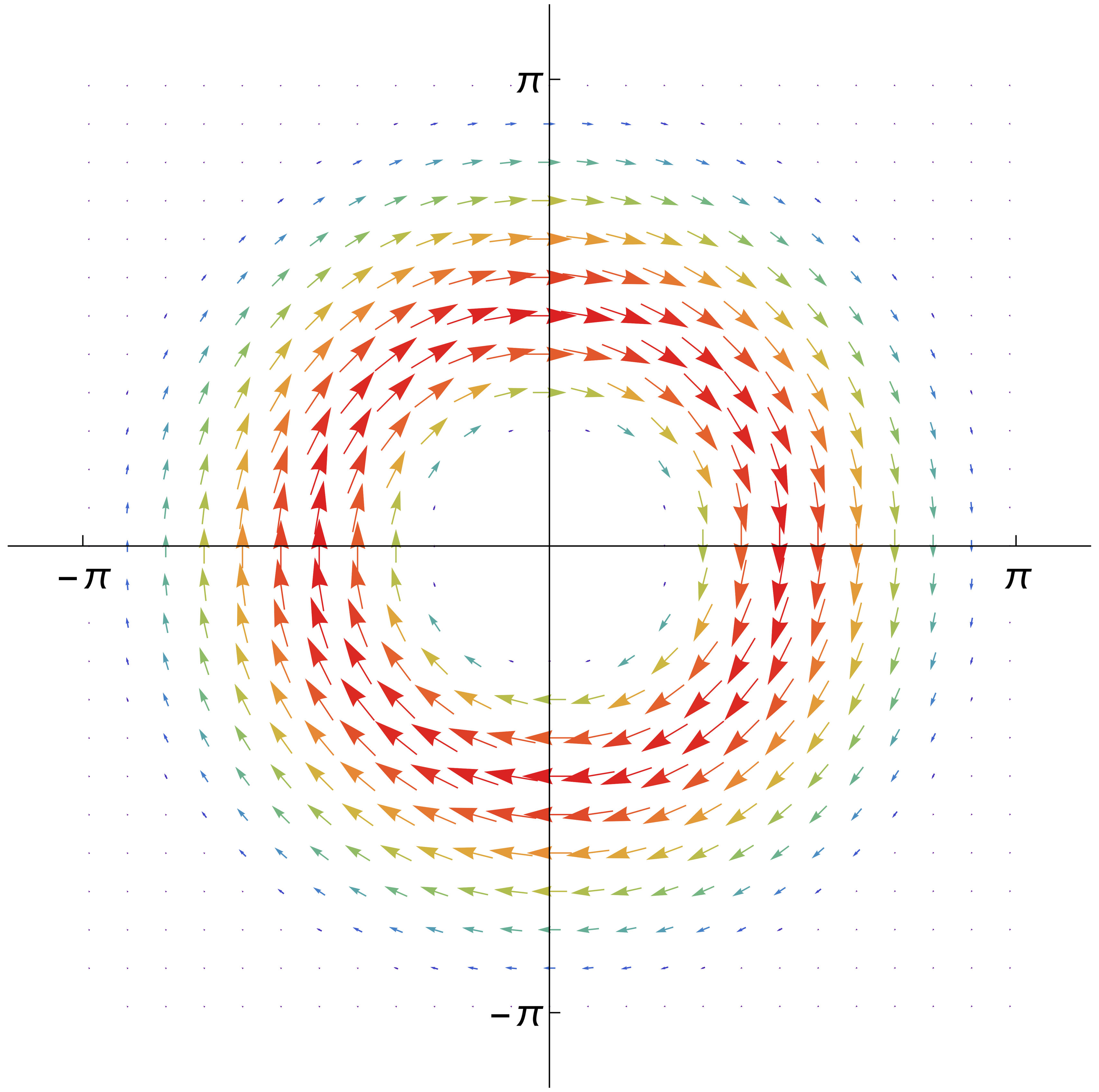
Jacobi-Anger formula: $$e^{i\langle \xi , x \rangle} = \sum_{m \in \mathbb{Z}} i^m e^{im(\theta_\xi - \theta_x)} J_m(|\xi| \, |x|)$$

Divergence free wavelets

$$\vec{\psi}(x) = h_1(|x|)\vec{e}_\theta$$

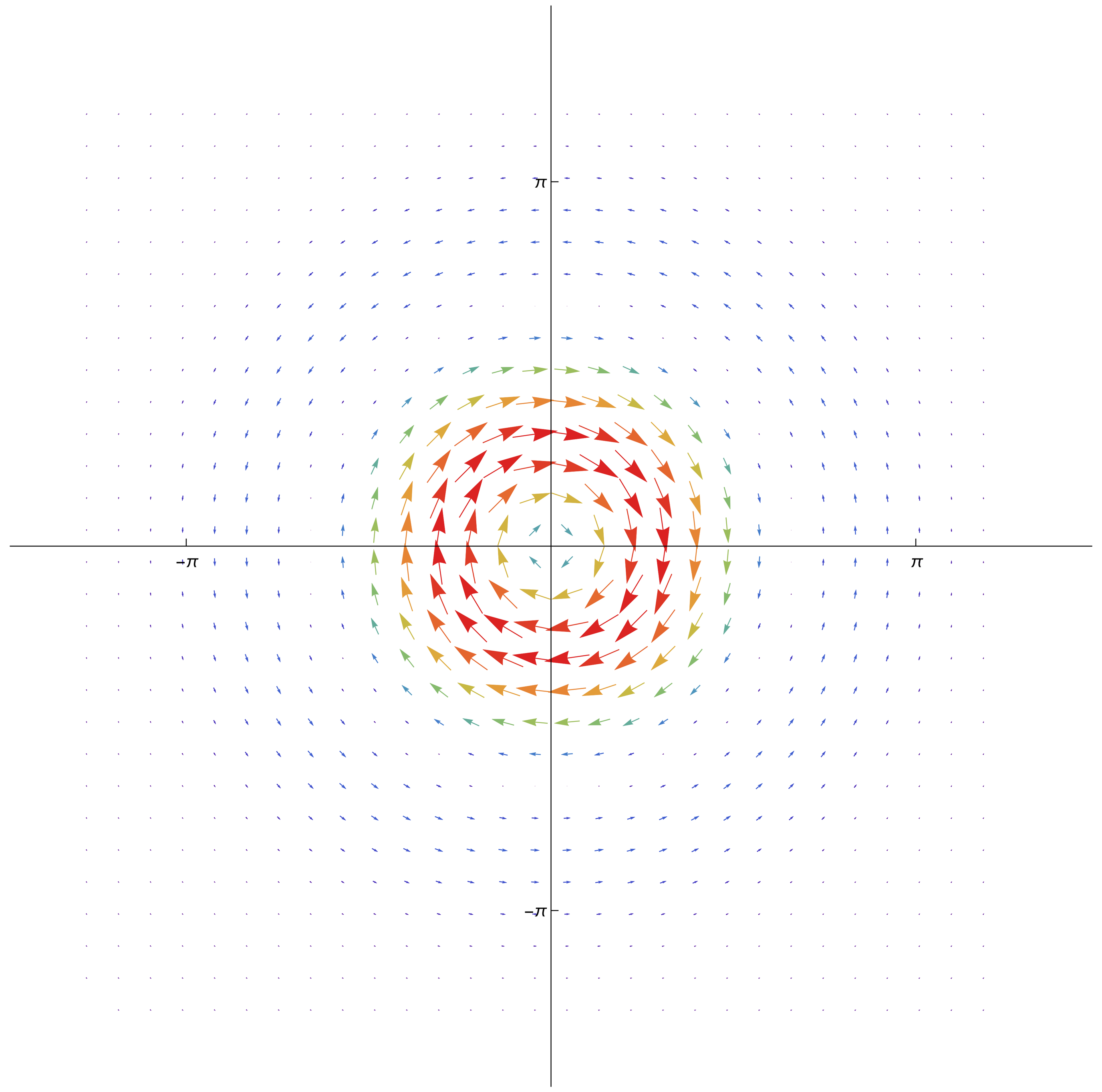
$$\hat{\vec{\psi}}(\xi) = -i\,h(|\xi|)\,\vec{e}_\theta$$





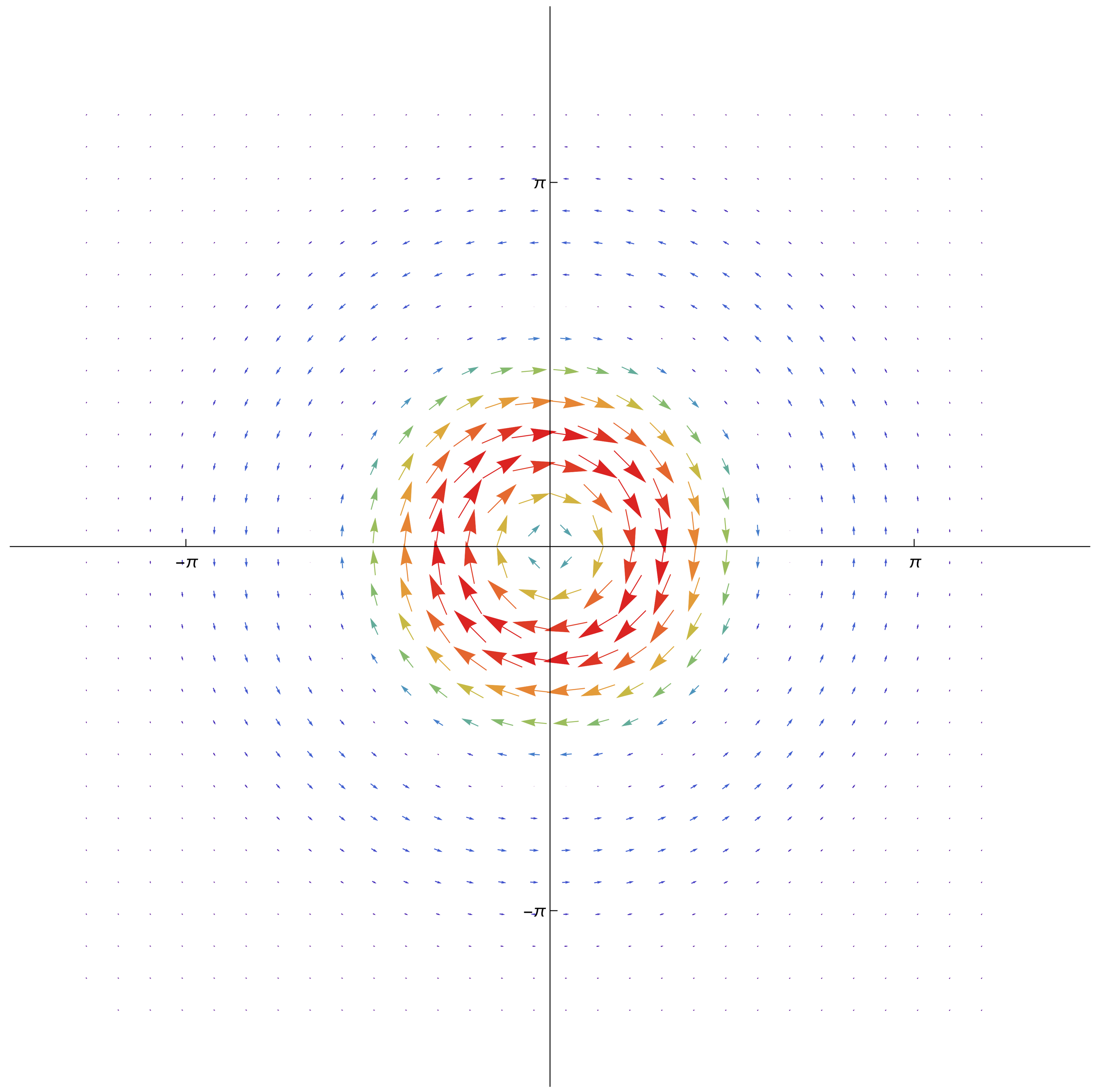
Divergence free wavelets

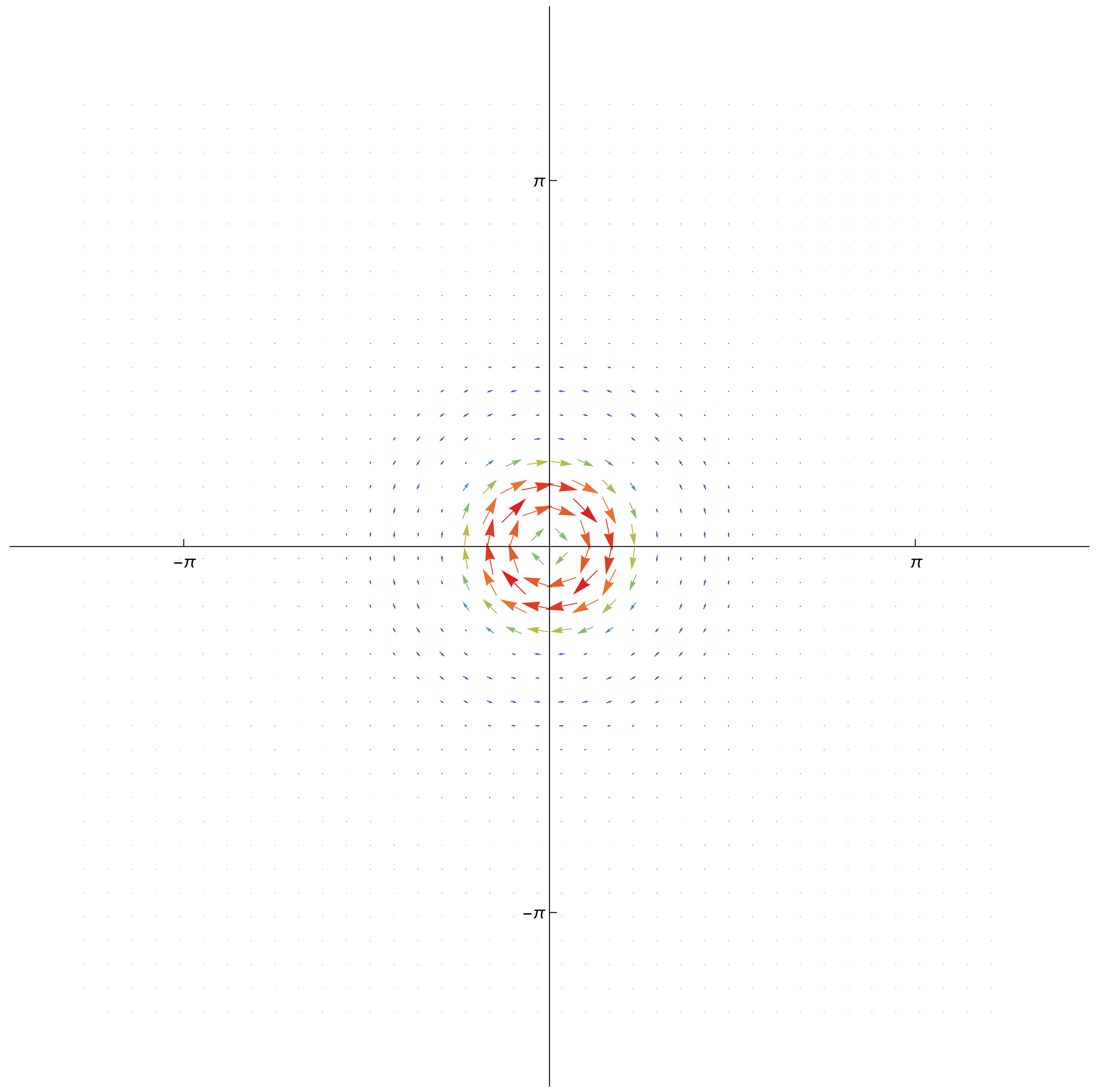
$$\vec{\psi}_0(x) = \vec{\psi}(2^0 x)$$



Divergence free wavelets

$$\vec{\psi}_0(x) = \vec{\psi}(2^0 x) \qquad \vec{\psi}_1(x) = \vec{\psi}(2^1 x)$$





Divergence free wavelets

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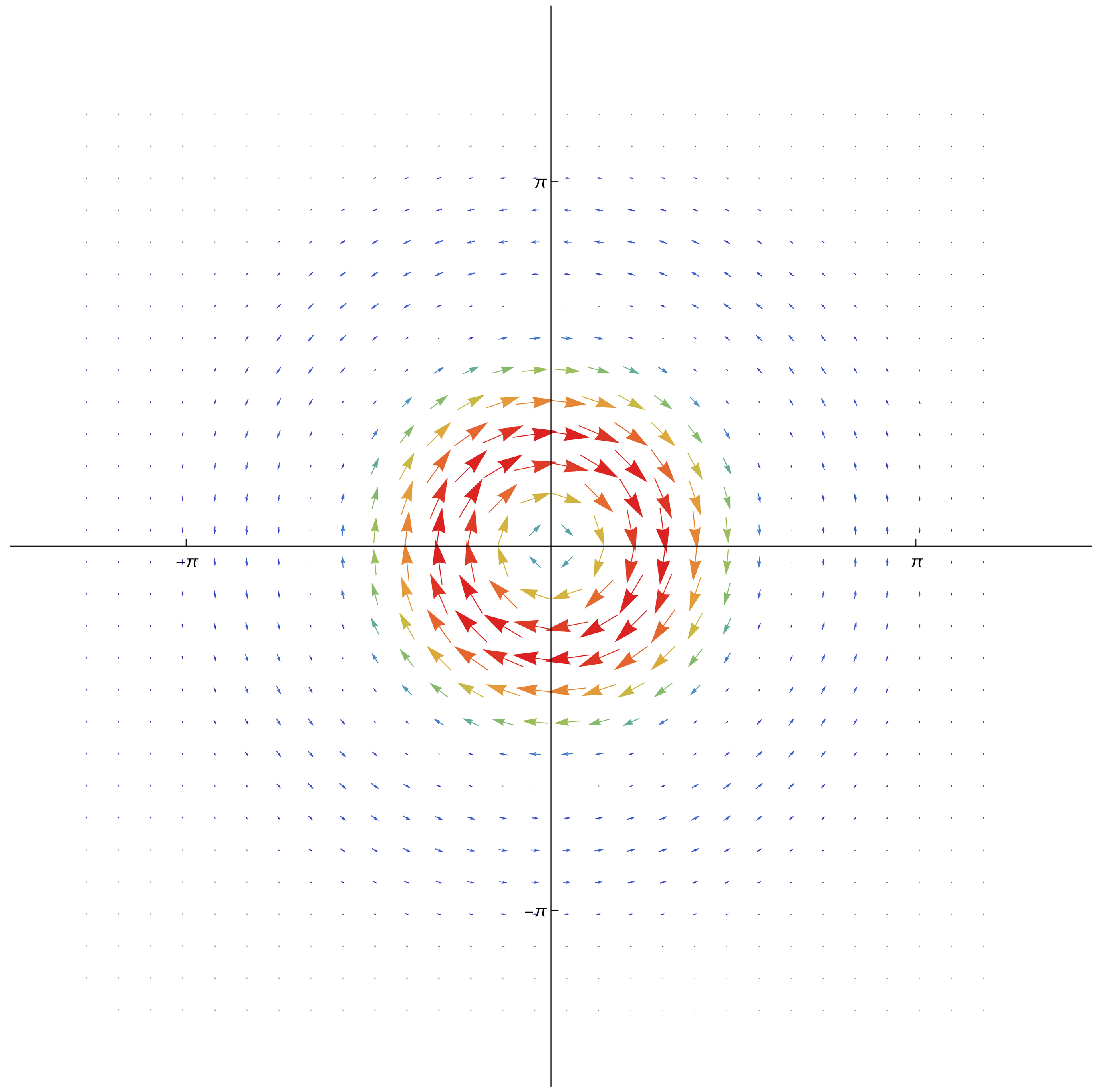
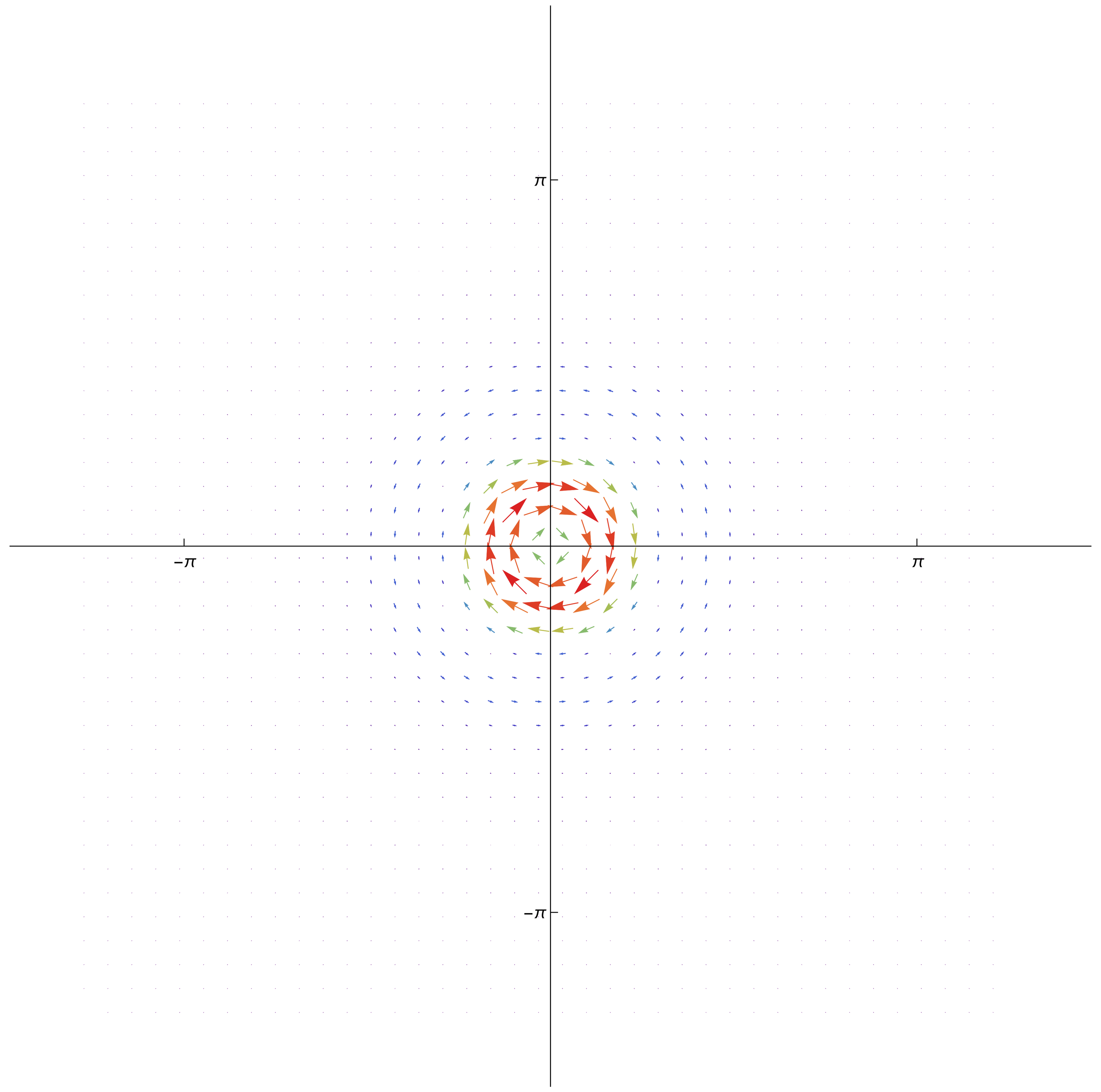
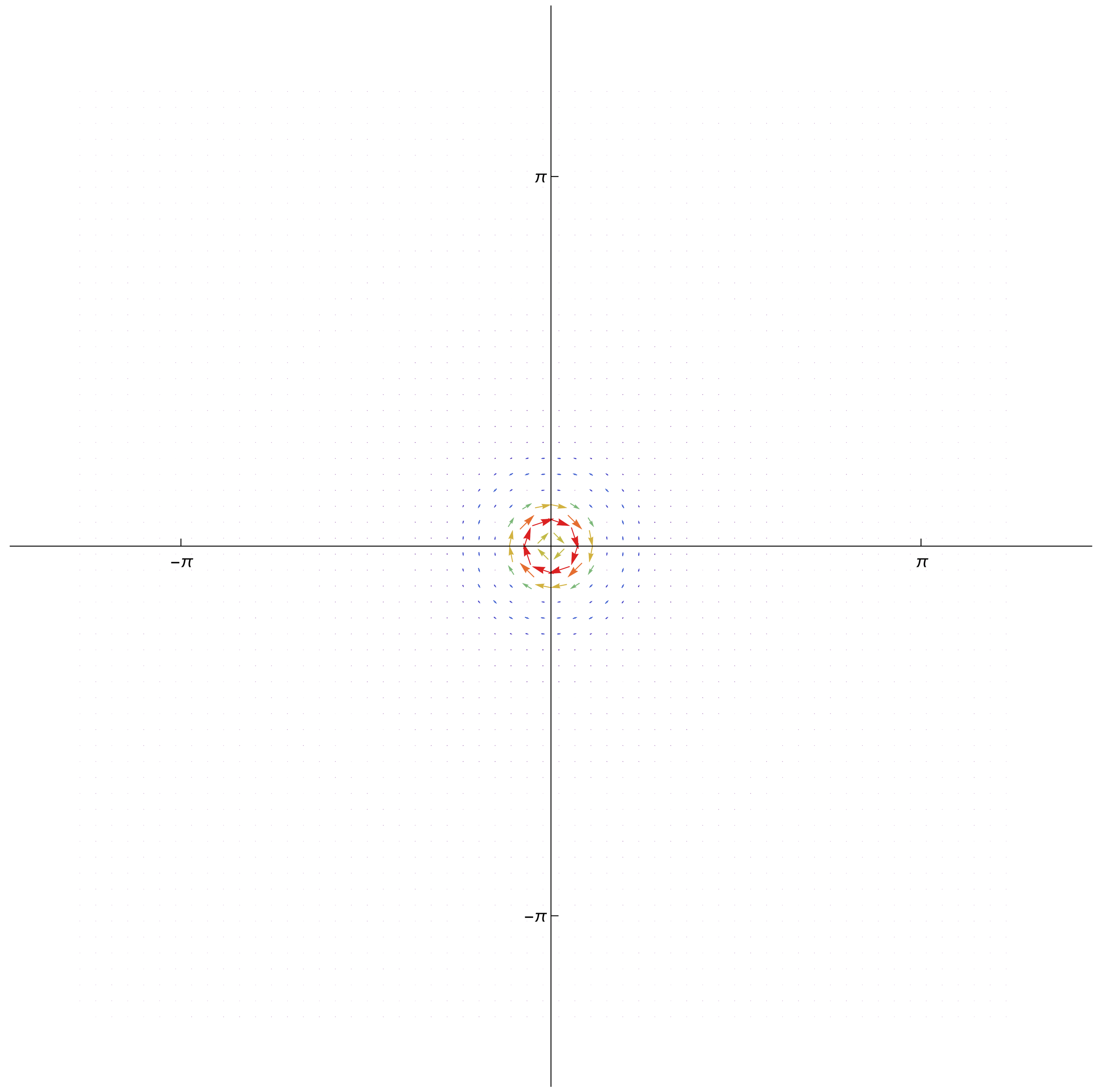


$\vec{\psi}_1(x) = \vec{\psi}(2^1 x)$



$\vec{\psi}_2(x) = \vec{\psi}(2^2 x)$



…

Divergence free wavelets

Lemma: $\vec{u} \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^2)$

$$\vec{u}(x) = \sum_{j=-1}^{\infty} \sum_{k \in \mathbb{Z}^2} \left\langle \vec{u}(y) \, , \, \vec{\psi}_{jk}(y) \right\rangle \vec{\psi}_{jk}(x)$$

$$\vec{\psi}_{jk}(x) = \frac{2^j}{2\pi} \vec{\psi}\left(2^j x - k\right)$$

Divergence free wavelets

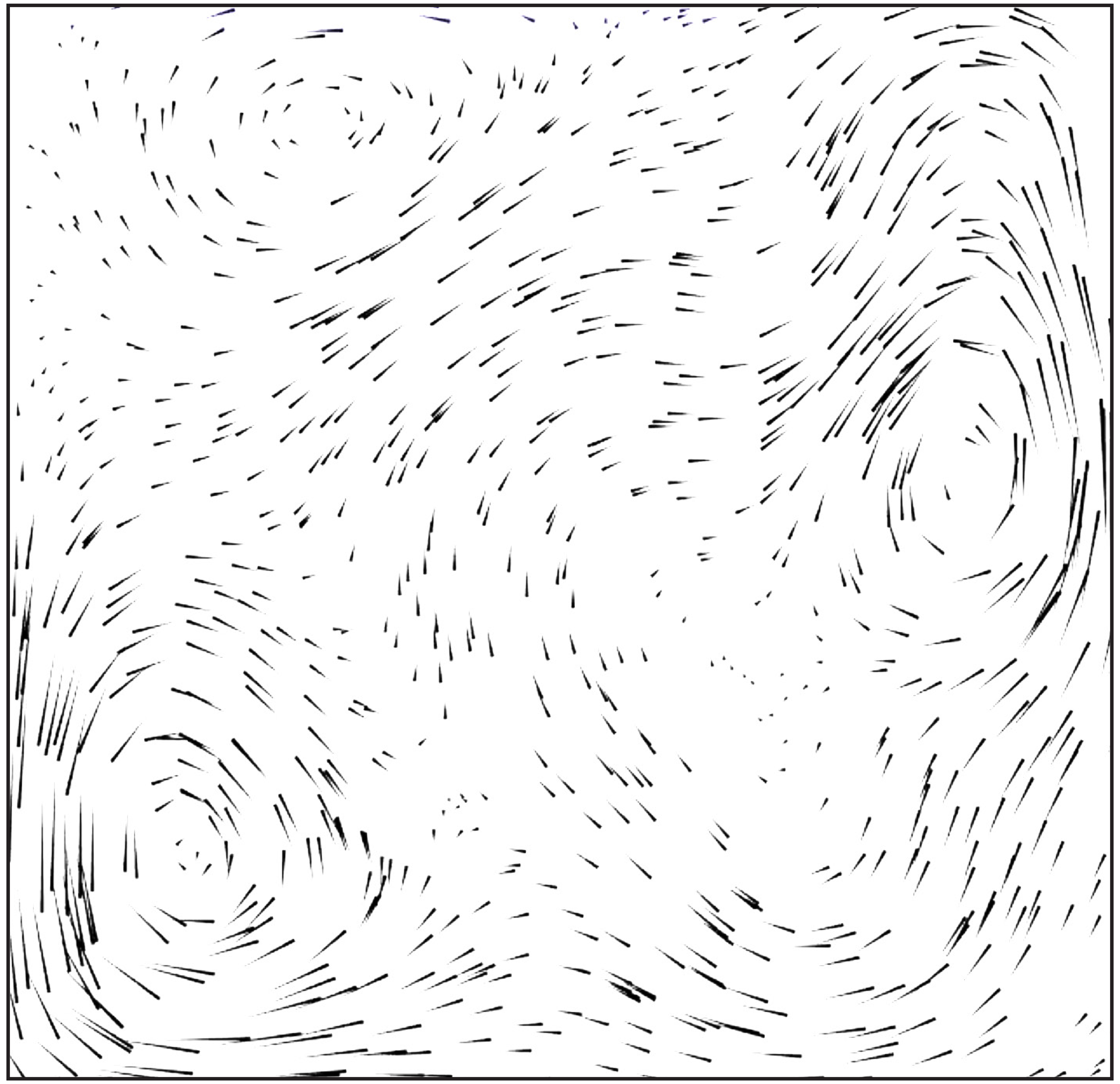
Lemma: $\vec{u} \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^2)$

Isolated vortices of different sizes on finer and finer grids

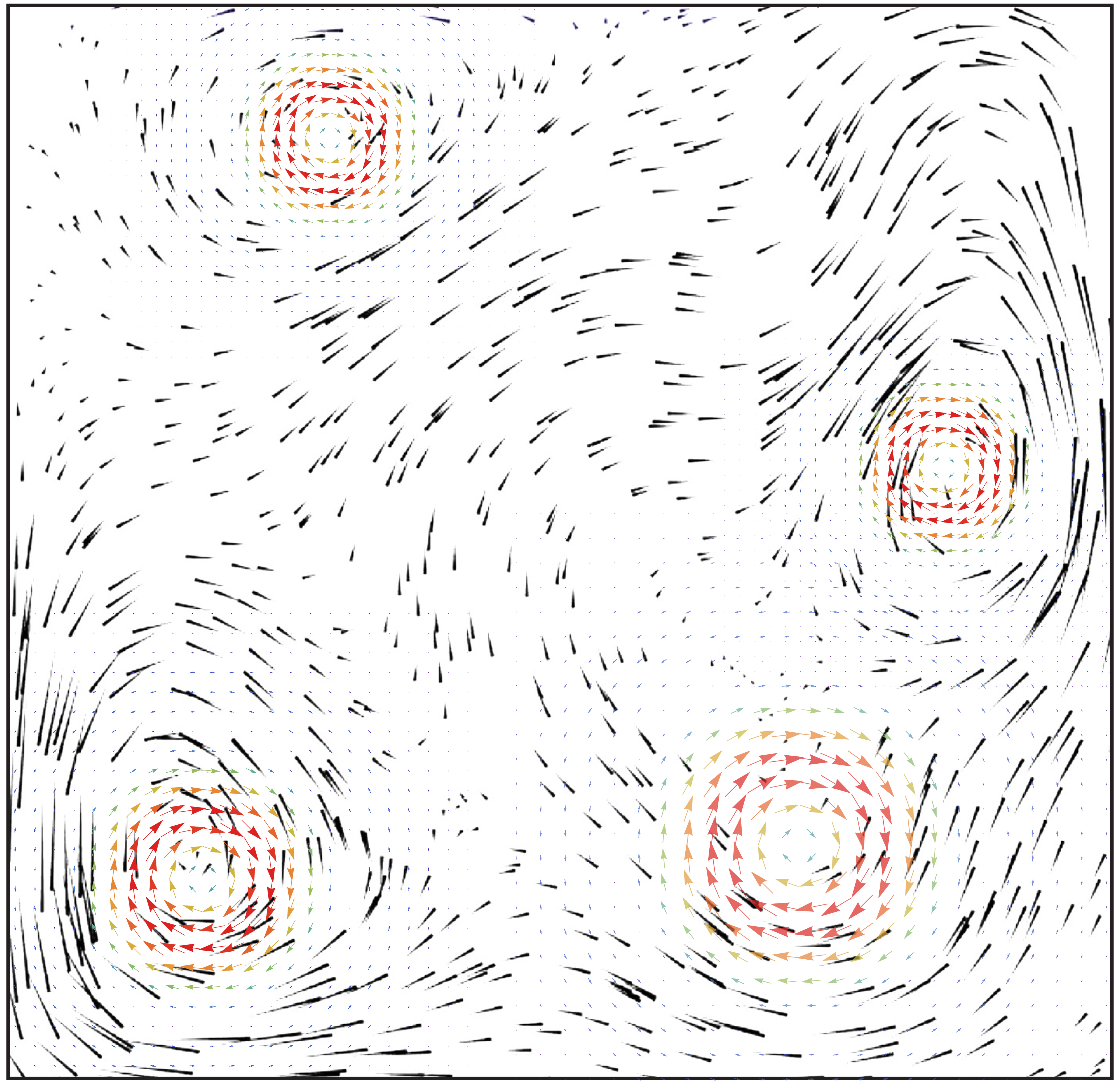
$$\vec{u}(x) = \sum_{j=-1}^{\infty} \sum_{k \in \mathbb{Z}^2} \left\langle \vec{u}(y) \, , \boxed{\vec{\psi}_{jk}(y)} \right\rangle \vec{\psi}_{jk}(x)$$

$$\vec{\psi}_{jk}(x) = \frac{2^j}{2\pi} \vec{\psi}\left(2^j x - k\right)$$

Divergence free wavelets

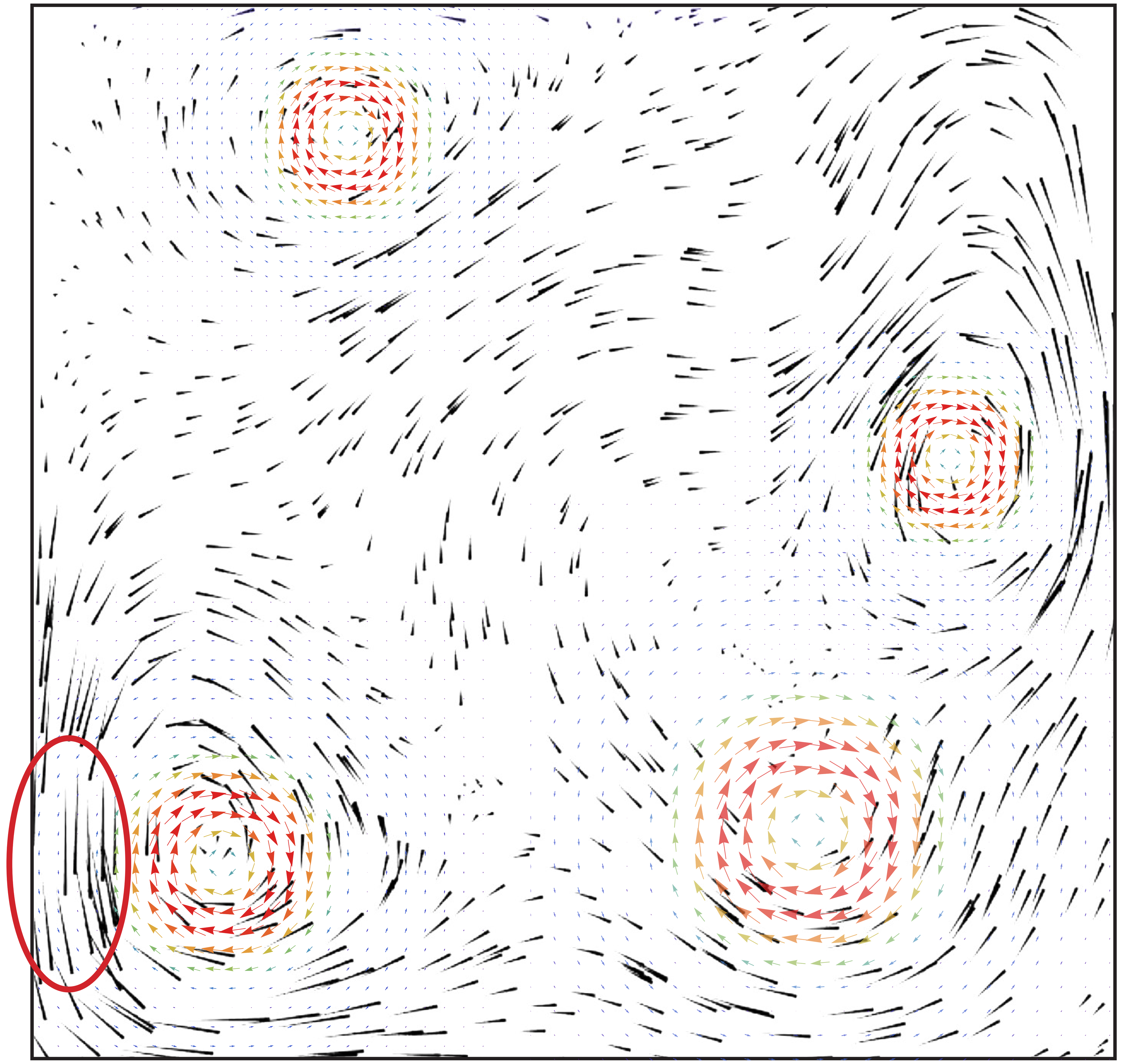


Divergence free wavelets



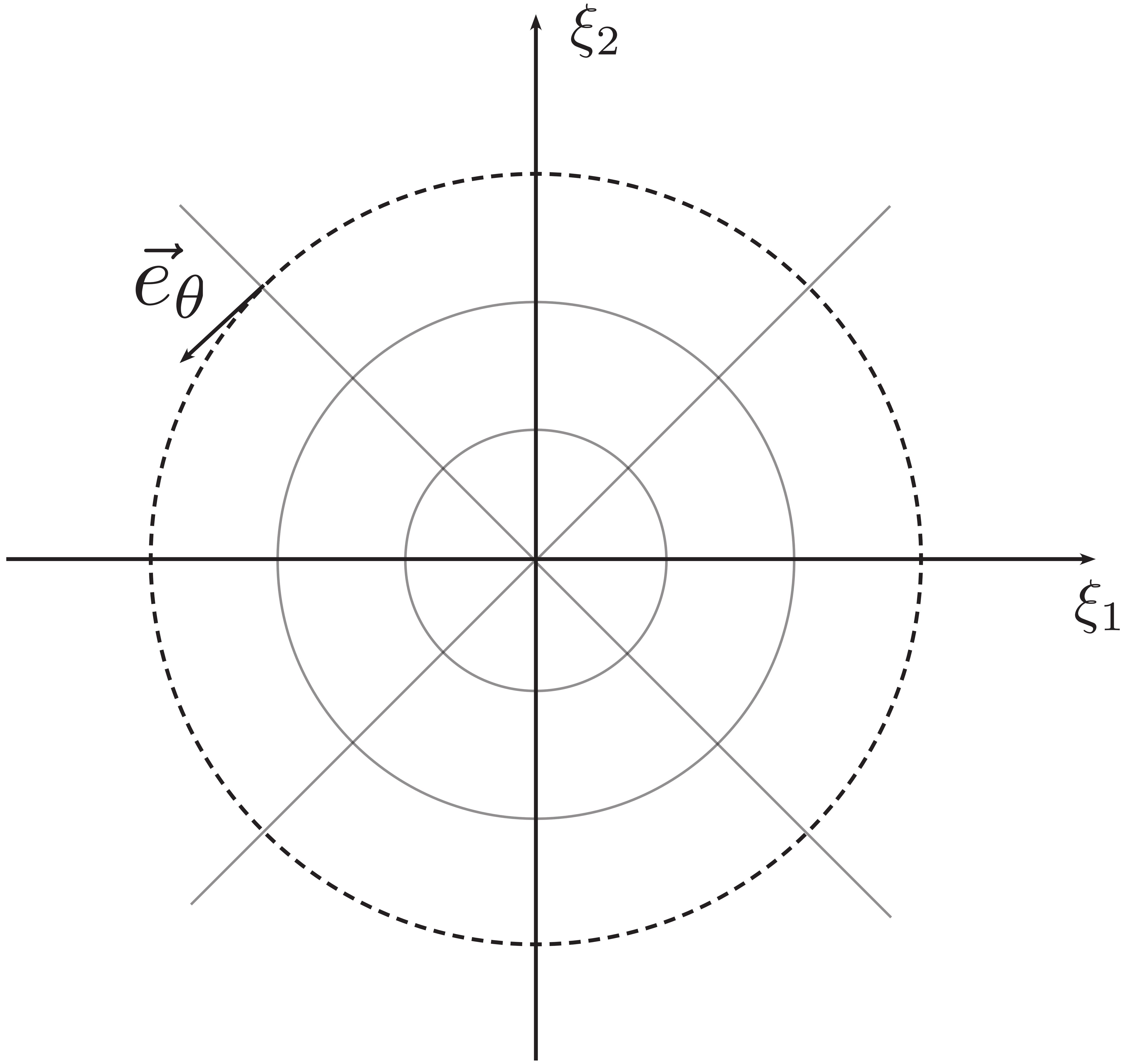
Divergence free wavelets

Isotropic vortices are not well suited for boundary flows



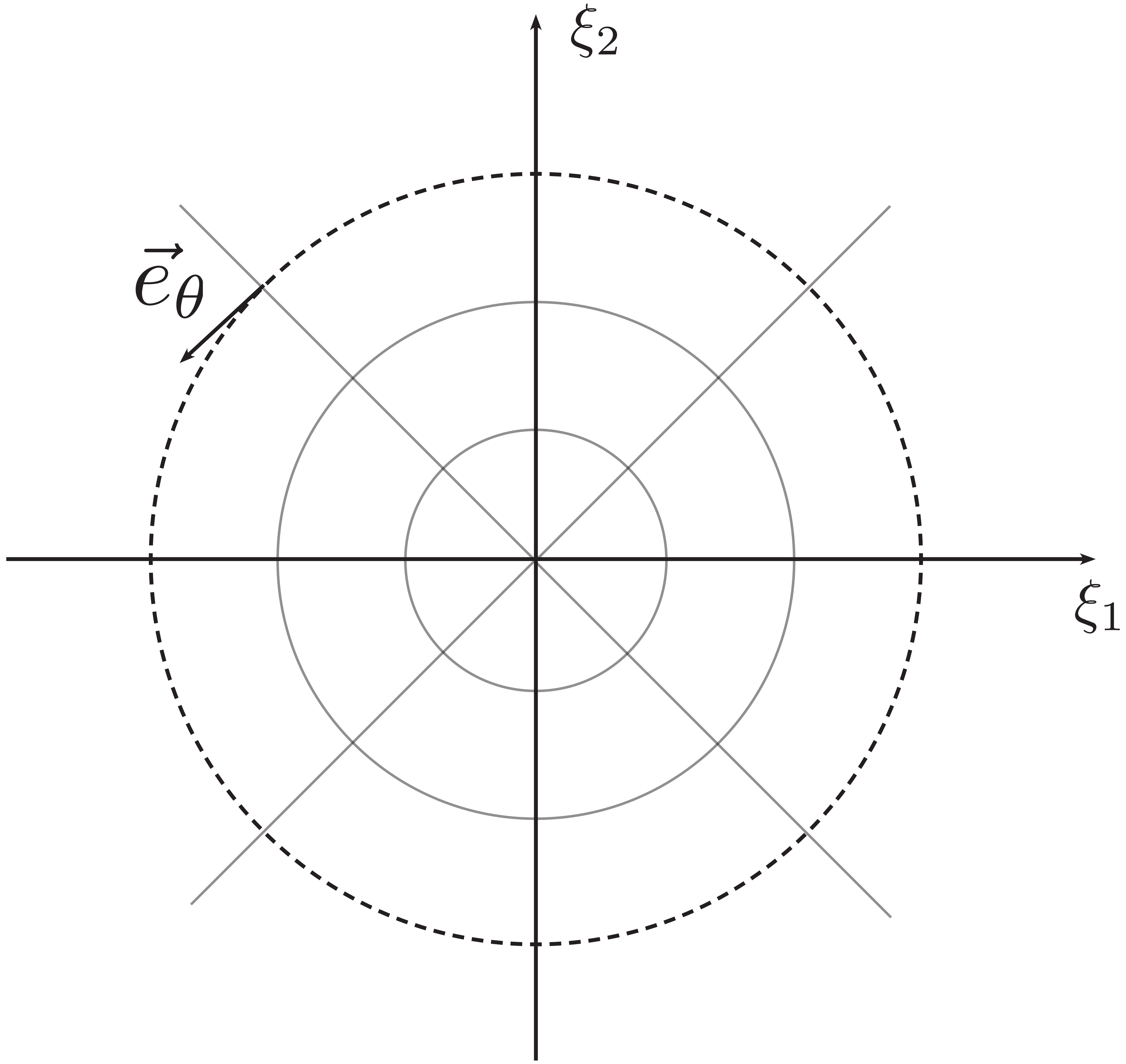
Divergence free wavelets

$$\hat{\vec{\psi}}(\xi) = \quad \textbf{...} \quad \vec{e}_{\theta}$$



Divergence free wavelets

$$\hat{\vec{\psi}}(\xi) = \underbrace{\quad \ldots \quad}_{\text{window function compatible with polar coordinates}} \vec{e}_\theta$$



Divergence free wavelets

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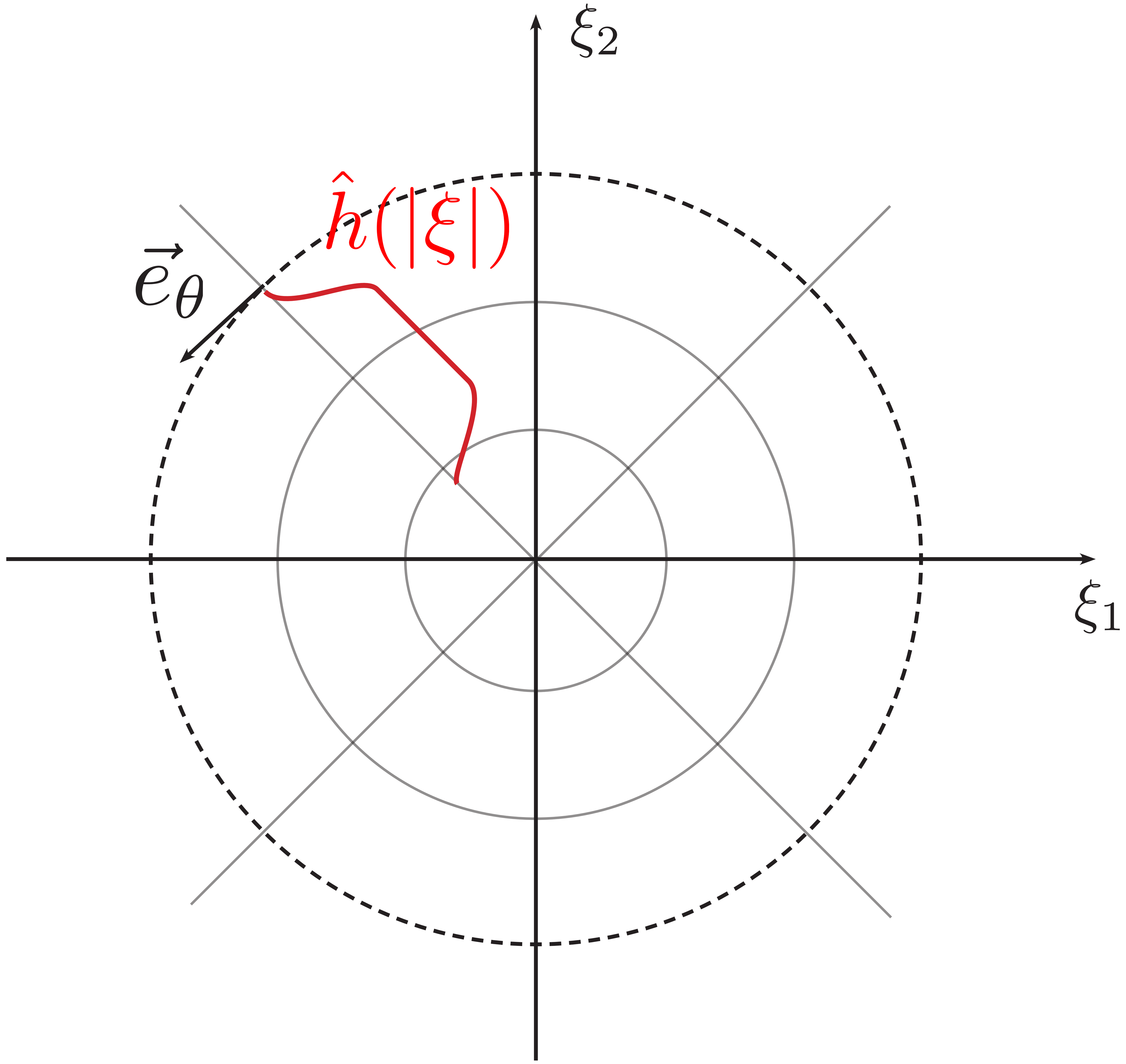

Divergence free wavelets

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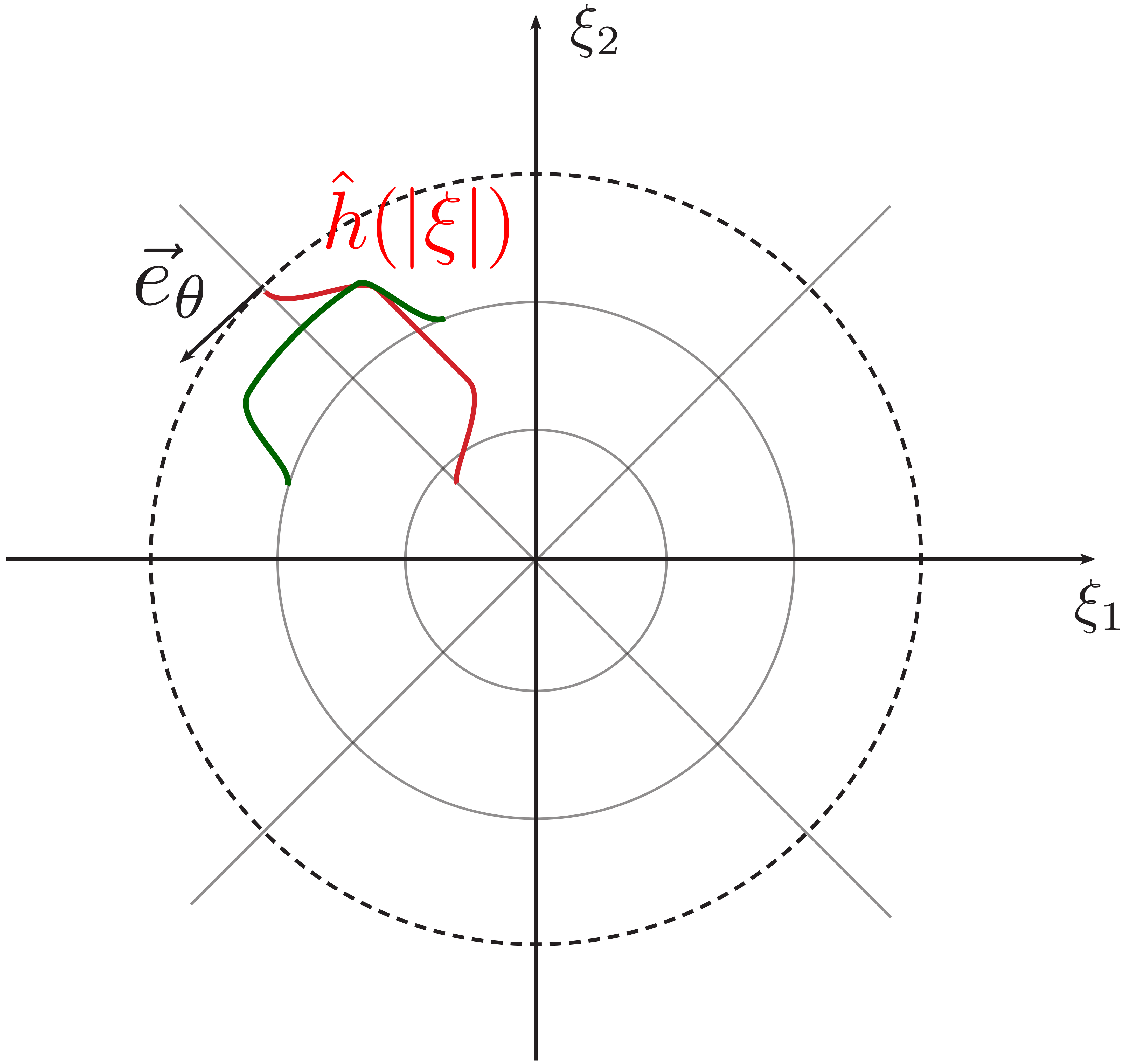
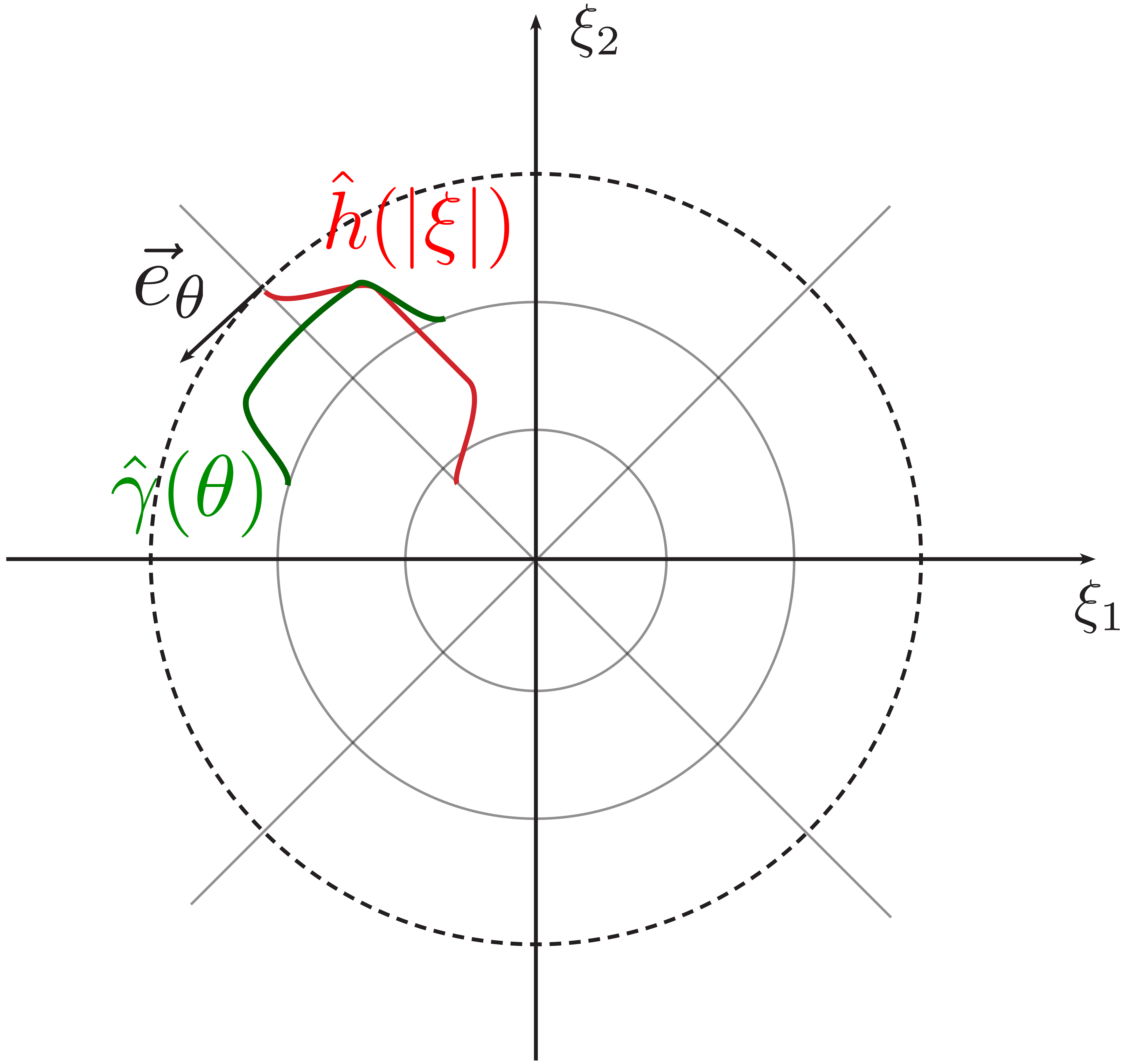


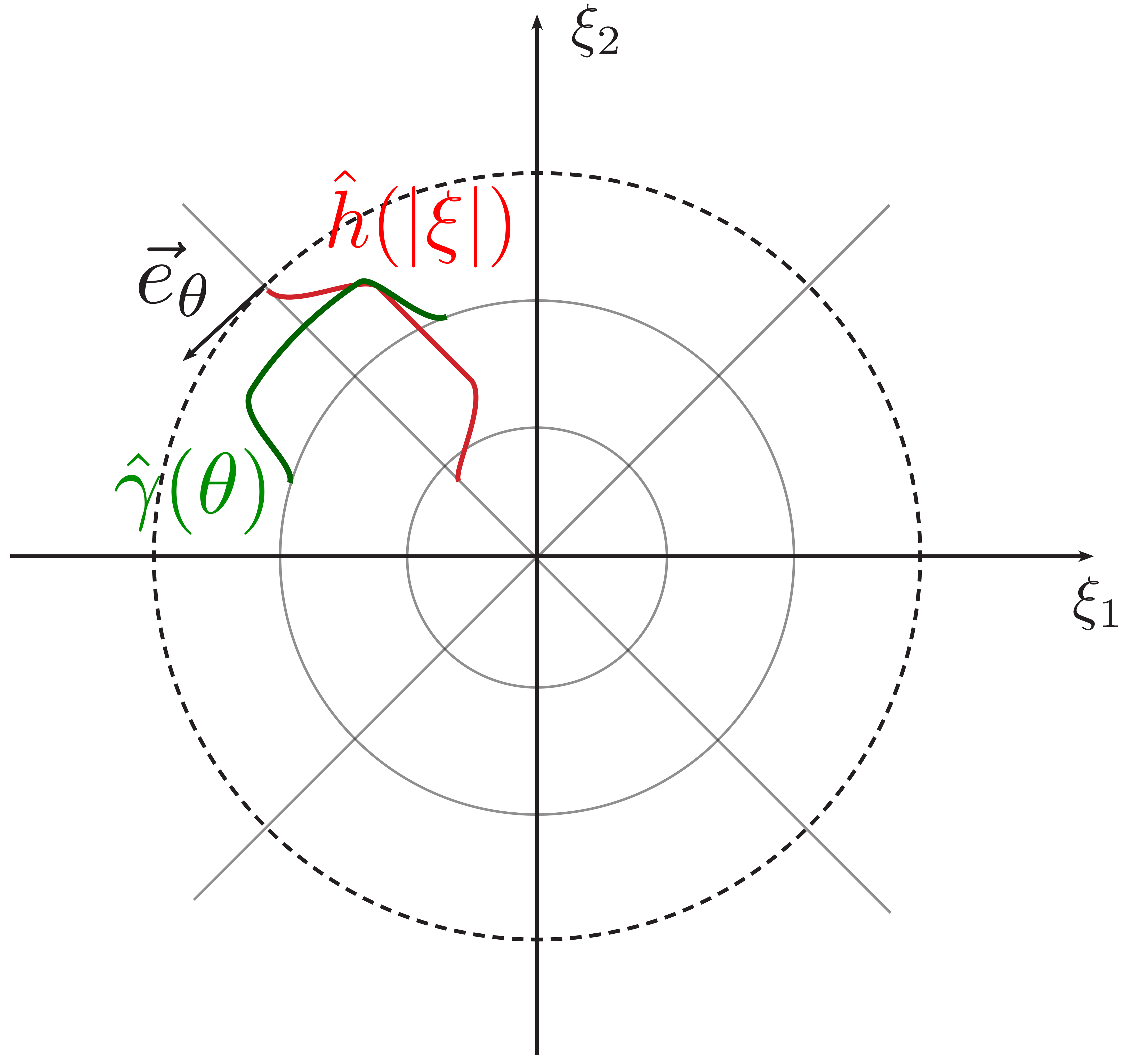
Divergence free wavelets

$$\hat{\vec{\psi}}(\xi) = -i\,\hat{\gamma}(\theta)\,\hat{h}(|\xi|)\,\vec{e}_\theta$$



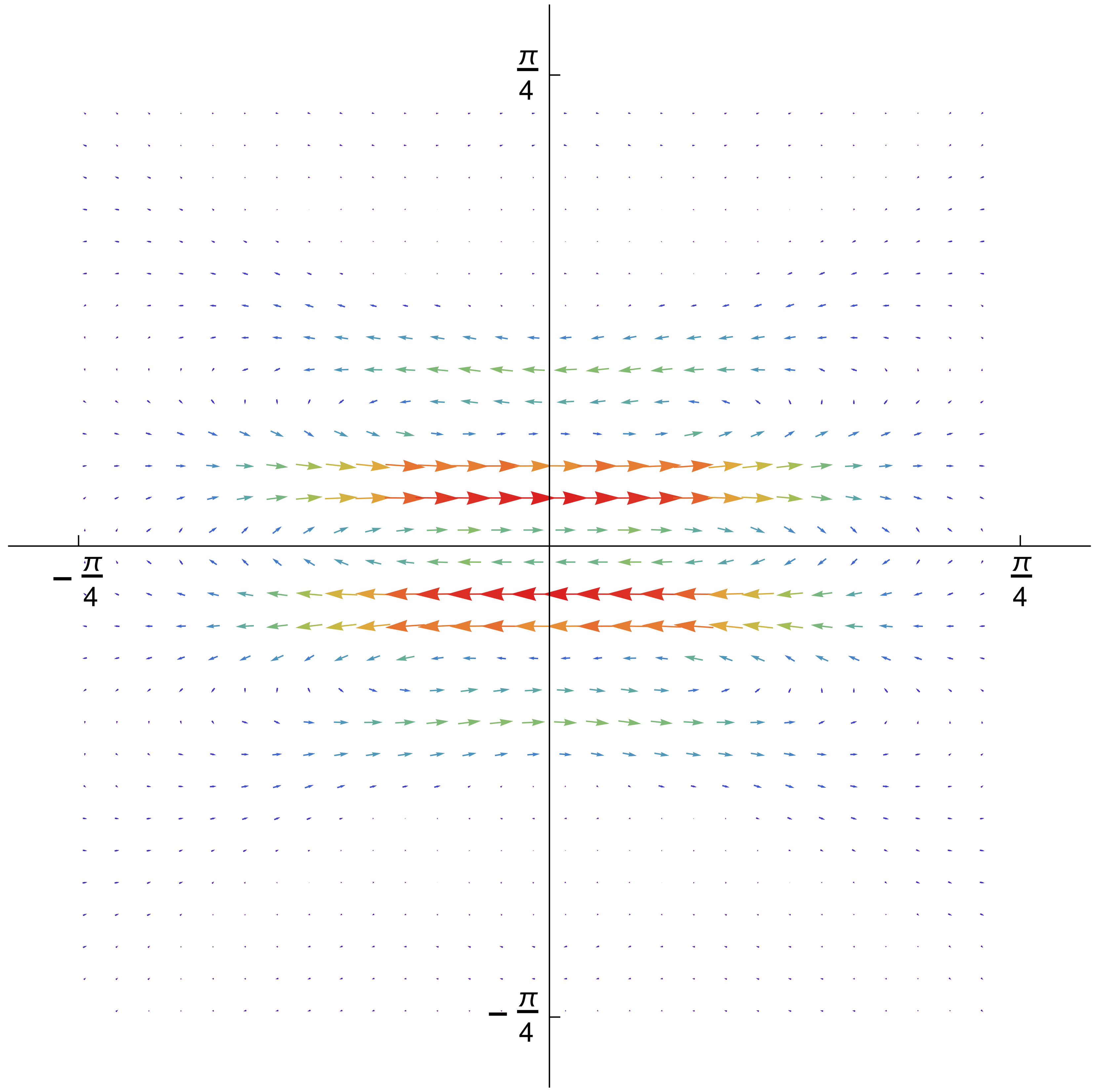
Divergence free wavelets

$$\hat{\vec{\psi}}(\xi) = -i\,\underbrace{\hat{\gamma}(\theta)}_{\hat{\gamma}(\theta) = \sum_m \beta_m\, e^{im\theta}}\,\hat{h}(|\xi|)\,\vec{e}_\theta$$

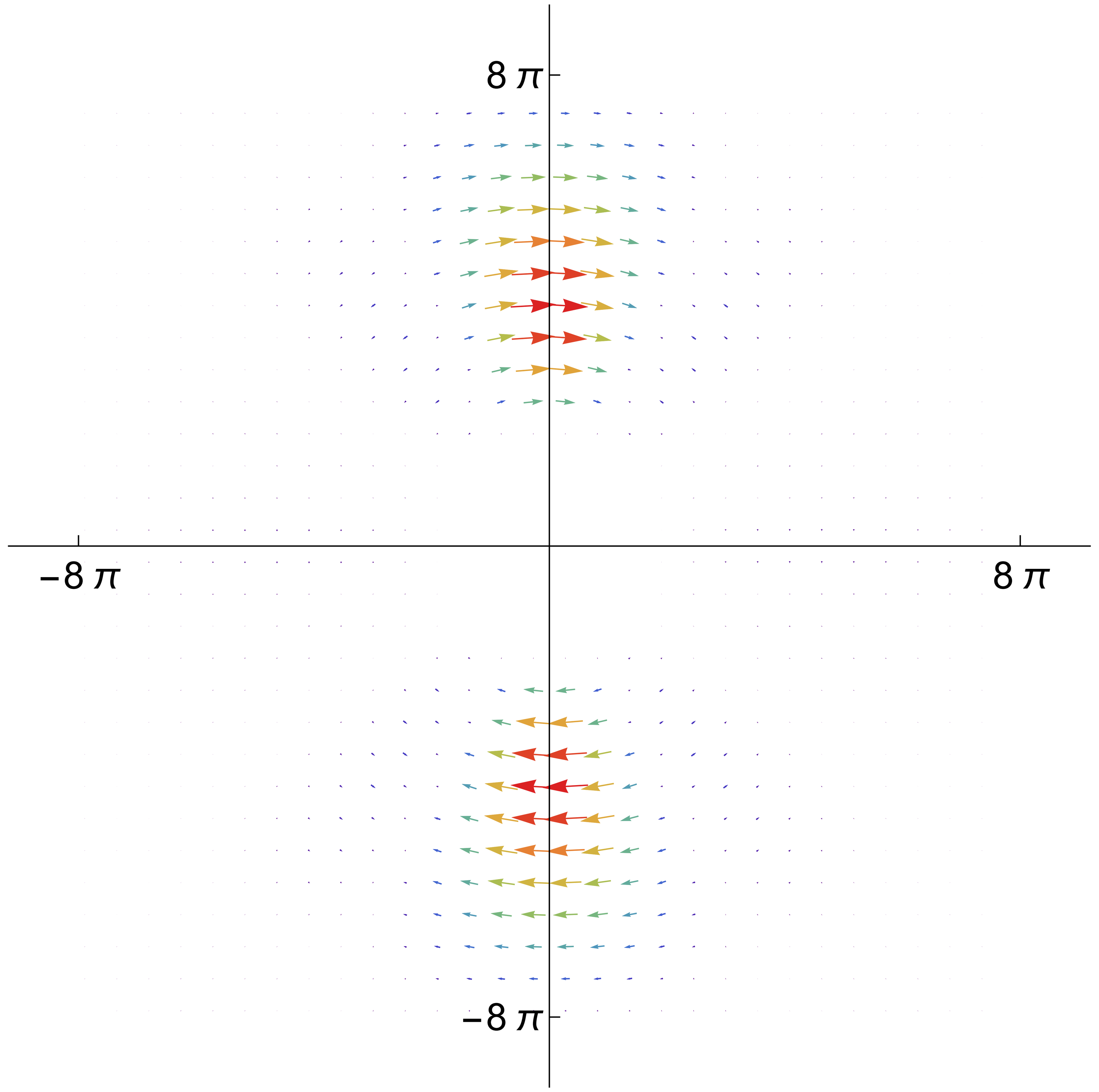


Divergence free wavelets

$$\vec{\psi}(x)$$



$$\hat{\vec{\psi}}(\xi)$$



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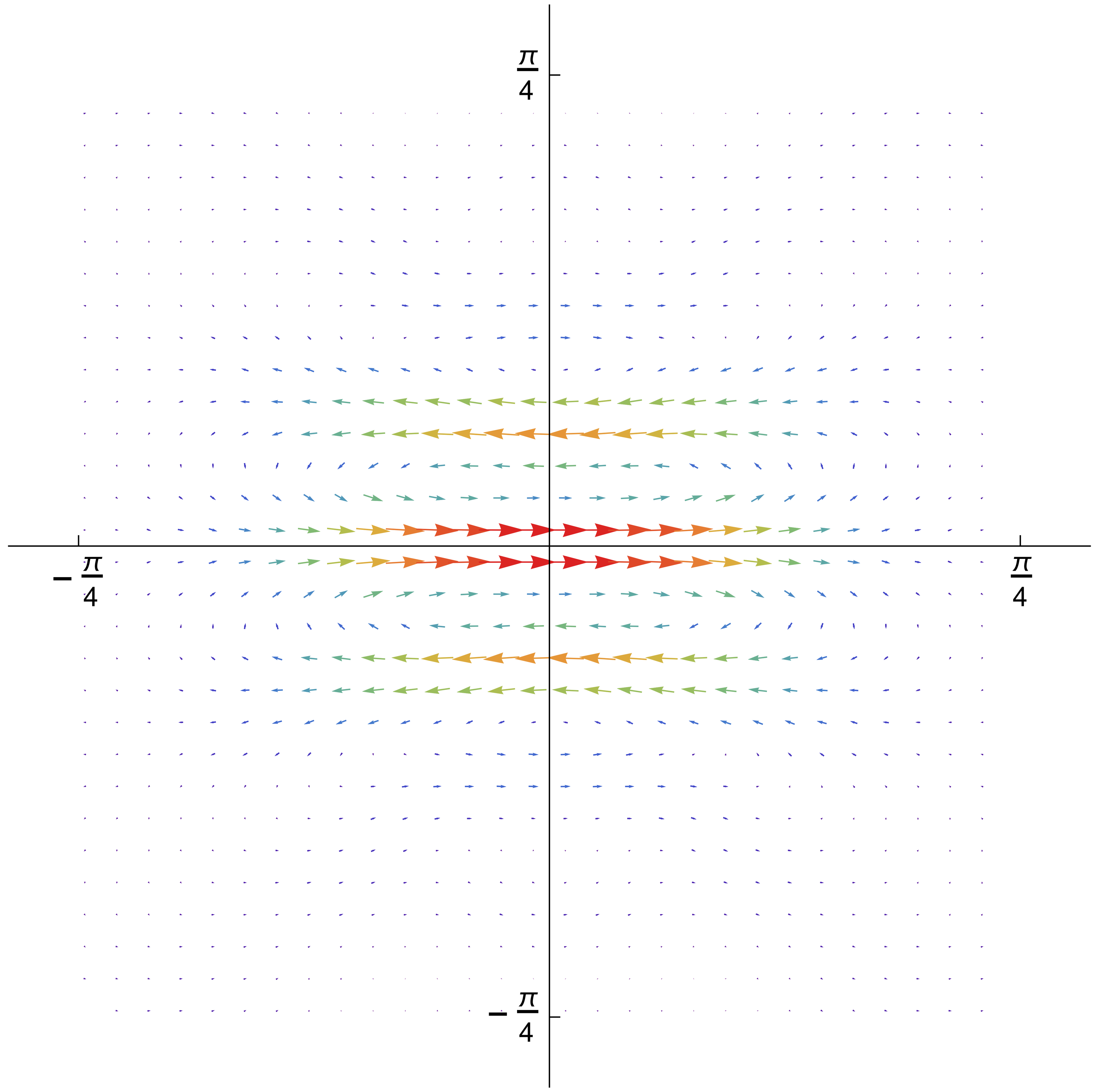
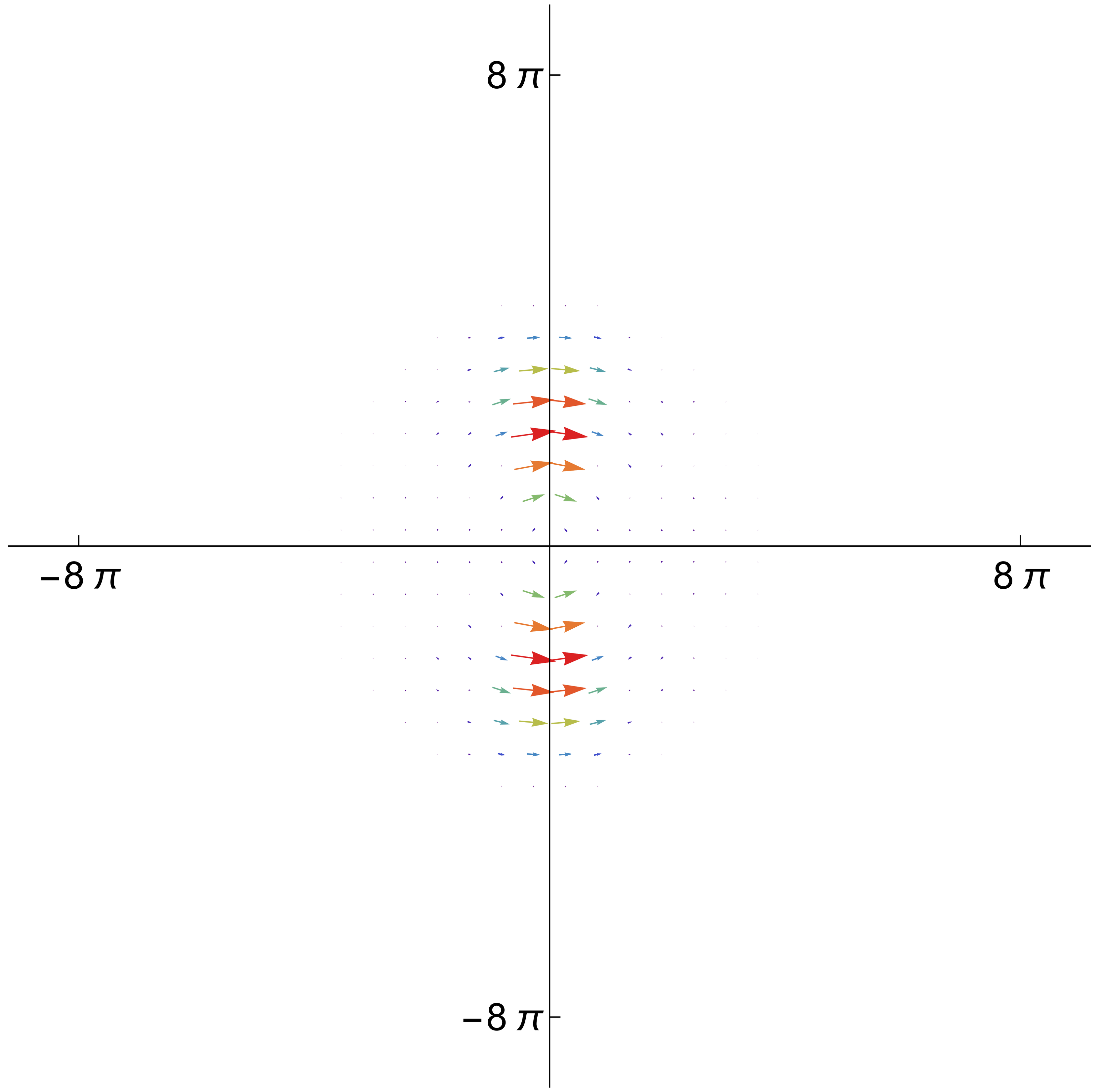
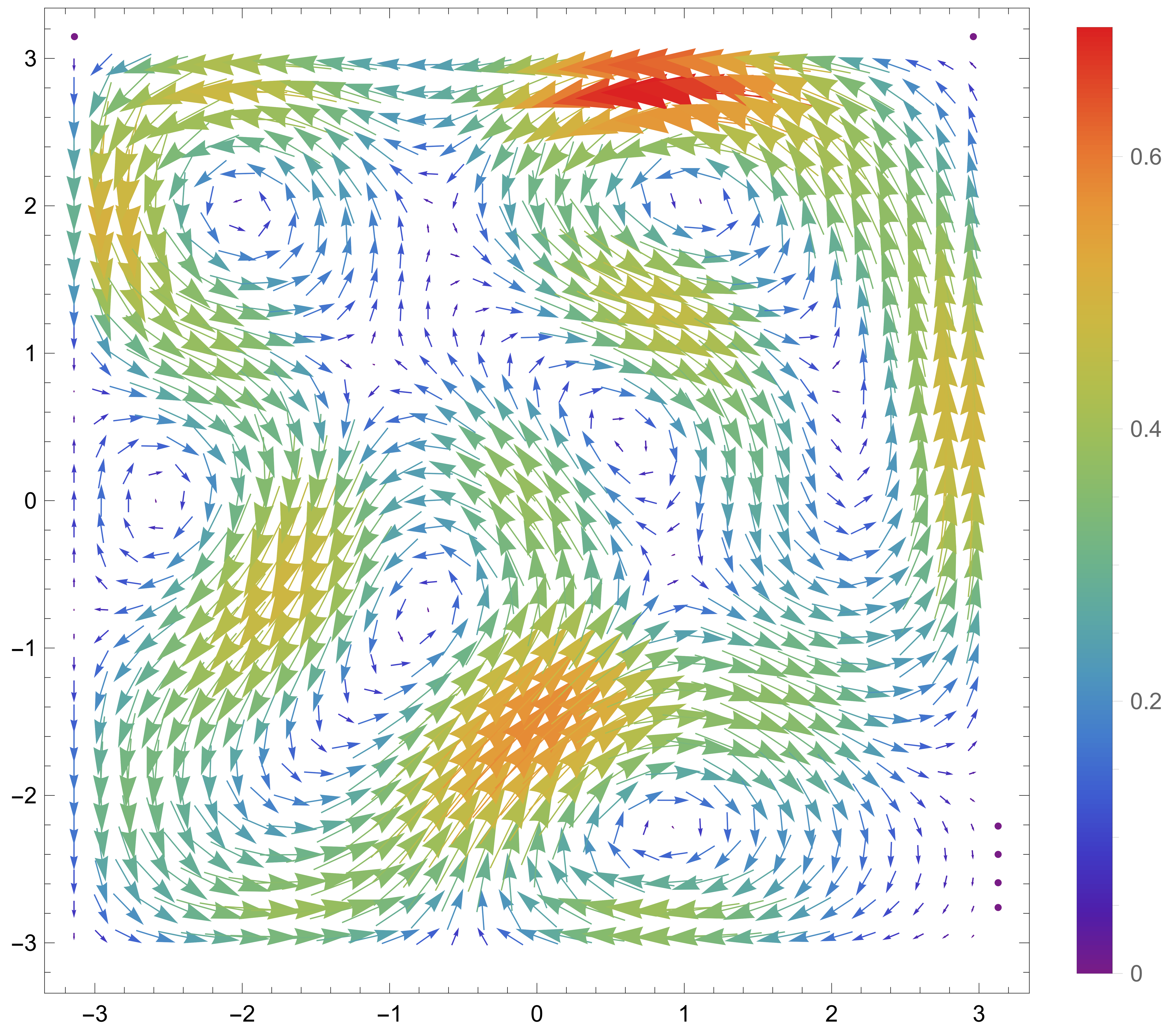


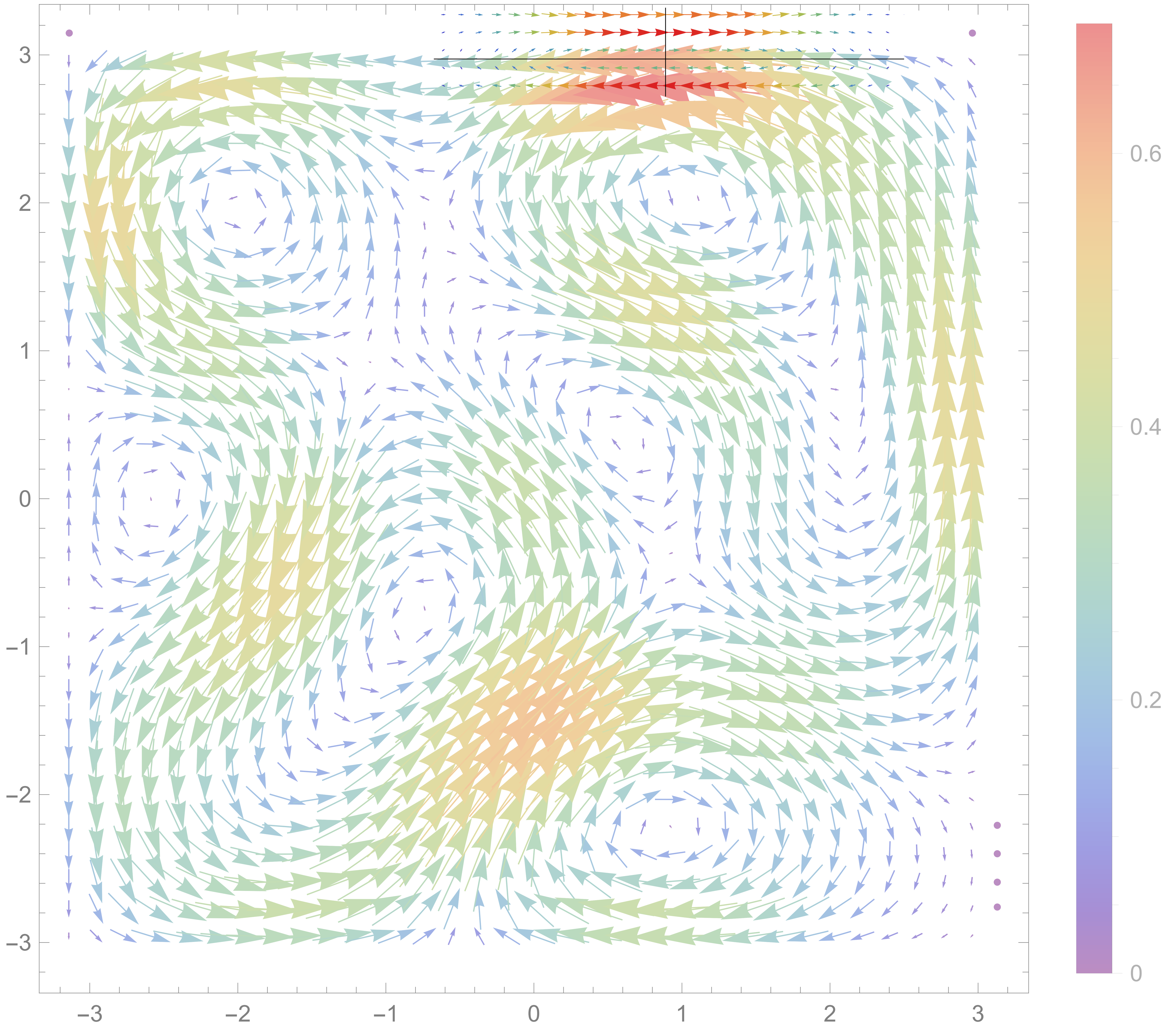
$$\hat{\vec{\psi}}(\xi)$$



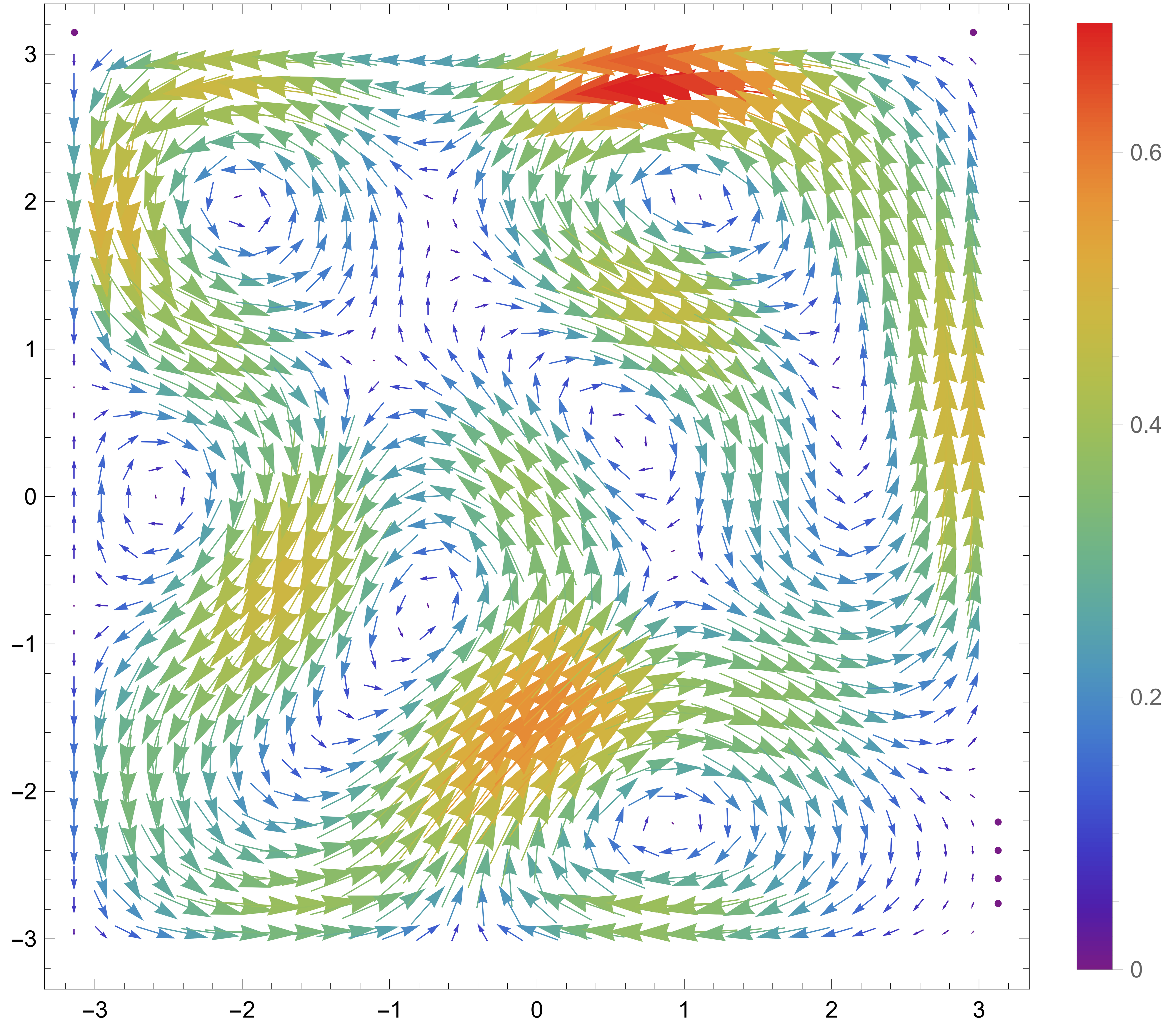
Divergence free wavelets



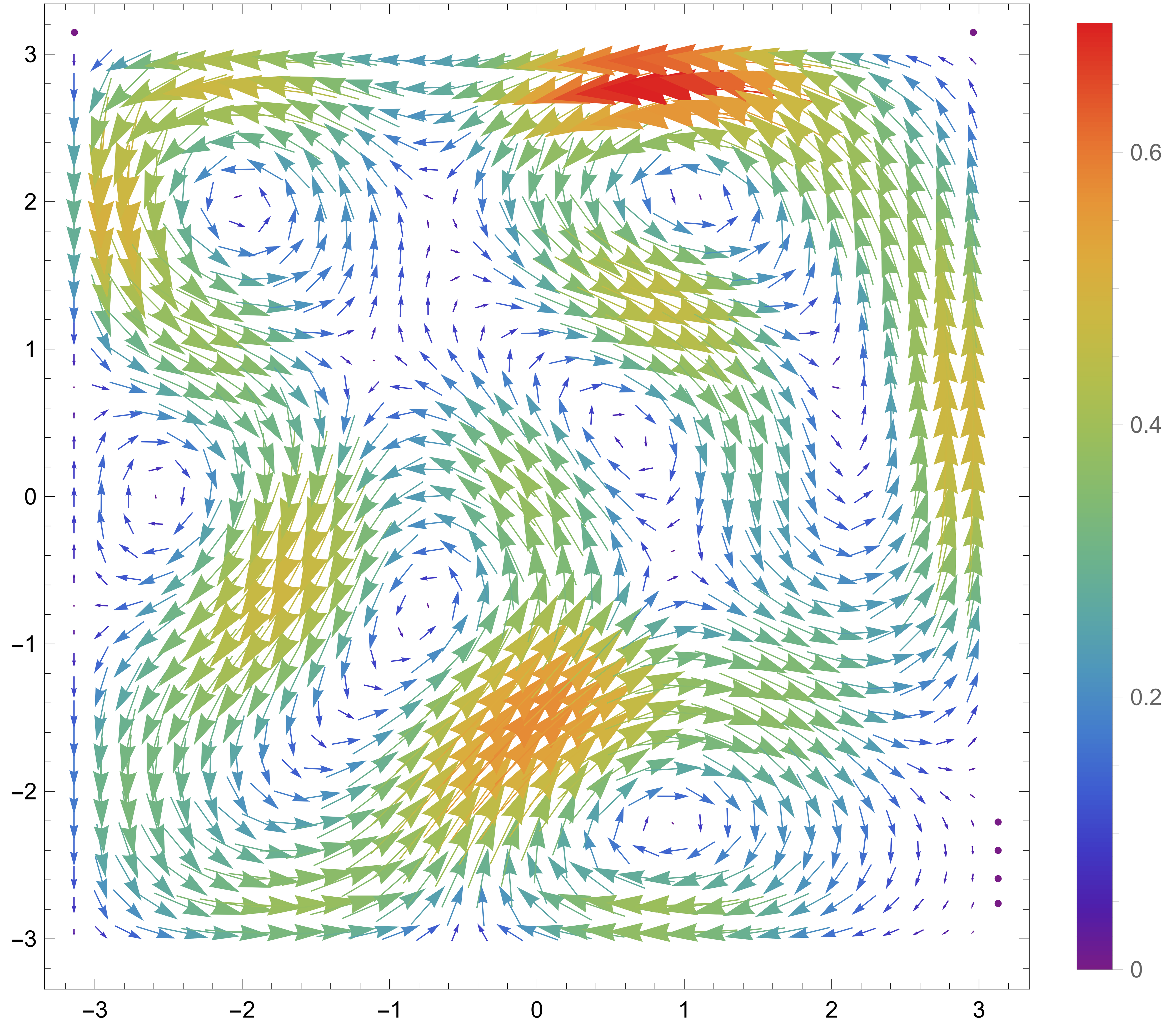
Divergence free wavelets

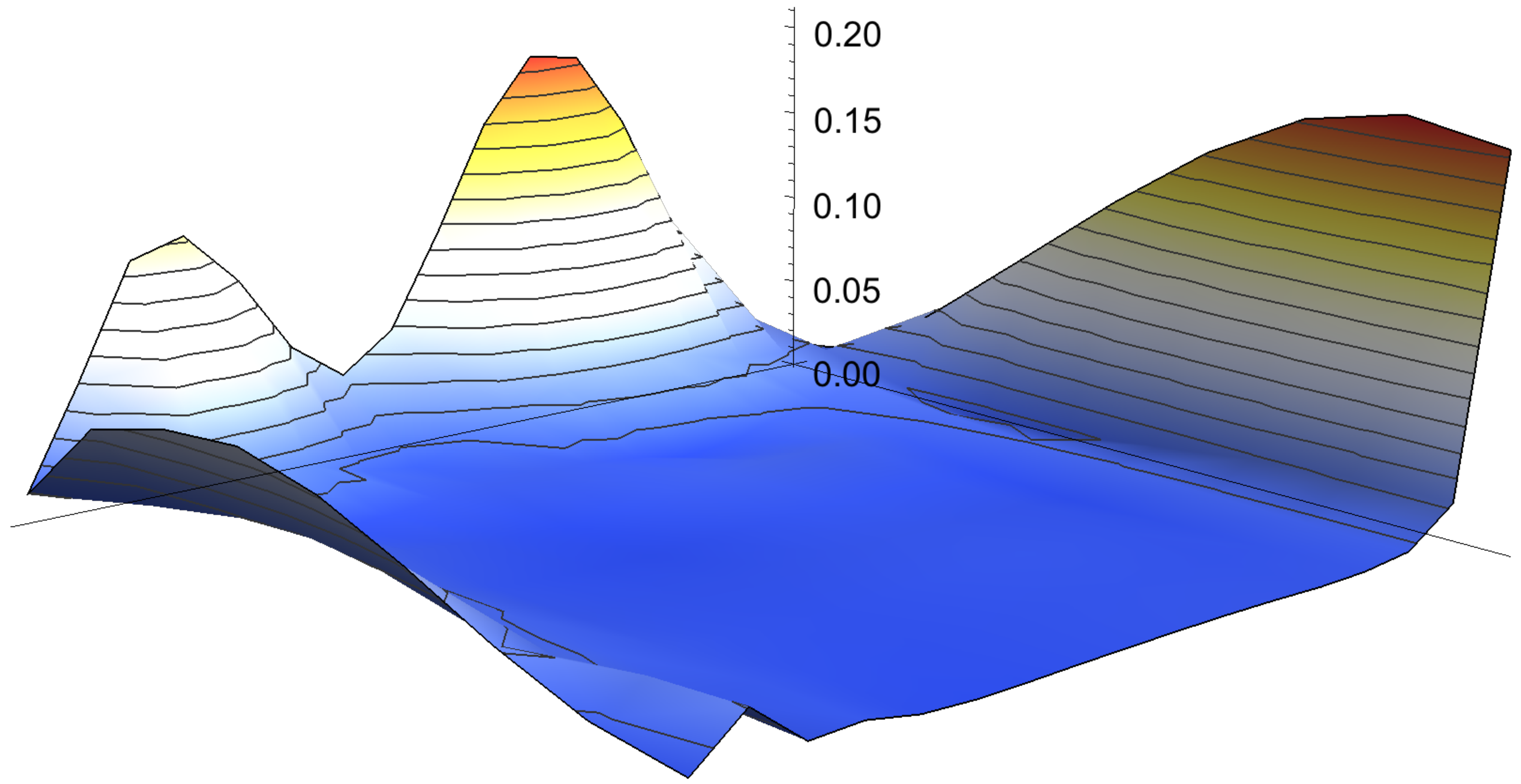


Divergence free wavelets



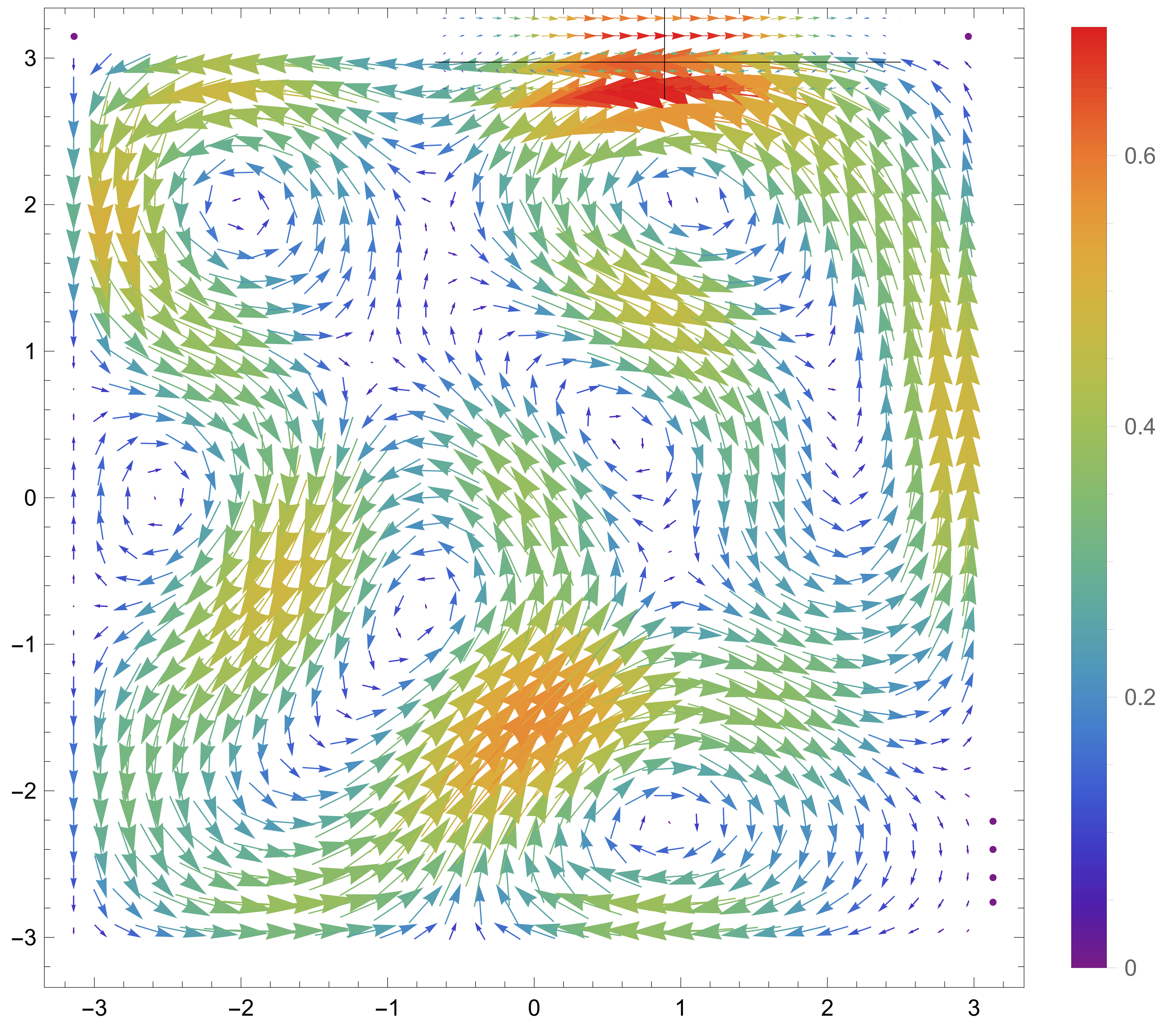
Divergence free wavelets

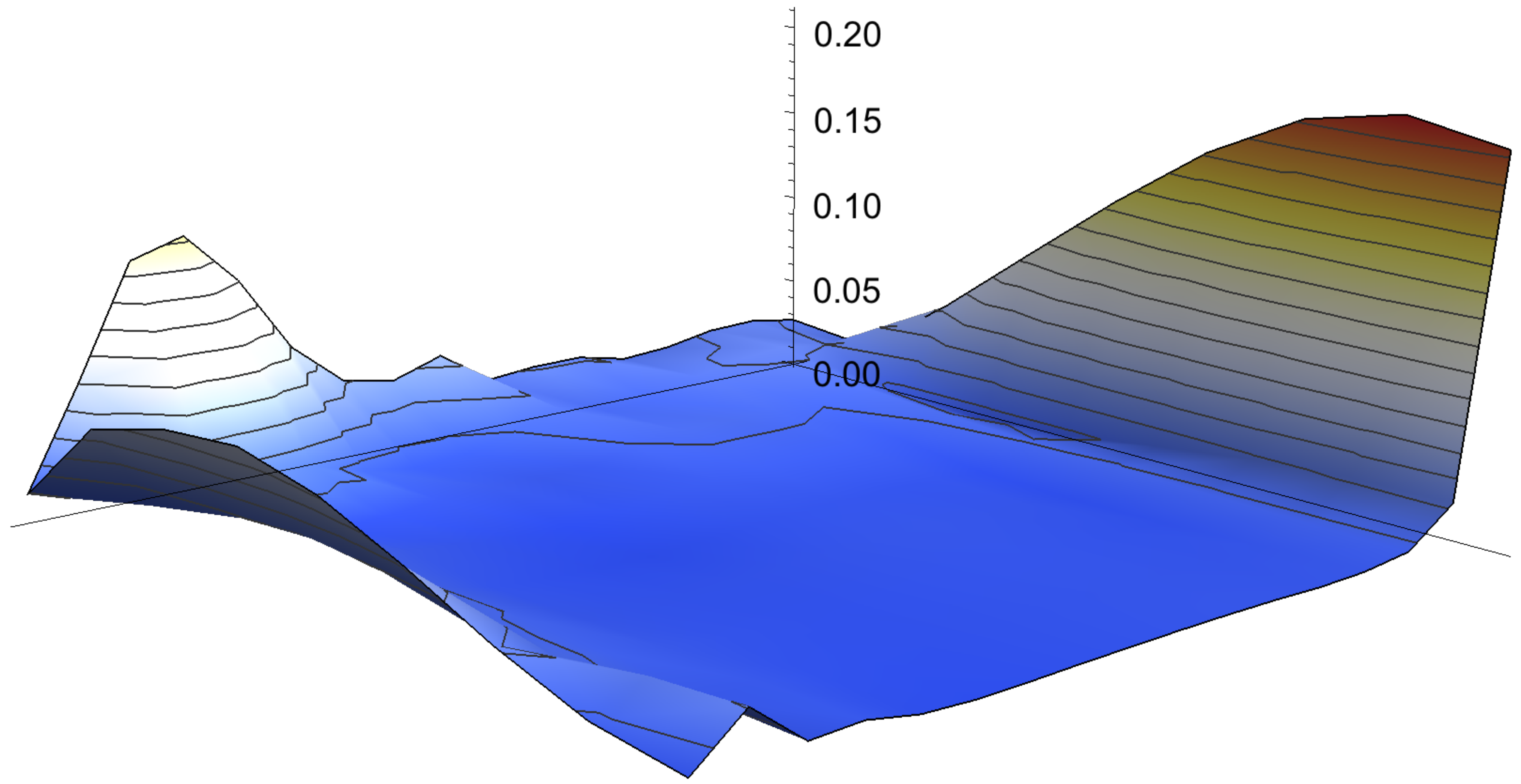




error for upper half without directional wavelets

Divergence free wavelets

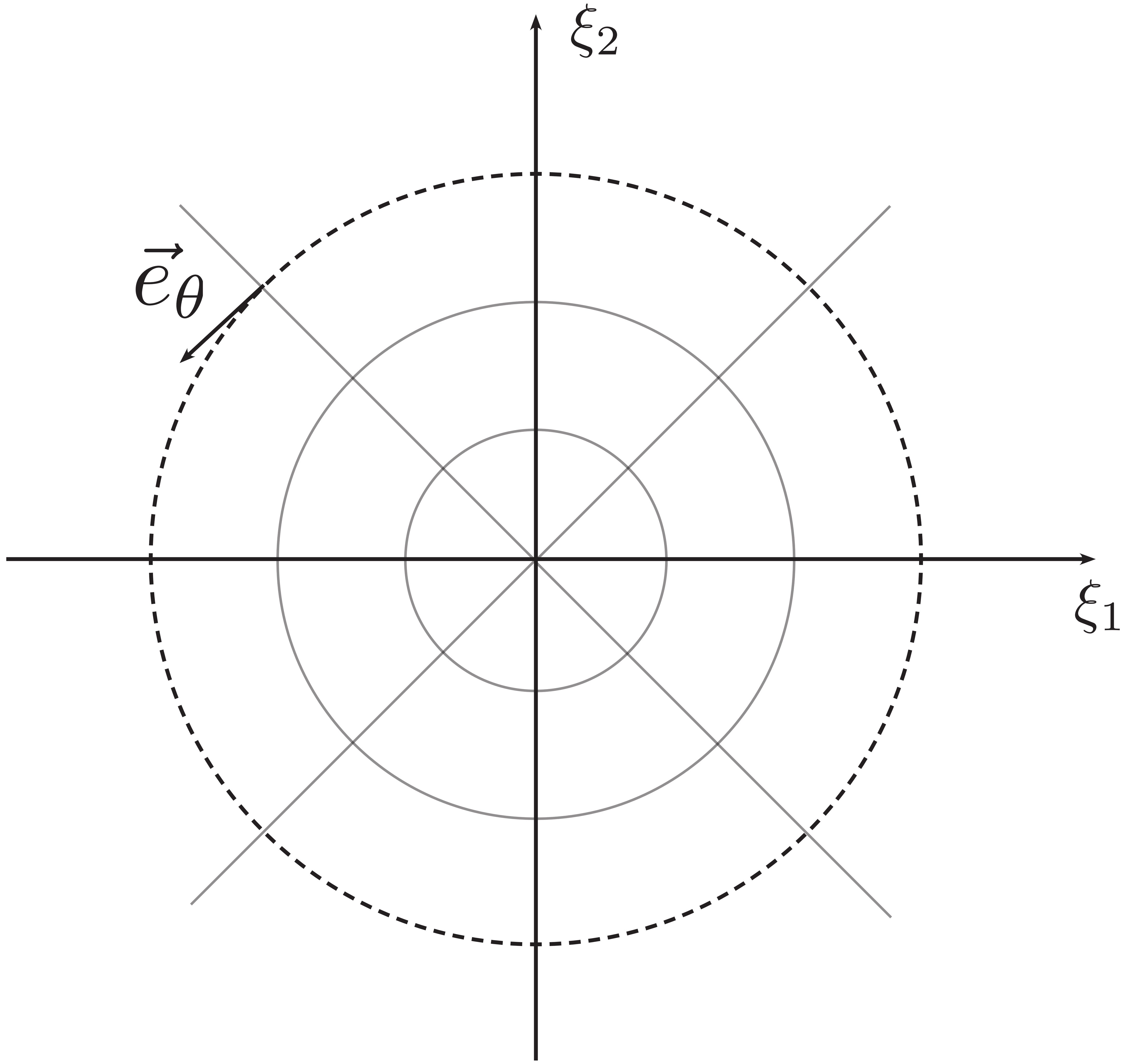




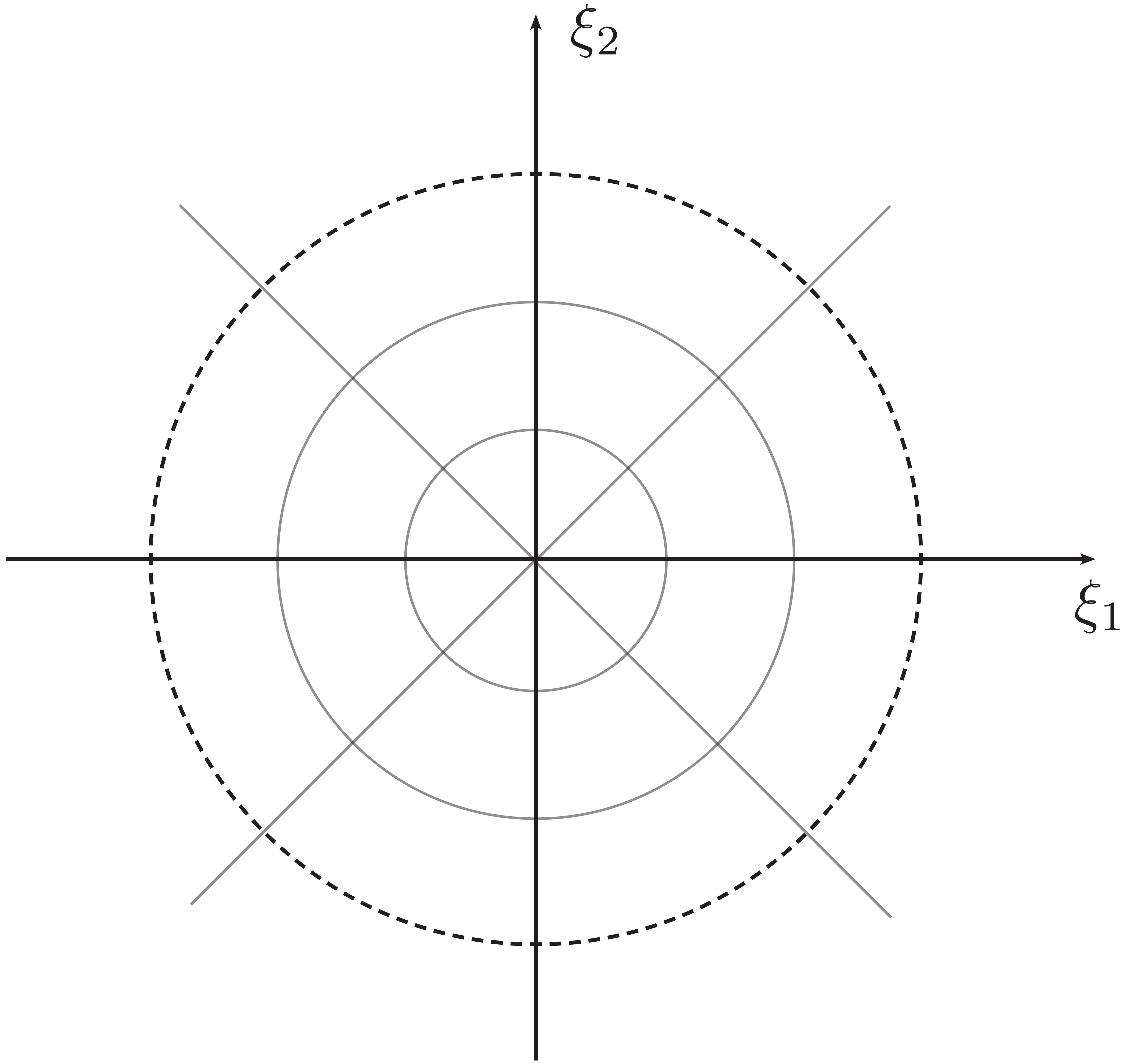
error for upper half with directional wavelets

Divergence free wavelets

$$\hat{\vec{\psi}}(\xi) = -i\,\hat{\gamma}(\theta)\,\hat{h}(|\xi|)\,\vec{e}_\theta$$

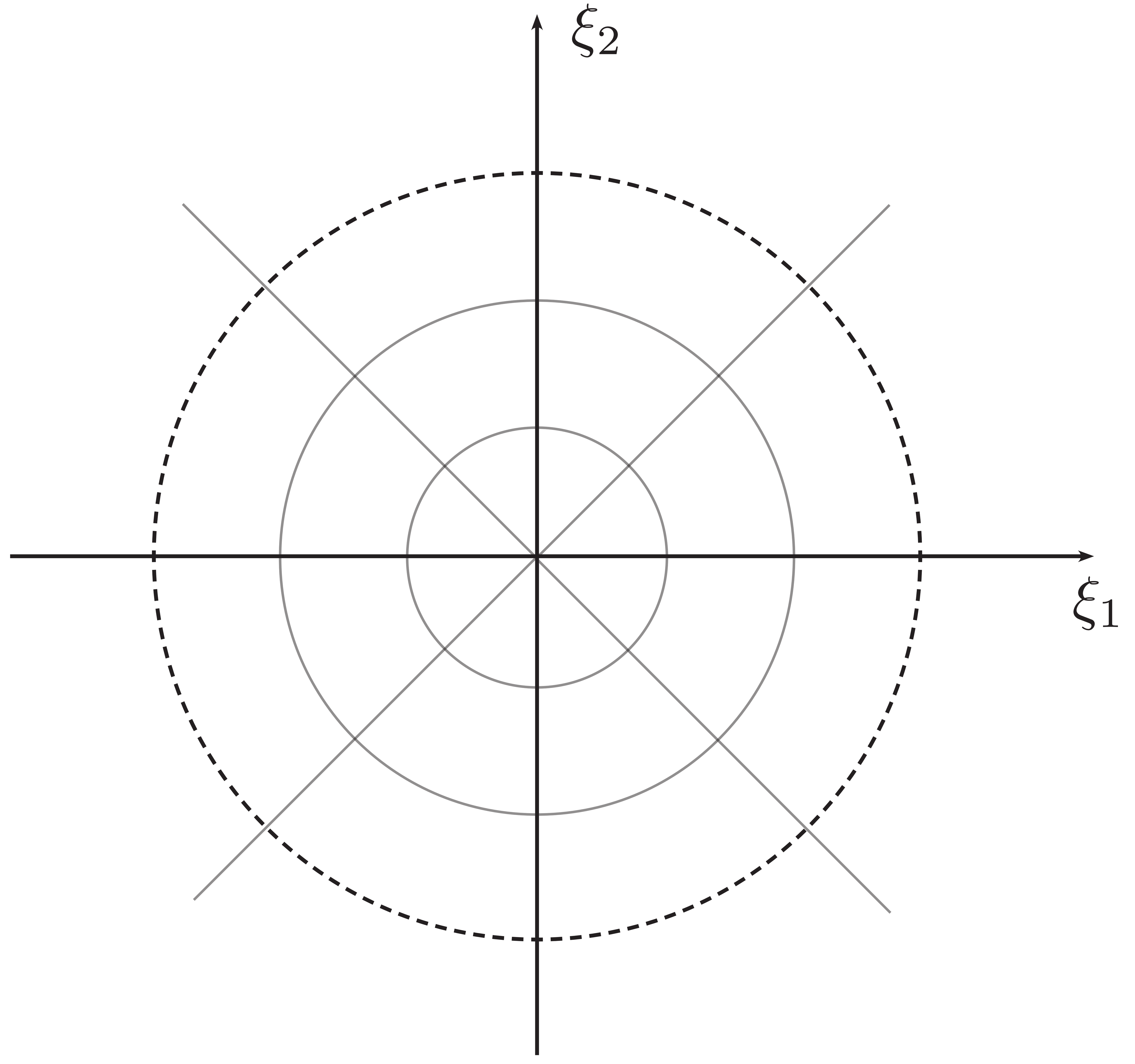


A more general perspective



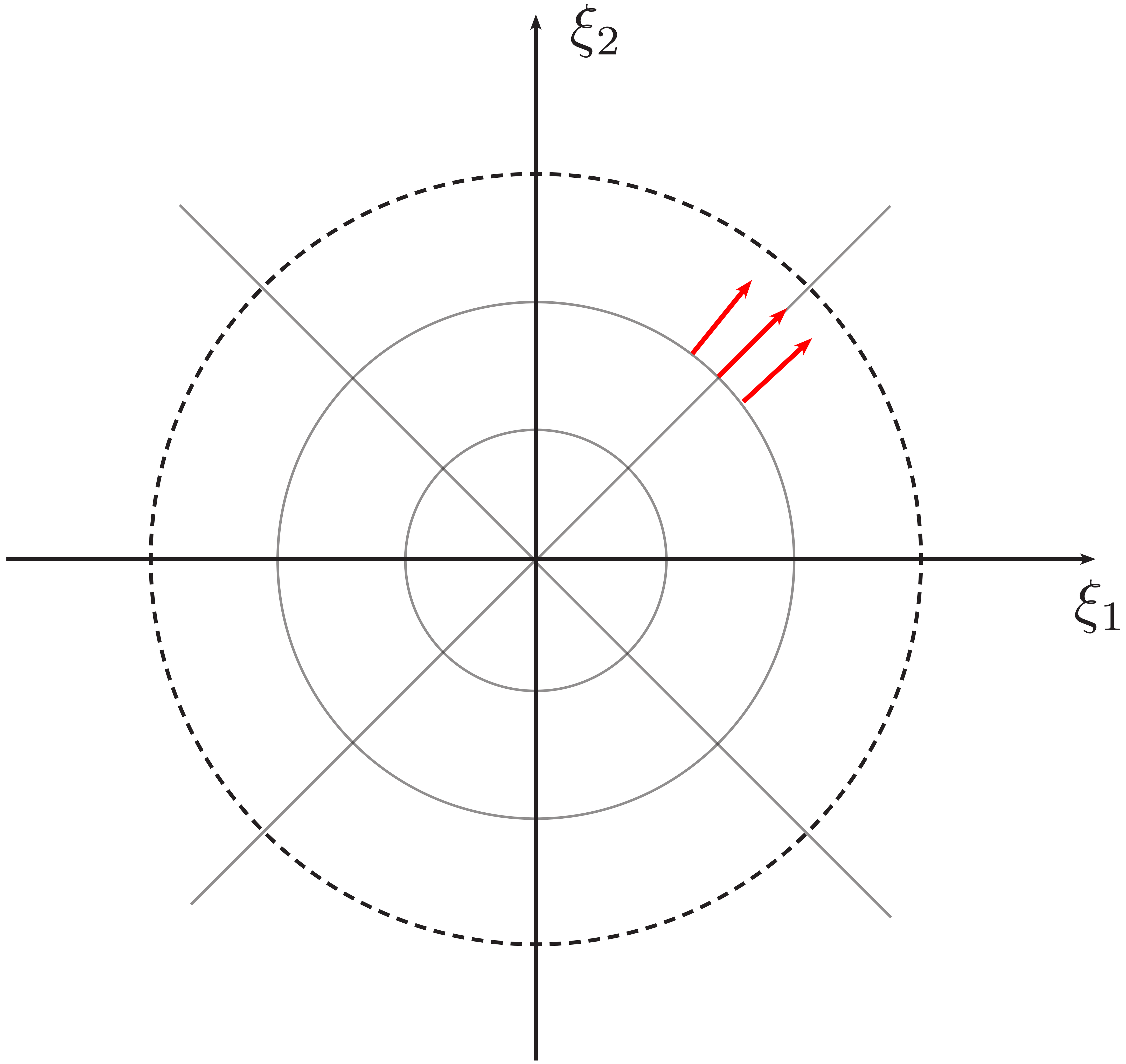
A more general perspective

∇f



A more general perspective

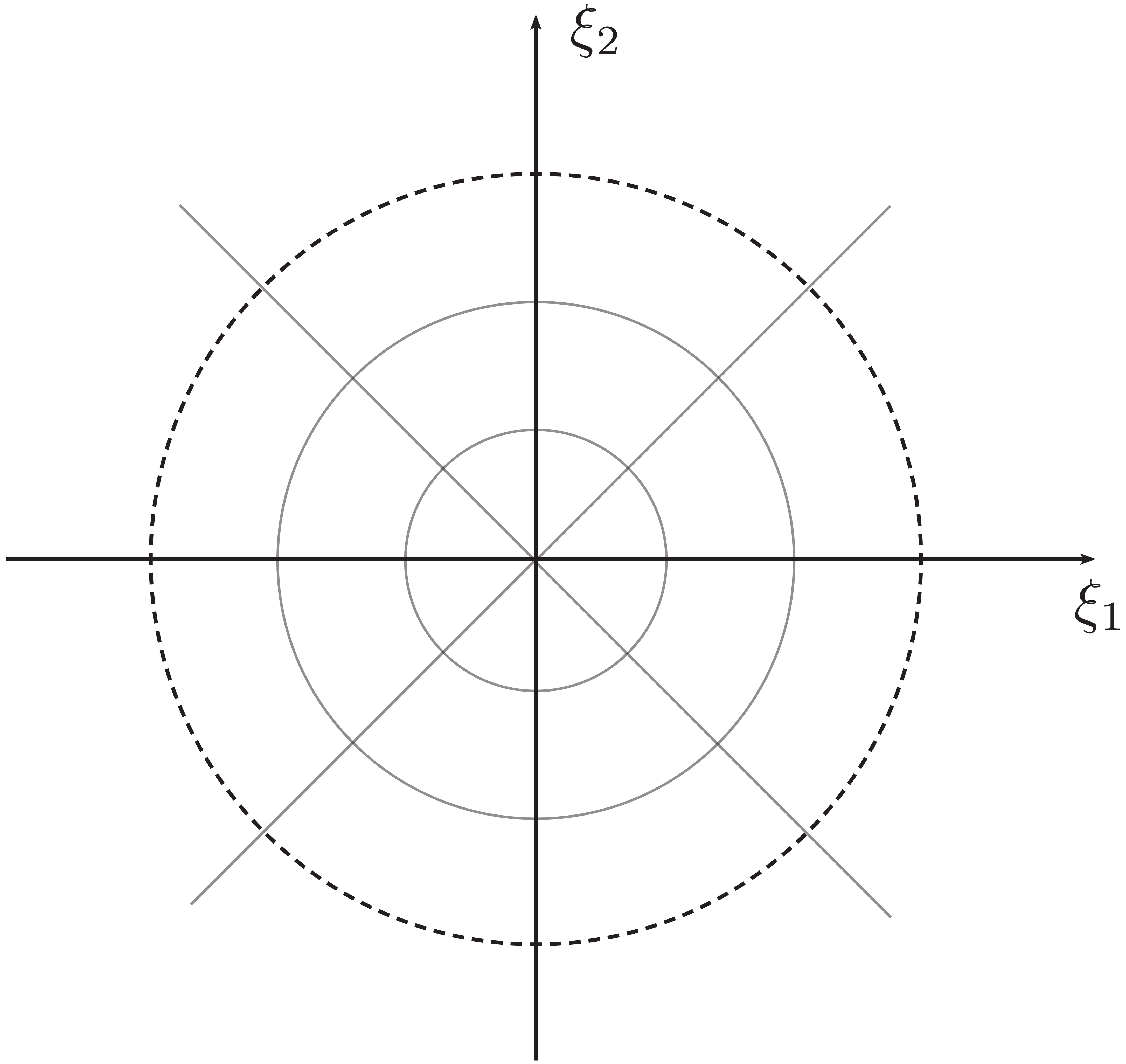
$$\widehat{\nabla f}(\xi) = -i\hat{f}(\xi)\,\xi$$



A more general perspective

$$\widehat{\nabla f}(\xi) = -i\hat{f}(\xi)\,\xi$$

$$\nabla \times \vec{v}$$



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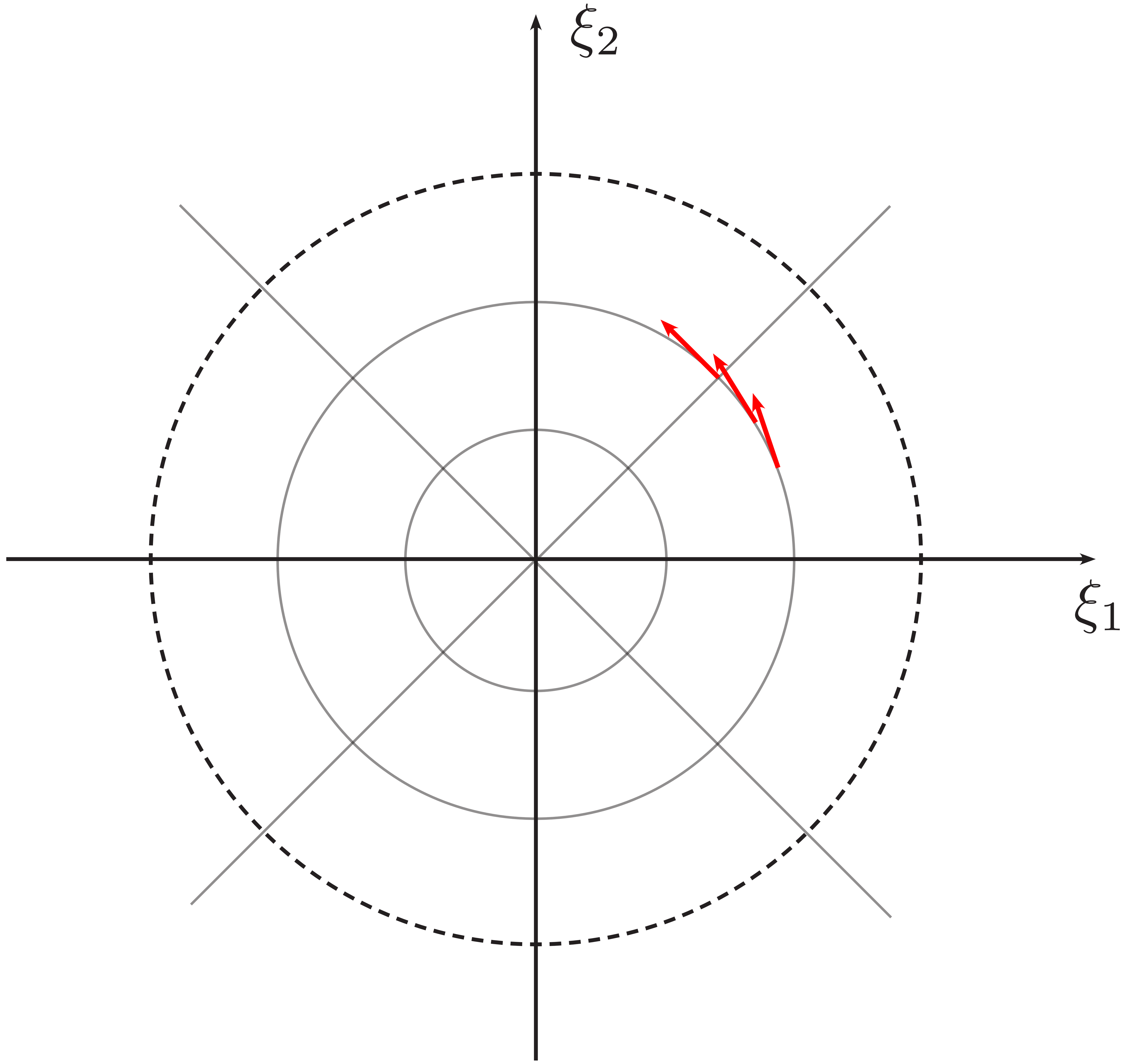
$$\widehat{\nabla f}(\xi) = -i\hat{f}(\xi)\,\xi$$

$$\widehat{\nabla \times \vec{v}}(\xi) = i\xi \times \hat{\vec{v}}(\xi)$$

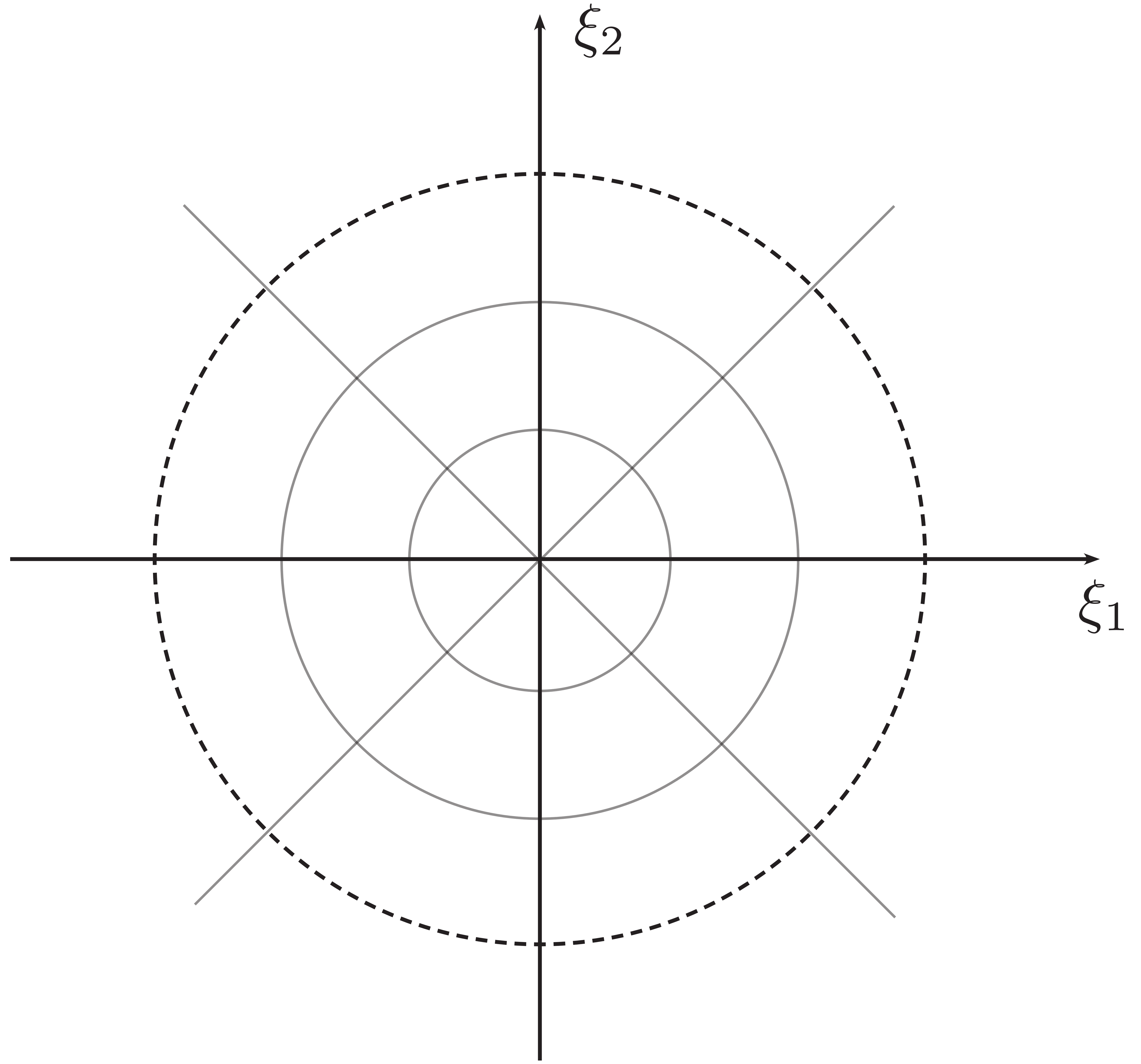


A more general perspective

$$\widehat{\nabla f}(\xi) = -i\hat{f}(\xi)\,\xi$$

$$\widehat{\nabla \times \vec{v}}(\xi) = i\xi \times \hat{\vec{v}}(\xi)$$

$$\nabla \cdot \vec{v}(\xi)$$



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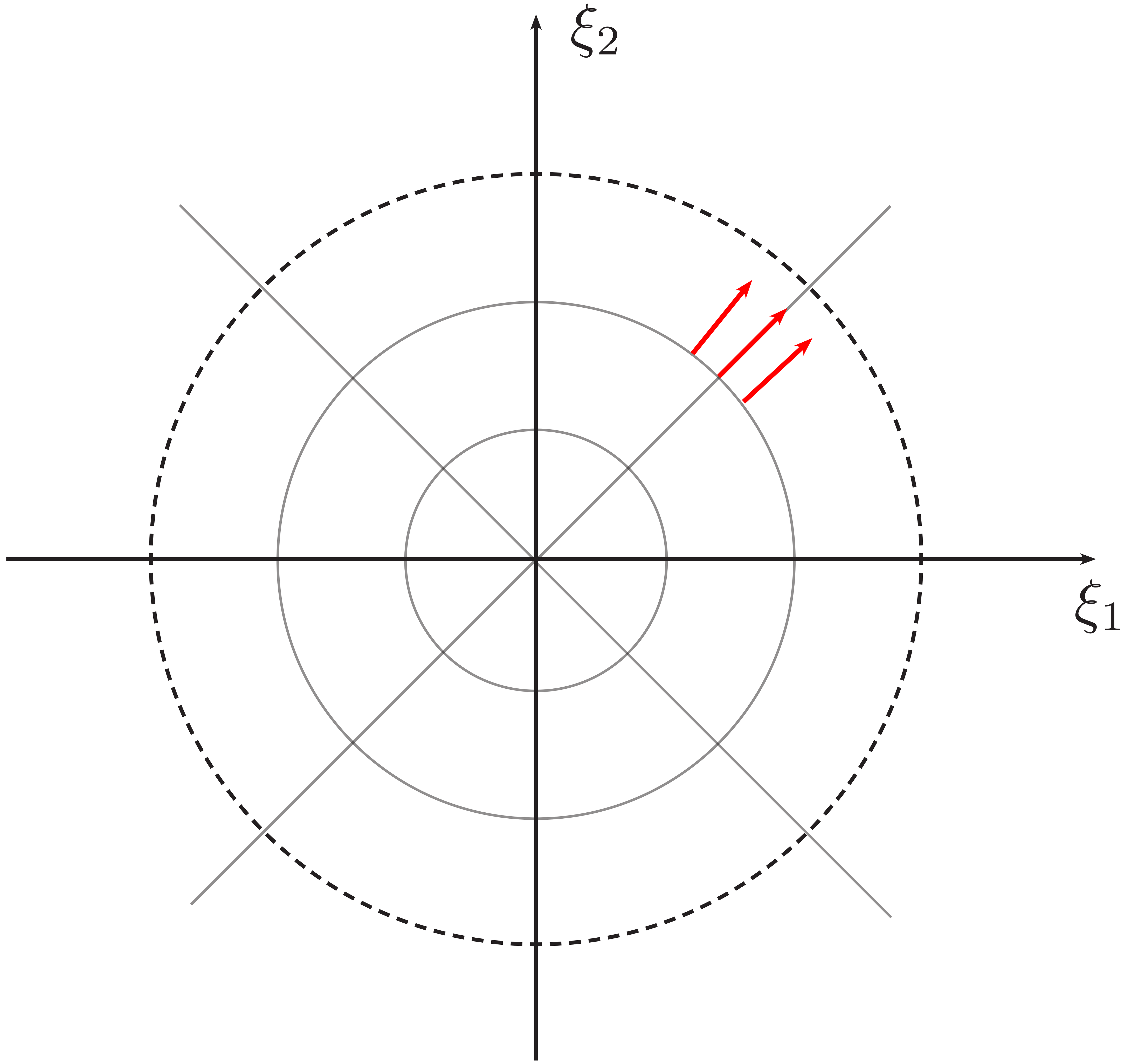

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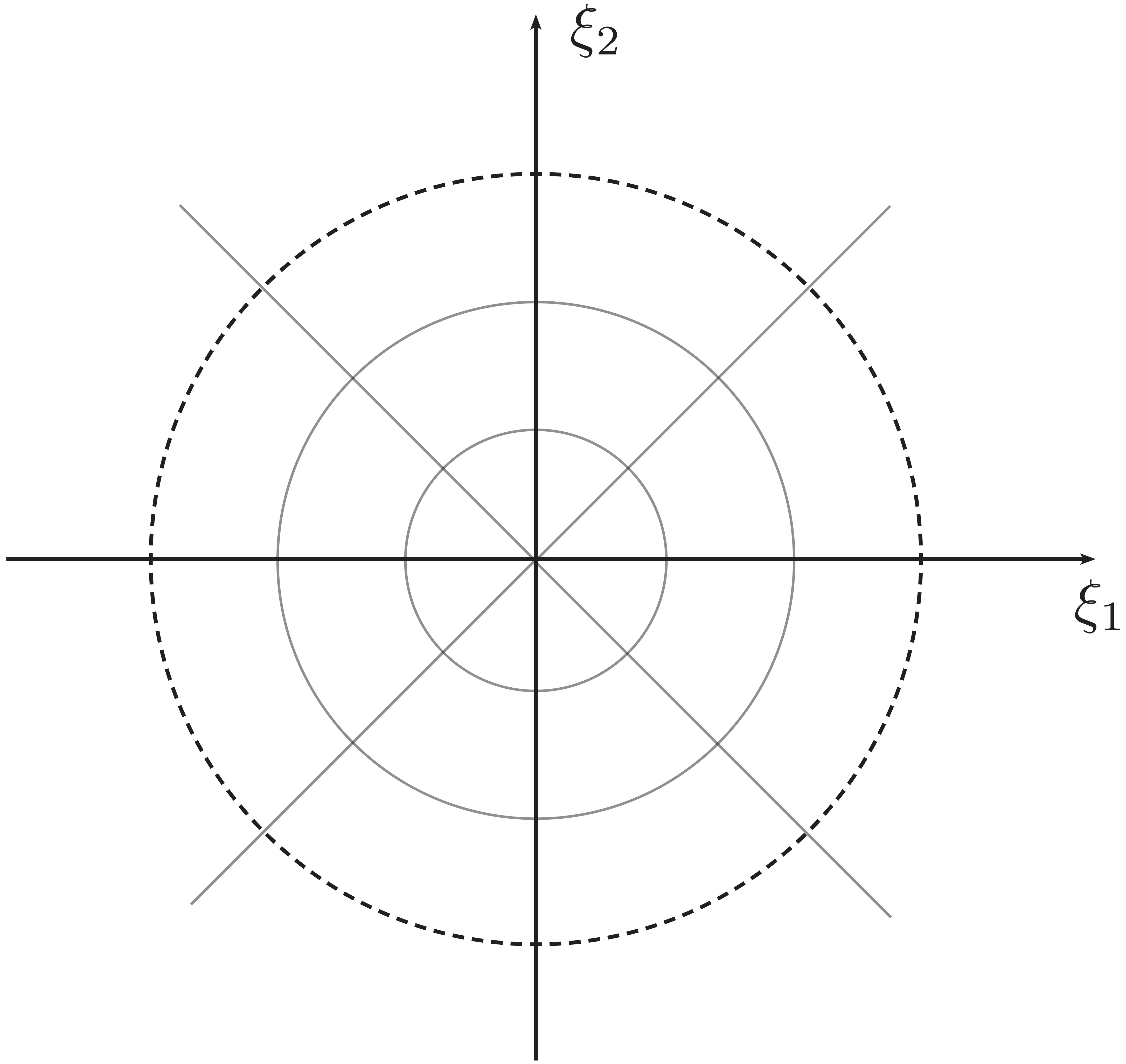
$$\Delta f(\xi)$$



A more general perspective

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$$\widehat{\nabla \cdot \vec{v}}(\xi) = \xi \cdot \hat{\vec{v}}(\xi)$$

$$\widehat{\Delta f}(\xi) = -|\xi|^2 \hat{f}(\xi)$$

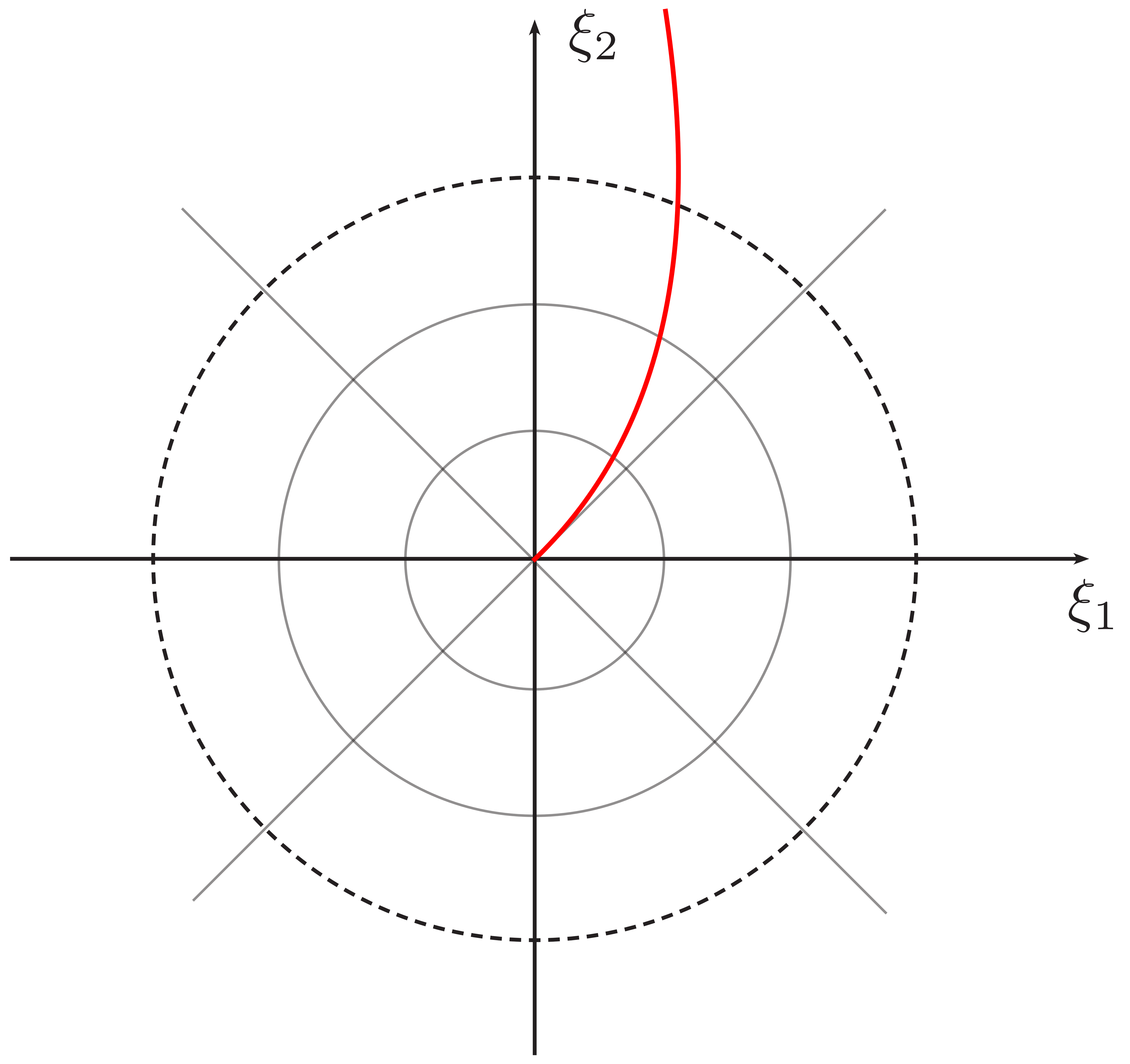

Image generation

Model: Hamza Cheggour

Image generation

Model: Hamza Cheggour

Image generation

- Classical perspective:
  - Geometric optics with transport of energy along rays

Image generation

Model: Hamza Cheggour

Image generation

Image generation

Image generation

Image generation

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Image generation

Model: Hamza Cheggour

Image generation

- Classical perspective:
  - Geometric optics with transport of energy along rays

Image generation

- Classical perspective:
  - Geometric optics with transport of energy along rays
  - Image as integral over all possible paths
  - Computed using Monte Carlo integration

Image generation

Build more effective techniques by
exploiting the system's structure

Image generation

Build more effective techniques by exploiting the system's structure

geometry

regularity

Image generation

Model: Hamza Cheggour

Image generation

Build more effective techniques by
exploiting the system's structure

geometry

regularity

Light transport theory

Maxwell's equations

$$\frac{\partial}{\partial t}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\epsilon}\nabla\times \\ \frac{1}{\mu}\nabla\times & 0 \end{pmatrix}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}$$

Light transport theory

Maxwell's equations

$$\frac{\partial}{\partial t}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\epsilon}\nabla\times \\ \frac{1}{\mu}\nabla\times & 0 \end{pmatrix}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}$$

$$\Big\downarrow \text{ short wavelength asymptotics}$$

Light transport equation

$$\dot{\ell} = -\{\ell, H\}$$

Light transport theory

Maxwell's equations

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Light transport equation

$$\dot{\ell} = -\{\ell, H\} \qquad \underbrace{\ell \in L_2(T^*\mathbb{R}^3)}_{\text{light field}}$$

Light transport theory

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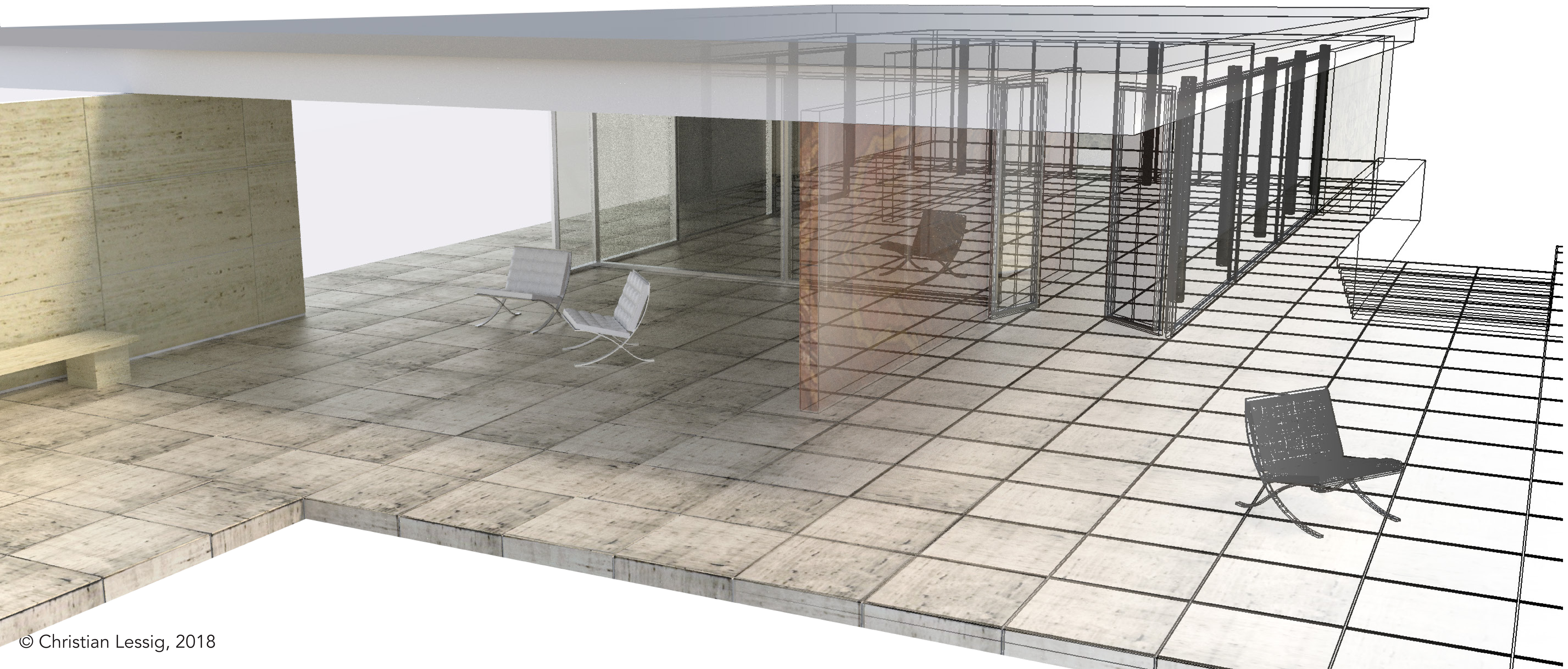
Light transport equation

$$\underbrace{\dot{\ell} = -\{\ell, H\}}_{\text{Hamiltonian p.d.e}} \qquad \underbrace{\ell \in L_2(T^*\mathbb{R}^3)}_{\text{light field}}$$

Light transport theory

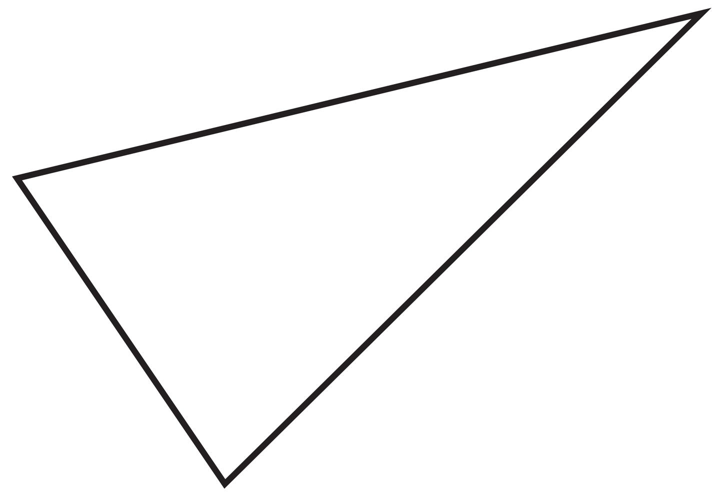
Theorem: The number of samples required to represent the light field is invariant under transport.

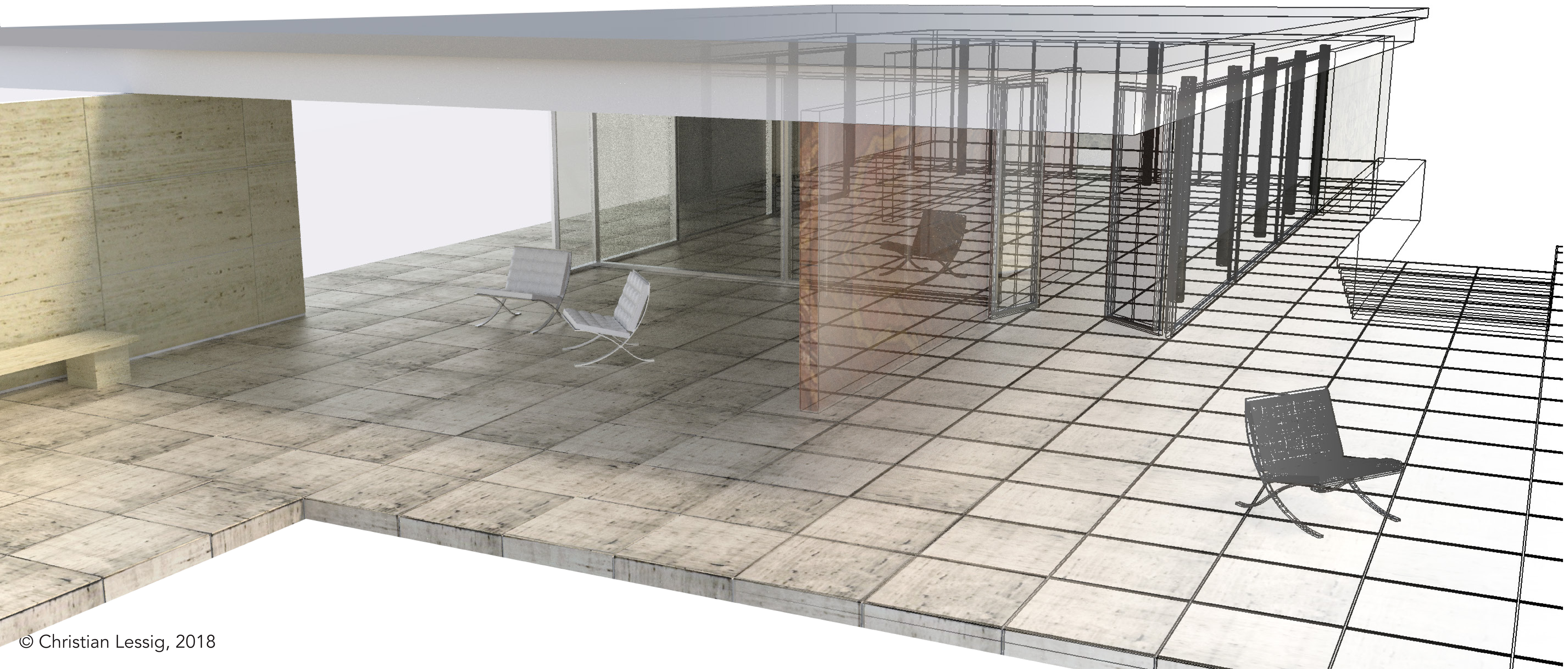
Light transport theory



lel: Hamza Cheggour

Light transport theory





Light transport theory

lel: Hamza Cheggour

Light transport theory

Theorem: The number of samples required to represent the light field is invariant under transport.

Light transport theory

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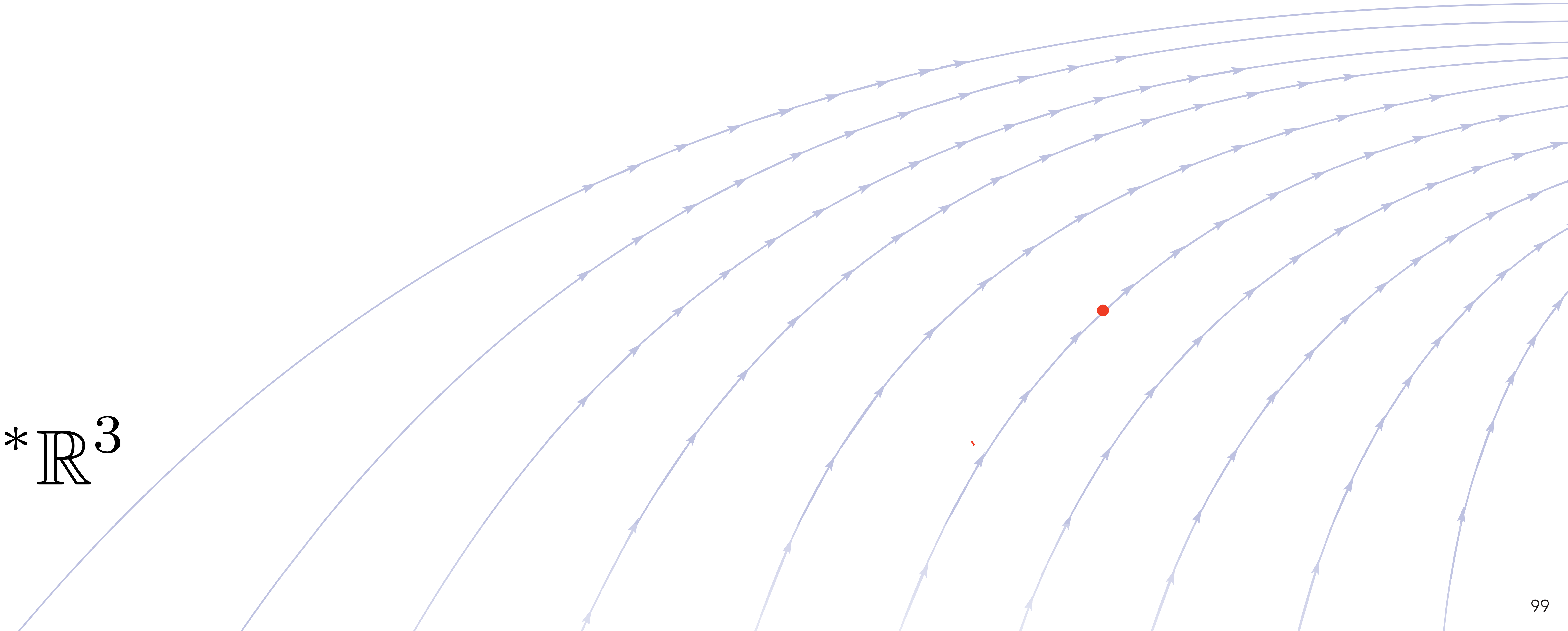
$T^*\mathbb{R}^3$



Light transport theory

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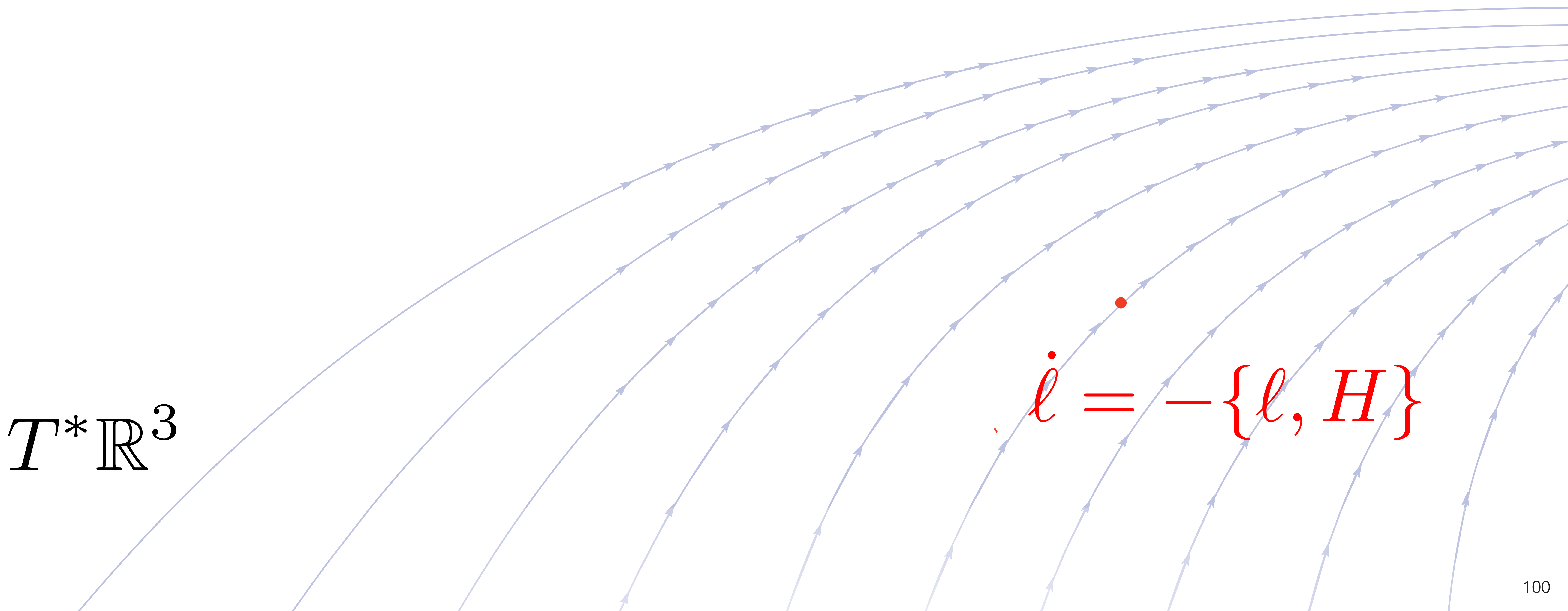

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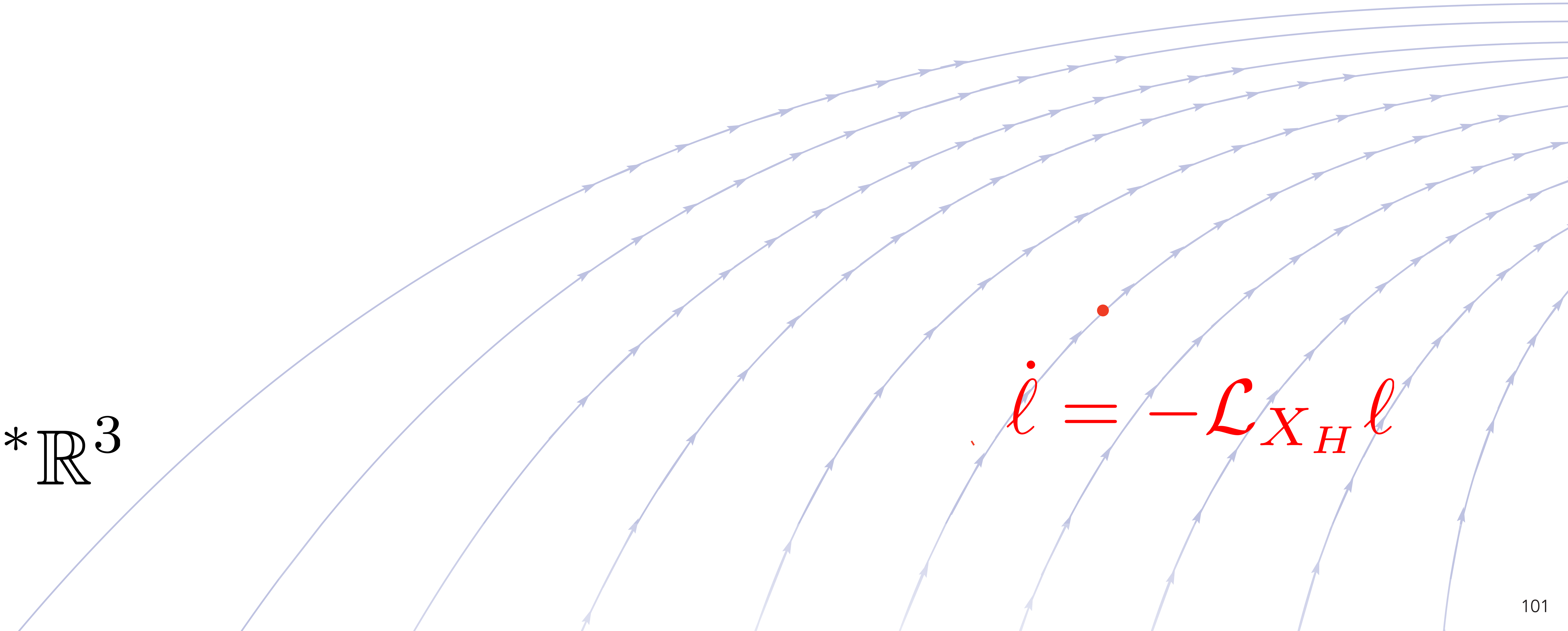
Theorem: The number of samples required to represent the light field is invariant under transport.

$T^*\mathbb{R}^3$



Light transport theory

- Let H be a Hilbert space with

$$f = \sum_{i \in \mathcal{I}} \langle f, \psi_i \rangle \, \psi_i = \sum_{i \in \mathcal{I}} f_i \, \psi_i$$

Light transport theory

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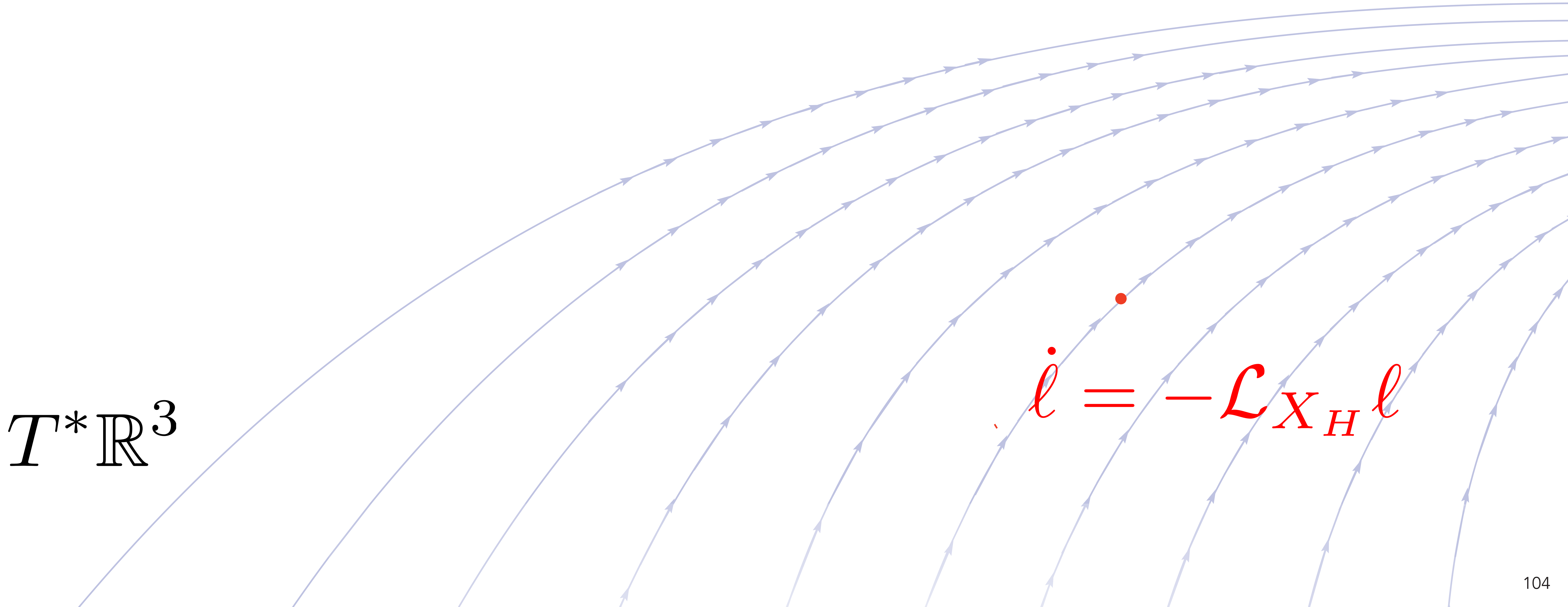
$$f = \sum_{i \in \mathcal{I}} \langle f, \psi_i \rangle \, \psi_i = \sum_{i \in \mathcal{I}} f_i \, \psi_i$$

Then if the point evaluation function δ_x is continuous also

$$f = \sum_{i \in \mathcal{I}} \langle f(y), k_{x_i}(y) \rangle \, \tilde{k}_i(x) = \sum_{i \in \mathcal{I}} f(x_i) \, \tilde{k}_i(x)$$

Light transport theory

Theorem: The number of samples required to represent the light field is invariant under transport.



Light transport theory

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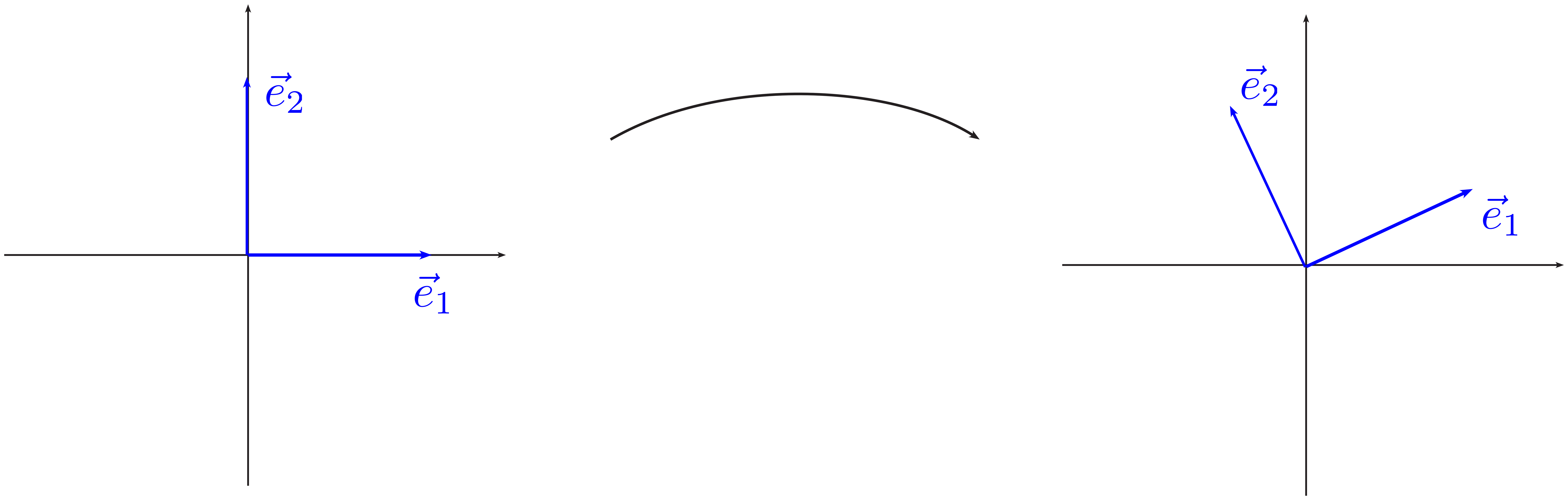

Light transport theory

From samples to images

- Task: reconstruct image from samples $\ell(x_i)$
  - Where should we place samples?
  - How to perform reconstruction?

$\Longrightarrow$ Regularity analysis

From samples to images

- If we know the (local) function space we can reconstruct from samples using a change of basis from a reproducing kernel basis (which is linear)
- We can bound the function space using regularity analysis and the invariance of the sample count

Thank you!

Many thanks to
Eugene Fiume, Marc Alexa, Mathieu Desbrun,
Philipp Herholz, Michal Jarzabek