Covariance of the Exterior Derivative

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The covariance of the exterior derivative is one of the central tenets of exterior calculus. The following note details how this requires the anti-symmetry of the covariant tensors involved.[1]

Setup. Let $\varphi : V \to V$ be a coordinate transformation of the linear space V with $\dim(V) = n < \infty$, i.e. φ is at least C^1 and has a C^1 inverse (when φ is the transition map between charts then the argument generalizes to manifolds). We will typically write $y = \varphi(x)$. The tangent map $T\varphi$ is in coordinates given by

$$J_i^j(x) = \frac{\partial \varphi^j(x)}{\partial x^i} = \frac{\partial y^j(x)}{\partial x^i} \tag{1}$$

and it provides the push-forward for vectors. For co-vectors the push-forward is in coordinates given by

$$\bar{J}_j^i(y(x)) = \frac{\partial (\varphi^{-1})^i(y(x))}{\partial y^j} = \frac{\partial x^i(y(x))}{\partial y^j}, \tag{2}$$

i.e. by the Jacobian of the inverse φ^{-1}, which, by the inverse function theorem, equals the inverse of $J_i^j(x)$.

The exterior derivative for functions. For functions, the exterior derivative, or co-differential, is defined as

$$\mathrm{d}f = \frac{\partial f}{\partial x^1} \mathrm{d}x^1 + \cdots + \frac{\partial f}{\partial x^n} \mathrm{d}x^n. \tag{3}$$

It arises naturally when one considers the infinitesimal transport of f along a flow η_t generated by a vector field X,

$$\frac{d}{dt}\left(\eta_{-t}^* f\right)(y) = \frac{d}{dt} f(\eta_{-t}(y)) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \underbrace{\frac{\partial x^i}{\partial t}}_{-X^i} \tag{4}$$

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[1]The argument is based on those by Mukhi and Mukunda [MM10, p.58].

where we used that the inverse flow $\eta_{-t} = (\eta^{-1})_t$ is generated by the "inverse" vector field $-X$. For Eq. 4 to be covariant, the pairing between the vector components $-X^i$ and the $\partial f/\partial x^i$ has to be coordinate independent, which holds exactly when the $\partial f/\partial x^i$ are the components of the co-vector in Eq. 3.

For Eq. 3, covariance should also hold in the sense that $\mathrm{d} \circ \varphi = \varphi \circ \mathrm{d}$, i.e. exterior derivative and coordinate transformations commute so that first computing the exterior derivative and then performing the coordinate transformation yields the same result as first changing coordinates and then applying d. For functions, the coordinate transformation is given by

$$\bar{f}(y) = f\big(\varphi^{-1}(y)\big) = (\varphi^{-1})^* f. \tag{5}$$

For the pairing $(\mathrm{d}f)(X) =$ to be covariant for some vector field $X \in \mathfrak{X}(V)$, we have to have

$$\frac{\partial f(x)}{\partial x^i} X^i(x) = \frac{\partial \bar{f}(y)}{\partial y^i} \bar{X}^i(y) \tag{6}$$

where $\bar{X}^i$ denotes the coordinates of X in the new coordinate system. By Eq. 1, these are given by

$$\bar{X}^i(y) = J_j^i(x) X^j(x). \tag{7}$$

For the right hand side of Eq. 6 we thus have

$$\frac{\partial \bar{f}(y)}{\partial y^i} \bar{X}^i(y) = \frac{\partial f\big(\varphi^{-1}(y)\big)}{\partial y^i} J_i^j(x) X^j(x) \tag{8}$$

which equals the left hand side of the equation only if the Jacobian $J_i^j(x)$ is cancelled. But writing

$$\frac{\partial f\big(\varphi^{-1}(y)\big)}{\partial y^i} = \frac{\partial}{\partial y^i} f\big(\varphi^{-1}(y)\big) \tag{9}$$

and using that $x = \varphi^{-1}(y)$ and Eq. 2 we have

$$\frac{\partial}{\partial y^i} f\big(\varphi^{-1}(y)\big) = \frac{\partial f(x)}{\partial x^n} \bar{J}_i^n(\varphi(x)) = \left(\bar{J}_i^n(\varphi(x)) \frac{\partial}{\partial x^n} \right) f(x). \tag{10}$$

Covariance thus indeed holds. The last equation can furthermore be interpreted in that $\partial/\partial x^n$ transforms like a co-vector, i.e.

$$\frac{\partial}{\partial y^i} = \bar{J}_i^n(\varphi(x)) \frac{\partial}{\partial x^n}. \tag{11}$$

The exterior derivative for co-vectors. Let $\alpha \in \mathfrak{X}^*(V)$ be a co-vector field. We are interested in the exterior derivative

$$\mathrm{d}\alpha \in \mathcal{T}_2^0(V) \tag{12}$$

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that is yet to be defined. By linearity (as any derivation, i.e. derivative-like operator, d has to be linear), d acts on the coordinate functions $\alpha_i(x)$ and then it has to match the exterior derivative for functions. This yields

$$\frac{\partial \alpha_1(x)}{\partial x^1}\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \cdots \frac{\partial \alpha_1(x)}{\partial x^n}\mathrm{d}x^1 \otimes \mathrm{d}x^n + \frac{\partial \alpha_2(x)}{\partial x^1}\mathrm{d}x^2 \otimes \mathrm{d}x^1 + \cdots \quad (13)$$

For our argument it suffices to consider an arbitrary term from the above expression. Evidently, for the exterior derivative to be well defined it has to be irrelevant in which coordinate frame it is computed. Thus

$$\mathrm{d}\alpha(X, Y) \quad (14)$$

for two arbitrary vector fields X and Y has to be the same if computed before or after applying the coordinate transformation φ, analogous to what we saw before for the exterior derivative of functions. We note that by Eq. 1 the coordinate transformations for X and Y are given by

$$\bar{X}^i = J_a^i X^a \qquad Y^j = J_c^j Y^c \quad (15)$$

Analogous to Eq. 8, the two Jacobians above hence have to be cancelled by the transformation law for $\mathrm{d}\alpha$ for covariance to hold.

Let $\bar{\alpha}_i$ denote the coordinates of α after the coordinate transformation. Then we have for the derivative of the i-th term of $\bar{\alpha}$ with respect to y^j that

$$\frac{\partial}{\partial y^j}\bar{\alpha}_i = \left(\bar{J}_j^n(y(x)) \frac{\partial}{\partial x^n} \right) \left(\bar{J}_i^k(y(x)) \bar{\alpha}_k \right) \quad (16)$$

where, by Eq. 11, $\partial/\partial x^n$ on the right transform likes a co-vector.

The derivative $\partial/\partial x^n$ acts on the product $\bar{J}_i^k(y(x)) \bar{\alpha}_k$. By the Leibniz rule we hence have

$$\frac{\partial}{\partial y^j}\bar{\alpha}_i = \bar{J}_j^n(y(x))\bar{J}_i^k(y(x))\frac{\partial \alpha_k}{\partial x^n} + \bar{J}_j^n(y(x))\left(\frac{\partial}{\partial x^n} \frac{\partial x^k(y(x))}{\partial y^i} \right)\alpha_k \quad (17)$$

where we used the definition of $\bar{J}_i^k(y(x))$. Computing the derivative in the second term by using the chain to resolve the x dependency in $\partial x^i(y(x))/\partial y^k$ we obtain

$$\frac{\partial}{\partial y^j}\bar{\alpha}_i = \bar{J}_j^n(y(x))\bar{J}_i^k(y(x))\frac{\partial \alpha_k}{\partial x^n} + \bar{J}_j^n(y(x))\left(\frac{\partial x^k(y(x))}{\partial y^i \partial y^m} \frac{\partial y^m}{\partial x^n} \right)\alpha_k \quad (18)$$

which equals

$$\frac{\partial}{\partial y^j}\bar{\alpha}_i = \bar{J}_j^n(y(x))\bar{J}_i^k(y(x))\frac{\partial \alpha_k}{\partial x^n} + \bar{J}_j^n(y(x))\frac{\partial x^k(y(x))}{\partial y^i \partial y^m} J_n^m(x)\,\alpha_k. \quad (19)$$

Since $\bar{J}_j^n(y(x))$ and $J_n^m(x)$ are Jacobians for φ and its inverse φ^{-1}, also the Jacobians are the inverses of each other, i.e.

$$\bar{J}_j^n(y(x))\, J_n^m(x) = \delta_j^m. \quad (20)$$

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Thus we have

$$\frac{\partial}{\partial y^j}\bar\alpha_i = \bar J^n_j(y(x))\bar J^k_i(y(x))\frac{\partial\alpha_k}{\partial x^n} + \frac{\partial x^k(y(x))}{\partial y^i \partial y^j}\,\alpha_k. \tag{21}$$

Considering now the pairing with $\bar X^i = J^i_a X^a$ and $Y^j = J^j_c Y^c$ we see that for the first term the Jacobians are indeed cancelled but this is not true for the second term. In other words, the second term does *not* transform like a tensor and the result of the pairing $d\alpha(X,Y)$ is not covariant.

To obtain a physically (i.e. for applications) relevant exterior derivative, we have to "remove" the second term that breaks covariance. Considering the term, we see that it contains the second derivative (Hessian) of the coordinate transformation, which is symmetric in i and j. We can hence eliminate it by using only the anti-symmetric part of Eq. 13. Considering again the derivative of the i-th term of $\bar\alpha$ with respect to y^j, the anti-symmetric part is given by (recall that for a matrix A the anti-symmetric part is $A - A^T$)

$$\begin{aligned}\frac{\partial}{\partial y^j}\bar\alpha_i - \frac{\partial}{\partial y^i}\bar\alpha_j &= \left(\bar J^n_j(y(x))\bar J^k_i(y(x))\frac{\partial\alpha_k}{\partial x^n} + \frac{\partial x^k(y(x))}{\partial y^i \partial y^j}\,\alpha_k\right)\\ &\quad -\left(\bar J^n_i(y(x))\bar J^k_j(y(x))\frac{\partial\alpha_k}{\partial x^n} + \frac{\partial x^k(y(x))}{\partial y^j \partial y^i}\,\alpha_k\right).\end{aligned} \tag{22}$$

Hence

$$\frac{\partial}{\partial y^j}\bar\alpha_i - \frac{\partial}{\partial y^i}\bar\alpha_j = \bar J^n_j(y(x))\bar J^k_i(y(x))\frac{\partial\alpha_k}{\partial x^n} - \bar J^n_i(y(x))\bar J^k_j(y(x))\frac{\partial\alpha_k}{\partial x^n} \tag{23}$$

which is now indeed covariant. Thus, the exterior derivative $d : \mathcal{T}^0_1(V) \to \mathcal{T}^0_2(V)$ is well defined on covariant tensors when these are anti-symmetric, i.e. on these the derivative is covariant. Such tensors hence play a distinguished role in the description of "physical" systems (albeit often in disguise) and are known as *differential forms*.

References

[MM10]   Sunil Mukhi and N Mukunda. *Lectures on advanced mathematical methods for physicists.* Singapore: World Scientific, 2010.